A note on Hardy spaces on quadratic CR manifolds

M. Calzi

Abstract. Given a quadratic CR manifold $M$ embedded in a complex space, and a holomorphic function $f$ on a tubular neighbourhood of $M$, we show that the $L^p$-norms of the restrictions of $f$ to the translates of $M$ is decreasing for the ordering induced by the closed convex envelope of the image of the Levi form of $M$.

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1. Introduction

Let $f$ be a holomorphic function on the upper half-plane $C_+ = \mathbb{R} + i\mathbb{R}_+^*$. If $f$ belongs to the Hardy space $H^p(C_+)$, that is, if $\sup_{y>0}\|f_y\|_{L^p(\mathbb{R})}$ is finite, where $f_y : x \mapsto f(x + iy)$, then it is well known that the function $y \mapsto \|f_y\|_{L^p(\mathbb{R})}$ is decreasing on $\mathbb{R}_+^*$ for every $p \in [0, \infty]$. Nonetheless, if $f$ is simply holomorphic, then the lower semicontinuous function $y \mapsto \|f_y\|_{L^p(\mathbb{R})}$ need not be decreasing. Actually, the set where it is finite may be any interval in $\mathbb{R}_+^*$, or even a disconnected set.

Now, replace the upper half-plane $C_+$ with a Siegel upper half-space

$$D := \{ (\zeta, z) \in \mathbb{C}^n \times \mathbb{C} : \text{Im} z - |\zeta|^2 > 0 \},$$

and define

$$f_h : \mathbb{C}^n \times \mathbb{R} \ni (\zeta, x) \mapsto f(\zeta, x + i|\zeta|^2 + h)$$
for every $h > 0$ and for every function $f$ on $D$. This definition is motivated by the fact that

$$bD := \{ (\xi, x + i|\xi|^2) : (\xi, x) \in \mathbb{C}^n \times \mathbb{R} \}$$

is the boundary of $D$, and the sets $bD + (0, ih)$, for $h > 0$, foliate $D$ as the sets $\mathbb{R} + iy$, for $y > 0$, foliate $\mathbb{C}_+$. If $f$ is holomorphic on $D$, then the mapping $h \mapsto ||f_h||_{L^p(\mathbb{C}^n \times \mathbb{R})}$ is always decreasing (though not necessarily finite), in contrast to the preceding case (cf. Theorem 3.1). This fact is closely related with the fact that every holomorphic function defined in a neighbourhood of $bD$ automatically extends to $D$. More precisely, if one observes that $bD$ has the structure of a CR submanifold of $\mathbb{C}^n \times \mathbb{C}$, one may actually prove that every CR function (of class $C^1$) is the boundary values of a unique holomorphic function on $D$ (cf. [2, Theorem 1 of Section 15.3]).

In this note, we show that an analogous property holds when $bD$ is replaced by a general quadratic, or quadric, CR submanifold of a complex space, and then discuss some examples of Šilov boundaries of (homogeneous) Siegel domains.

2. Preliminaries

We fix a complex hilbertian space $E$ of dimension $n$, a real hilbertian space $F$ of dimension $m$, and a hermitian map $\Phi : E \times E \rightarrow F_C$. Define

$${\mathcal{M}} := \{ (\xi, x + i\Phi(\xi)) : \xi \in E, x \in F \} = \{ (\xi, z) \in E \times F_C : \text{Im } z - \Phi(\xi) = 0 \},$$

where $F_C$ denotes the complexification of $F$, while $\Phi(\xi) := \Phi(\xi, \cdot)$ for every $\xi \in E$. We define

$$\rho : E \times F_C \ni (\xi, z) \mapsto \text{Im } z - \Phi(\xi) \in F.$$ 

We endow $E \times F_C$ with the product

$$(\xi, z)(\xi', z') := (\xi + \Phi(\xi', \cdot), z + z' + 2i(\Phi(\xi'), \cdot))$$

for every $(\xi, z), (\xi', z') \in E \times F_C$, so that $E \times F_C$ becomes a 2-step nilpotent Lie group, and $\mathcal{M}$ a closed subgroup of $E \times F_C$. In particular, the identity of $E \times F_C$ is $(0, 0)$ and $(\xi, z)^{-1} = (-\xi, -z + 2i(\Phi(\xi)))$ for every $(\xi, z) \in E \times F_C$. It will be convenient to identify $\mathcal{M}$ with the 2-step nilpotent Lie group $\mathcal{N} := E \times F$, endowed with the product

$$(\xi, x)(\xi', x') := (\xi + \xi', x + x' + 2\text{Im } \Phi(\xi, \xi'))$$

for every $(\xi, x), (\xi', x') \in \mathcal{N}$, by means of the isomorphism

$$\iota : \mathcal{N} \ni (\xi, x) \mapsto (\xi, x + i\Phi(\xi)) \in E \times F_C.$$ 

In particular, the identity of $\mathcal{N}$ is $(0, 0)$ and $(\xi, x)^{-1} = (-\xi, -x)$ for every $(\xi, x) \in \mathcal{N}$. Notice that, in this way, $\mathcal{N}$ acts holomorphically (on the left) on $E \times F_C$. Given a function $f$ on $E \times F_C$, we shall define

$$f_h : \mathcal{N} \ni (\xi, x) \mapsto f(\xi, x + i\Phi(\xi) + ih) \in C$$

for every $h \in F$. 

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Theorem 3.1. Let \( \Omega \) be an open subset of \( F \) such that \( \Omega = \Omega + \overline{C} \), and set \( D := \rho^{-1}(\Omega) \). Then, for every \( f \in \text{Hol}(D) \), for every \( p \in ]0, \infty[ \), for every \( h \in \Omega \) and for every \( h' \in \overline{C} \),

\[
\| f_{h+h'} \|_{L^p(\Omega)} \leq \| f_h \|_{L^p(\Omega)}.
\]

The proof is based on the ‘analytic disc technique’ presented in [2, Section 15.3].

Observe that the assumption that \( \Omega = \Omega + \overline{C} \) is not restrictive. Indeed, if \( \Omega \) is connected and \( C \) has a non-empty interior \( \text{Int} \ C \), then every function which is holomorphic on \( \rho^{-1}(\Omega) \) extends (uniquely) to a holomorphic function on \( \rho^{-1}(\Omega + \text{Int} C \cup \{ 0 \}) \) by [2, Theorem 1 of Section 15.3], and \( \Omega + \text{Int} C \cup \{ 0 \} = \Omega + \overline{C} \) since \( \Omega \) is open and \( \overline{C} = \text{Int} C \) by convexity. The case in which \( \text{Int} C = \emptyset \) may be treated directly using similar techniques.

We also mention that, if \( p < \infty \) and either \( \Phi \) is degenerate or the polar of \( \Phi(E) \) has an empty interior (that is, the closed convex envelope of \( \Phi(E) \) contains a non-trivial vector subspace), then either \( f_h = 0 \) or \( f_h \not\in L^p(\Omega) \) (at least for \( p \geq 1 \) when \( \Phi \) is non-degenerate). Cf. [6] for more details in a similar case.

**Proof.** For every \( v = (v_j) \in E^m \), consider

\[
A_v : C \ni w \mapsto \left( \sum_{j=1}^m v_j w^j, i \sum_{j=1}^m \Phi(v_j) + 2i \sum_{k<j} \Phi(v_j, v_k) w^{j-k} \right) \in E \times F_C,
\]

and

\[
\Psi(v) := \sum_{j=1}^m \Phi(v_j) \in C,
\]
and observe that the following hold:

- $A_v(0) = (0, i\Psi(v))$;
- $\Psi(E^m)$ is the convex envelope $C$ of $\Phi(E)$, thanks to [12, Corollary 17.1.2];
- $\rho(A_v(w)) = 0$ for every $w \in T$;
- the mapping $A : E^m \ni v \mapsto A_v \in \text{Hol}(C ; E \times F_C)$ is continuous (actually, polynomial).

Now, take $h \in \Omega$. By continuity, there is $\epsilon > 0$ such that $A_v(U) + ih \subseteq D$ for every $v \in B_{E^m}(0, \epsilon)$, where $U$ denotes the unit disc in $C$, and $U$ its closure. Then, $A_v(U) + ih' \subseteq D$ for every $v \in B_{E^m}(0, \epsilon)$ and for every $h' \in h + \overline{C}$. For every $h' \in \Psi(B_{E^m}(0, \epsilon))$, denote by $\nu_{h'}$ the image of the normalized Haar measure on $T$ under the mapping $\pi \circ A_v$, for some $v \in B_{E^m}(0, \epsilon) \cap \Psi^{-1}(h')$, where $\pi : E \times F_C \ni (\zeta, z) \mapsto (\zeta, \text{Re} z) \in \mathcal{N}$. Observe that, for every $(\zeta, x) \in \mathcal{N}$ and for every $h'' \in h + \overline{C}$, the mapping

$$\overline{U} \ni w \mapsto f((\zeta, x + i\Phi(\zeta)) \cdot [A_v(w) + (0, ih'')]) \in \mathbb{C}$$

is continuous and holomorphic on $U$, so that, by subharmonicity (cf., e.g., [13, Theorem 15.19]),

$$|f(\zeta, x + i\Phi(\zeta) + i(h' + h''))|^{\min(1, p)}$$

$$\leq \int_T |f((\zeta, x + i\Phi(\zeta)) \cdot [A_v(w) + (0, ih'')]|^{\min(1, p)} \, dw$$

$$= \int_{\mathcal{N}} |f_{h''}((\zeta, x)(\zeta', x'))|^{\min(1, p)} \, d\nu_{h'}(\zeta', x')$$

$$= (|f_{h''}|^{\min(1, p)} \ast \tilde{\nu}_{h'})(\zeta, x),$$

where $\tilde{\nu}_{h'}$ denotes the reflection of $\nu_{h'}$, while $v$ is a suitable element of $B_{E^m}(0, \epsilon) \cap \Psi^{-1}(h')$. Since $\nu_{h'}$ is a probability measure, by Young’s inequality (cf., e.g., [4, Chapter III, § 4, No. 4]) we then infer that

$$\|f_{h' + h''}\|_{LP(\mathcal{N})} = \|f_{h' + h''}|^{\min(1, p)}\|^{1/\min(1, p)}_{L^{\max(1, p)}}$$

$$\leq \|f_{h''}|^{\min(1, p)}\|^{1/\min(1, p)}_{L^{\max(1, p)}} = \|f_{h''}\|_{LP(\mathcal{N})}$$

for every $h' \in \Psi(B_{E^m}(0, \epsilon))$ and for every $h'' \in h + \overline{C}$. Since every element of $C$ may we written as a finite sum of elements of $\Psi(B_{E^m}(0, \epsilon))$, the arbitrariness of $h''$ shows that

$$\|f_{h + h'}\|_{LP(\mathcal{N})} \leq \|f_h\|_{LP(\mathcal{N})}$$

for every $h' \in C$, hence for every $h' \in \overline{C}$ by lower semi-continuity. The proof is complete. \hfill \square

**Corollary 3.2.** Assume that $C$ has a non-empty interior $\Omega$, and set $D := \rho^{-1}(\Omega)$. Then, for every $p \in [0, \infty]$ and $f \in \text{Hol}(D)$,

$$\sup_{h \in \Omega} \|f_h\|_{LP(\mathcal{N})} = \lim_{h \to 0} \inf_{h \in \Omega} \|f_h\|_{LP(\mathcal{N})}.$$
In particular, if we define the Hardy space $H^p(D)$ as the set of $f \in \text{Hol}(D)$ such that $\sup_{h \in \Omega} ||f_h||_{H^p(D)}$ is finite, the preceding result states that $H^p(D)$ may be equivalently defined as the set of $f \in \text{Hol}(D)$ such that $\liminf_{h \to 0, h \in \Omega} ||f_h||_{L^p(N)}$ is finite. This result should be compared with [3], where the boundary values of the elements of $H^p(D)$ are characterized as the CR elements of $L^p(N)$, for $p \in [1, \infty]$. In particular, Corollary 3.2 could be deduced from the results of [3], when $p \in [1, \infty]$, though at the expense of some further technicalities. This result extends [7, Corollary 1.43].

4. Examples

We shall now present some examples of homogeneous Siegel domains $D = \rho^{-1}(\Omega)$ for which $\Omega$ is the closed convex envelope of $\Phi(E)$, so that Corollary 3.2 applies.

We recall that $D$ is said to be a Siegel domain if $\Omega$ is an open convex cone not containing affine lines, $\Phi$ is non-degenerate, and $\Phi(E) \subseteq \overline{\Omega}$. In addition, $D$ is said to be homogeneous if the group of its biholomorphisms acts transitively on $D$. It is known (cf., e.g., [5, Proposition 1]) that $D$ is homogeneous if and only if there is a triangular Lie subgroup $T_+$ of $GL(F)$ which acts simply transitively on $\Omega$, and for every $t \in T_+$ there is $g \in GL(E)$ such that $t \Phi = \Phi(g \times g)$.

If $T'_+$ is another Lie subgroup of $GL(F)$ with the same properties as $T_+$, then $T_+$ and $T'_+$ are conjugated by an automorphism of $F$ preserving $\Omega$. Thanks to this fact, we may use the results of [7] even if a different $T_+$ is chosen. In particular, there is a surjective (open and) continuous homomorphism of Lie groups

$$\Delta : T_+ \rightarrow (\mathbb{R}^*_+)^r$$

for some $r \in \mathbb{N}$, called the rank of $\Omega$, so that

$$\Delta^s = \Delta_{t_1}^{s_1} \cdots \Delta_{t_r}^{s_r},$$

$s \in \mathbb{C}^r$, are the characters of $T_+$. Once a base point $e_\Omega \in \Omega$ has been fixed, $\Delta^s$ induces a function $\Delta^s_\Omega$ on $\Omega$, setting $\Delta^s_\Omega(t(e_\Omega)) = \Delta^s(t)$ for every $t \in T_+$.

Up to modifying $\Delta$, we may then assume that the functions $\Delta^s_\Omega$ are bounded on the bounded subsets of $\Omega$ if and only if $\text{Re} \ s \in \mathbb{R}^*_+$ (cf. [7, Lemma 2.34]). In particular, there is $b \in \mathbb{R}^*_-$ such that $\Delta^{-b}(t) = |\text{det} C_g|^2$ for every $t \in T_+$ and for every $g \in GL(E)$ such that $t \Phi = \Phi(g \times g)$ (cf. [7, Lemma 2.9]), and one may prove that $b \in (\mathbb{R}^*_+)^r$ if and only if $\Phi(E)$ generates $F$ as a vector space, in which case $\Omega$ is the interior of the convex envelope of $\Phi(E)$ (cf. [7, Proposition 2.57 and its proof, and Corollary 2.58]). Therefore, we are interested in finding examples of homogeneous Siegel domains for which $b \in (\mathbb{R}^*_+)^r$.

Notice, in addition, that if $b \not\in (\mathbb{R}^*_+)^r$, then $\Phi(E)$ is contained in a hyperplane, so that the interior of its convex envelope is empty.

The Siegel domain $D$ is said to be symmetric if it is homogeneous and admits an involutive biholomorphism with a unique fixed point (equivalently, if for every $(\zeta, z) \in D$ there is an involutive biholomorphism of $D$ for which $(\zeta, z)$ is
an isolated (or the unique) fixed point). The domain $D$ is said to be irreducible if it is not biholomorphic to the product of two non-trivial Siegel domains.

It is well known that every symmetric Siegel domain is biholomorphic to a product of irreducible ones, and that the irreducible symmetric Siegel domains can be classified in four infinite families plus two exceptional domains (cf., e.g., [1, §§ 1, 2]). In particular, for an irreducible symmetric Siegel domain, either $b = 0$ (that is, $E = \{0\}$, in which case $D$ is ‘of tube type’), or $b \in (\mathbb{R}^*)^r$ (cf., e.g., [7, Example 2.11]). Hence, when $D$ is a symmetric Siegel domain, $\overline{\Omega}$ is the closed convex envelope of $\Phi(E)$ if and only if none of the irreducible components of $D$ is of tube type. Note that these domains can be also characterized as those which do not admit any non-constant rational inner functions, thanks to [8].

We now present some examples of (homogeneous) Siegel domains.

**Example 4.1.** Let $K$ be either $\mathbb{C}$ or the division ring of the quaternions. In addition, fix $r, k, p \in \mathbb{N}$ with $p \leq r$, and define

- $E$ as the space of $k \times r$ matrices over $K$ whose $j$-th columns have zero entries for $j = p + 1, \ldots, r$;
- $F$ as the space of self-adjoint $r \times r$ matrices over $K$;
- $\Omega$ as the cone of non-degenerate positive self-adjoint $r \times r$ matrices over $K$;
- $\Phi : E \times E \ni (\zeta, \zeta') \mapsto \frac{1}{2}[(\zeta^* \zeta + \zeta^* \zeta') + i(\zeta^* i \zeta' - \zeta^* i \zeta)] \in F_\mathbb{C}$;
- $T_+ \Delta : T_+ \ni t \mapsto (t_{1,1}, \ldots, t_{r,r}) \in (\mathbb{R}^*_n)^r$.

Then, $\Omega$ is an irreducible symmetric cone\(^1\) of rank $r$ on which $T_+$ acts simply transitively by [7, Example 2.6]. In addition, $\Phi$ is well defined, since $\zeta^* \zeta + \zeta^* \zeta' + \zeta^* i \zeta' - \zeta^* i \zeta \in F$ for every $\zeta, \zeta' \in E$, and clearly $\Phi(\zeta) \in \overline{\Omega}$ and

$$t \cdot \Phi(\zeta) = t \cdot (\zeta^* \zeta) = (\zeta t^*)^r(\zeta t^*) = \Phi(\zeta t^*)$$

for every $t \in T_+$ and for every $\zeta \in E$ (with $\zeta t^* \in E$), so that $D$ is homogeneous. Then, $b = (b_j)$, with $b_j = -k \dim_c K$ for $j = 1, \ldots, p$ and $b_j = 0$ for $j = p + 1, \ldots, r$. Consequently, $\overline{\Omega}$ is the closed convex envelope of $\Phi(E)$ if and only if $p = r$ and $k > 0$.

Notice that $D$ is irreducible since $\Omega$ is irreducible (cf. [9, Corollary 4.8]), and that $D$ is symmetric if $kp = 0$ or if $p = r$ and $K = \mathbb{C}$ (cf. [7, Examples 2.14 and 2.15]). If $kp(r - p) > 0$, or if $K \neq \mathbb{C}, r \geq 3$, and $k \geq 2$, then $D$ cannot be symmetric.

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\(^1\)A cone is said to be homogeneous if the group of its linear automorphisms acts transitively on it. It is said to be symmetric if, in addition, it is self-dual for some scalar product. A convex cone is said to be irreducible if it is not isomorphic to a product of non-trivial convex cones.
Example 4.2. Take $k, p, q \in \mathbb{N}$, $p \leq 2$. Define:

- $E$ as the space of formal $k \times 2$ matrices whose entries of the first column belong to $\mathbb{C}$ (and are 0 if $p = 0$), and whose entries of the second column belong to $\mathbb{C}^q$ (and are 0 if $p \leq 1$);
- $F$ as the space of formally self-adjoint $2 \times 2$ matrices whose diagonal entries belong to $\mathbb{R}$, and whose non-diagonal entries belong to $\mathbb{C}^q$;
- $\Omega$ as the cone of $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in F$ with $a, c > 0$, $b \in \mathbb{C}^q$, and $ac - |b|^2 > 0$;
- $\Phi$ so that

\[
\Phi\left(\begin{array}{c} a_1 \\ \vdots \\ a_k \\ b_1 \\ \vdots \\ b_k \end{array}\right) = \begin{pmatrix} \sum_j |a_j|^2 & \sum_j \overline{a_j} b_j \\ \sum_j a_j \overline{b}_j & \sum_j |b_j|^2 \end{pmatrix}
\]

for every $\left(\begin{array}{c} a_1 \\ \vdots \\ a_k \\ b_1 \\ \vdots \\ b_k \end{array}\right) \in E$;
- $T_+$ as the group of formal $2 \times 2$ upper triangular matrices with diagonal entries in $\mathbb{R}_+^k$ and non-diagonal entries in $\mathbb{C}^q$, with the action

\[
\left(\begin{array}{cc} a & b \\ 0 & c \end{array}\right) \cdot \left(\begin{array}{cc} a' & b' \\ b & c' \end{array}\right) = \left(\begin{array}{cc} a'a^2 + c' |b|^2 + 2a \text{Re} \langle b, b' \rangle & abc' + cc' b \\ abc' + cc' b & c'c' \end{array}\right).
\]

- $\Delta : T_+ \ni t \mapsto (t_{1,1}, t_{2,2})$.

Then, $\Omega$ is an irreducible symmetric cone of rank 2 on which $T_+$ acts simply transitively (cf. [7, Example 2.7]). In addition, $\Phi(\xi) \in \overline{\Omega}$ for every $\xi \in E$, and

$\Phi(\xi - \Phi(\xi^*))$ for every $\xi \in E$ (with $\xi^* \in E$), provided that $p \leq 1$. Then, $D$ is an irreducible Siegel domain, and it is homogeneous if $p \leq 1$ (it is symmetric if $p = 0$). In addition, $b = 0$ if $p = 0$, while $b = (k, 0)$ if $p = 1$. Further, if $p = 2$, then $\Phi(E)$ contains the boundary of $\Omega$, since

\[
\left(\begin{array}{cc} a & b \\ b & c \end{array}\right) = \Phi\left(\begin{array}{cc} a^{1/2} & 0 \\ 0 & 0 \end{array}\right).
\]

for every $a > 0$, for every $c \geq 0$ and for every $b \in \mathbb{C}^q$ such that $|b|^2 = ac$ (the case $a = 0$, $b = 0$, $c \geq 0$ is treated similarly). Then, $\overline{\Omega}$ is the closed convex envelope of $\Phi(E)$ if and only if $p = 2$.

References


2Formally, $(\begin{array}{cc} a' & b' \\ b & c' \end{array}) \cdot (\begin{array}{cc} a & b \\ b & c \end{array}) = (\begin{array}{cc} a' & b' \\ b & c' \end{array})(\begin{array}{cc} a & b \\ b & c \end{array})^*.$
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(M. Calzi) DIAPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA C. SALDINI 50, 20133 MILANO, ITALY
mattia.calzi@unimi.it

This paper is available via http://nyjm.albany.edu/j/2022/28-64.html.