Strongly continuous composition semigroups on analytic Morrey spaces

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Abstract. For a semigroup $(\varphi_t)_{t \geq 0}$ consisting of analytic self-maps from the unit disk $\mathbb{D}$ to itself, a strongly continuous composition semi-group $(C_t)_{t \geq 0}$ induced by $(\varphi_t)_{t \geq 0}$ on analytic Morrey spaces $H^{2,\lambda}$, $0 < \lambda < 1$, is investigated. By the weak compactness of resolvent operator, we give a complete characterization of $H^{2,\lambda} = [\varphi_t, H^{2,\lambda}]$ for $0 < \lambda < 1$ in terms of the infinitesimal generator if the Denjoy-Wolff point of $(\varphi_t)_{t \geq 0}$ is in $\mathbb{D}$.

1. Introduction

Recall that a family $(\varphi_t)_{t \geq 0}$ of analytic self-maps of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ is said to be a semigroup if:

(i) $\varphi_0$ is the identity map $I$, i.e. $\varphi_0(z) = z, z \in \mathbb{D}$;
(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $t, s \geq 0$;
(iii) for each $z \in \mathbb{D}$, $\varphi_t(z) \to z$ as $t \to 0^+$.

A semigroup $(\varphi_t)_{t \geq 0}$ is said to be trivial if each $\varphi_t$ is the identity of $\mathbb{D}$. By [12], every non-trivial semigroup $(\varphi_t)_{t \geq 0}$ has a unique common fixed point $b \in \mathbb{D}$ with $|\varphi_t'(b)| \leq 1$ for all $t \geq 0$, called the Denjoy-Wolff point (DW point) of $(\varphi_t)_{t \geq 0}$. The infinitesimal generator of $(\varphi_t)_{t \geq 0}$ is the function

$$G(z) = \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t} = \frac{\partial \varphi_t(z)}{\partial t} \Big|_{t=0^+}, \quad z \in \mathbb{D}.$$
This convergence holds uniformly on compact subsets of \( \mathbb{D} \), so \( G \in \mathcal{H}(\mathbb{D}) \), the set of all analytic functions on \( \mathbb{D} \). Moreover, \( G \) has a unique representation
\[
G(z) = (bz - 1)(z - b)P(z), \quad z \in \mathbb{D},
\]
where \( b \) is the DW point of \( (\varphi_t)_{t \geq 0} \) and \( P \in \mathcal{H}(\mathbb{D}) \) with \( \text{Re}(P(z)) \geq 0 \) for \( z \in \mathbb{D} \). For every non-trivial semigroup \( (\varphi_t)_{t \geq 0} \) with the infinitesimal generator \( G \), there exists a unique univalent function \( h \), the Koenigs function of \( (\varphi_t)_{t \geq 0} \) on \( \mathbb{D} \), corresponding to \( (\varphi_t)_{t \geq 0} \). If the DW point \( b \in \mathbb{D} \), then \( h(b) = 0 \), \( h'(b) = 1 \) and
\[
h(\varphi_t(z)) = e^{G(b)t}h(z), \quad z \in \mathbb{D}, \quad t \geq 0.
\]
If the DW point \( b \in \partial\mathbb{D} = \{ z : |z| = 1 \} \), then \( h(0) = 0 \) and
\[
h(\varphi_t(z)) = h(z) + it, \quad z \in \mathbb{D}, \quad t \geq 0.
\]
Without loss of generality, the DW point \( b \in \mathbb{D} \) or \( b \notin \partial\mathbb{D} \) can be written as \( b = 0 \) or \( b = 1 \). See [5] and [12] for more results about the composition semigroups.

For a given semigroup \( (\varphi_t)_{t \geq 0} \) and a Banach space \( X \) consisting of analytic functions on \( \mathbb{D} \), we say that \( (\varphi_t)_{t \geq 0} \) generates a strongly continuous composition semigroup \( (C_t)_{t \geq 0} \) on \( X \) if \( C_t \) is bounded on \( X \) for \( t \geq 0 \) and

\[
\lim_{t \to 0^+} \|C_t(f) - f\|_X = 0 \quad \text{for all } f \in X,
\]
where \( C_t(f) = f \circ \varphi_t \) for \( f \in \mathcal{H}(\mathbb{D}) \). Here \( C_0 \) is the identity operator and \( C_{t+s} = C_t \circ C_s \) for \( t, s \geq 0 \). Denote by \( [\varphi_t, X] \) the maximal subspace of \( X \) on which \( (\varphi_t)_{t \geq 0} \) generates a strongly continuous composition semigroup \( (C_t)_{t \geq 0} \). Note that \( [\varphi_t, X] \subset X \) is obvious. By [2, 10, 11], we know that every semigroup \( (\varphi_t)_{t \geq 0} \) generates a strongly continuous composition semigroup \( (C_t)_{t \geq 0} \) on the Hardy space \( H^p \), \( 1 \leq p < \infty \), the Bergman space \( A^p \), \( 1 \leq p < \infty \), and the Dirichlet space \( D \), respectively. In our notation, \( [\varphi_t, H^p] = H^p \), \( [\varphi_t, A^p] = A^p \) for \( 1 \leq p < \infty \) and \( [\varphi_t, D] = D \). However, not all analytic function spaces admit the property that the corresponding composition semigroups are strongly continuous on them. For this situation, we choose \( X = H^\infty \), the Bloch space \( B \), the spaces \( \Omega_p \) and \( \Omega_K \), for examples. See [3, 9, 15] for the details.

The authors of [6] considered the same problems for the analytic Morrey spaces \( H^{2,\lambda} \), \( 0 \leq \lambda \leq 1 \). Let \( H^2 \) be the Hardy space of all analytic functions \( f \) on \( \mathbb{D} \) for which

\[
\sup_{0 \leq \theta < 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{d\vartheta}{2\pi} < \infty.
\]

Note that for \( f \in H^2 \), the function \( f(z) \) converges nontangentially to an \( L^2 \) function \( f(t) \) almost everywhere on \( \partial\mathbb{D} \). For \( 0 \leq \lambda \leq 1 \), the analytic Morrey space \( H^{2,\lambda} \) consisting of those functions \( f \in H^2 \) such that

\[
\|f\|_{H^{2,\lambda}} := \sup_{t \in \partial\mathbb{D}} \left( \frac{1}{|t|^\lambda} \int_{|t|} |f(t) - f_1|^2 \frac{dt}{2\pi} \right)^{1/2} < \infty,
\]
where \( f_t \) denotes the average of \( f \) over the arc \( I \subset \partial \mathbb{D} \) and \(|I|\) denotes the arc length of \( I \subset \partial \mathbb{D} \). It is clear that for \( \lambda = 0 \) or \( \lambda = 1, \) \( H^{2,\lambda} \) reduces to \( H^2 \) or \( \text{BMOA} \), the set of analytic functions in \( \mathbb{D} \) with boundary values of bounded mean oscillation. It is known (cf.\([14]\)), that \( \|f\|_{H^{2,\lambda}}^2 \) is equivalent to

\[
\sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^2} \int_{S(I)} |f'(z)|^2(1 - |z|^2)dm(z),
\]

(2)

where \( S(I) \) is the Carleson box and \( dm(z) \) is the normalized Lebesgue area measure on \( \mathbb{D} \).

It was shown in [6] that for every non-trivial semigroup \((\varphi_t)_{t \geq 0}\),

\[
\text{BMOA} \subsetneq H_0^{2,\lambda} \subset [\varphi_t, H^{2,\lambda}] \subsetneq H^{2,\lambda}, \quad 0 < \lambda < 1.
\]

(3)

Here, \( H_0^{2,\lambda} \) is the closure of all polynomials in \( H^{2,\lambda} \). [6, Theorem 3.1], the analogue of Sarason's characterization of a function in \( \text{VMOA} \), showed that \( H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}] \) for \( \varphi_t(z) = e^{-t}z \) with the DW point \( b = 0 \). However, by choosing

\[
\varphi_t(z) = \left( e^{-t}((1+z)\frac{1+i}{1-z} - 1) + 1 \right)\frac{2}{1+i} - 1, \quad 0 < \lambda < 1,
\]

with the DW point \( b = 0 \), we find that the function

\[
f_{\lambda}(z) = \left( \frac{1+z}{1-z} \right)^{\frac{1+i}{1}} - 1 \in H^{2,\lambda}\setminus H_0^{2,\lambda}, \quad 0 < \lambda < 1.
\]

Since

\[
\|f_{\lambda} \circ \varphi_t - f_{\lambda}\|_{H^{2,\lambda}} = (1 - e^{-t})\|f_{\lambda}\|_{H^{2,\lambda}} \to 0
\]

as \( t \to 0 \), \( f_{\lambda} \in [\varphi_t, H^{2,\lambda}] \). It means that \( H_0^{2,\lambda} \neq [\varphi_t, H^{2,\lambda}] \) holds for the semigroup \((\varphi_t)_{t \geq 0}\). In addition, we are able to find a semigroup \((\varphi_t)_{t \geq 0} = (e^{-t}z + 1 - e^{-t})_{t \geq 0} \) with the DW point \( b = 1 \), for example, such that \( H_0^{2,\lambda} \neq [\varphi_t, H^{2,\lambda}] \).

A natural problem is to characterize the semigroup \((\varphi_t)_{t \geq 0} \) such that \( H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}] \) holds. The authors of [6] obtained a sufficient condition for \( H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}] \) in terms of the infinitesimal generator of \((\varphi_t)_{t \geq 0} \) as follows.

**Theorem A** ([6]). Let \((\varphi_t)_{t \geq 0} \) be a semigroup of analytic self-maps of \( \mathbb{D} \) with the infinitesimal generator \( G \) and \( 0 < \lambda < 1 \). If

\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} \frac{1 - |z|}{|G(z)|^2} dm(z) = 0,
\]

(4)

then \( H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}] \).

They also gave a necessary condition on the infinitesimal generator of a semigroup with the DW point \( b \in \mathbb{D} \) such that \( H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}] \).
**Theorem B** ([6]). Let \((\varphi_t)_{t \geq 0}\) be a semigroup of analytic self-maps of \(\mathbb{D}\) with the DW point \(b \in \mathbb{D}\) and the infinitesimal generator \(G\). If for some \(\lambda \in (0, 1)\) we have \(H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}]\), then
\[
\lim_{|z| \to 1} \frac{(1 - |z|)^{3-\lambda}}{G(z)} = 0.
\]

The following result, **Theorem 1.1**, is our main result in this paper which gives a sufficient and necessary condition for \(H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}]\) in terms of the weakly compactness of the resolvent operator when the semigroup \((\varphi_t)_{t \geq 0}\) has a DW point in \(\mathbb{D}\). Moreover, this shows that when (5) holds, condition (4) in Theorem A is also necessary for \(H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}]\).

**Theorem 1.1.** Suppose \(0 < \lambda < 1\) and \((\varphi_t)_{t \geq 0}\) is a non-trivial semigroup of analytic self-maps of \(\mathbb{D}\) with the DW point \(b = 0\) and the infinitesimal generator \(G\). Denote by \(\Gamma\) the infinitesimal generator of the corresponding composition semigroup \((S_t)_{t \geq 0}\) on \(H^{2,\lambda}_0\) and denote by \(R(\sigma, \Gamma) = (\sigma - \Gamma)^{-1}\) the resolvent operator for \(\sigma \in \rho(\Gamma)\), the resolvent set of \(\Gamma\). Then \(H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}]\) if and only if the resolvent operator \(R(\sigma, \Gamma)\) is weakly compact on \(H^{2,\lambda}_0\). Moreover, if
\[
\sup_{I \subset \mathbb{D}} \frac{1}{|I|} \int_{s(I)} \frac{1 - |z|}{|G(z)|^2} \, dm(z) < \infty,
\]
then \(H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}]\) if and only if
\[
\lim_{|z| \to 1} \frac{1}{|I|} \int_{s(I)} \frac{1 - |z|}{|G(z)|^2} \, dm(z) = 0.
\]

Throughout the paper, the symbol \(A \asymp B\) means that \(A \preceq B \preceq A\). We say that \(A \preceq B\) if there exists a constant \(C > 0\) such that \(A \leq CB\).

**2. Lemmas**

For \(g \in \mathcal{H}(\mathbb{D})\), the Volterra type operator \(V_g\) on \(H^{2,\lambda}\) is defined by
\[
V_g(f)(z) = \int_0^z f(\xi)g'(\xi) \, d\xi, \quad f \in H^{2,\lambda}.
\]

The following Lemma 2.1 and Lemma 2.2 are extensions of the related results in [3].

**Lemma 2.1.** Let \(0 < \lambda < 1\) and \(g \in \mathcal{H}(\mathbb{D})\). Then the following are equivalent:

(i) \(V_g\) is bounded on \(H^{2,\lambda}\).

(ii) \(V_g\) is bounded on \(H^{2,\lambda}_0\).
Proof. (i) $\Rightarrow$ (ii). Suppose $V_g$ is bounded on $H^{2,\lambda}$. By [8], $g \in H^{2,\lambda}_0$ since $BMOA \subset H^{2,\lambda}_0$ for $0 < \lambda < 1$. A simple computation shows that

$$V_g(z^n) = \int_0^z \xi^n g'(\xi) d\xi$$

belong to $H^{2,\lambda}_0$ for all integers $n \geq 1$, and then $V_g(P) \in H^{2,\lambda}_0$ for all polynomials $P$. Thus, for $f \in H^{2,\lambda}_0$, $V_g(f)$ can be approximated by $H^{2,\lambda}_0$ functions since $H^{2,\lambda}_0$ is the closure of all polynomials in $H^{2,\lambda}$. Bearing in mind that $H^{2,\lambda}_0$ is closed and the assertion follows.

(ii) $\Rightarrow$ (i). Suppose $V_g$ is bounded on $H^{2,\lambda}_0$. From [13], we know that the second dual of $H^{2,\lambda}_0$ is isomorphic to $H^{2,\lambda}$ under the pairing:

$$\langle f, h \rangle = \frac{1}{2\pi} \int_{\partial \mathbb{D}} f(\xi) \overline{h(\xi)} |d\xi|$$

for $f \in H^{2,\lambda}_0$ and $h \in (H^{2,\lambda}_0)^*$. Let $V^*_g$ be the adjoint of $V_g$ acting on the dual space $(H^{2,\lambda}_0)^*$ under (2.1), and let $V^{**}_g$ be the adjoint of $V^*_g$ acting on $H^{2,\lambda}$. Thus, by the definition of the adjoint operator,

$$\langle V_g(f), h \rangle = \langle f, V^*_g(h) \rangle = \langle V^*_g(h), f \rangle = \langle h, V^{**}_g(f) \rangle = \langle V^{**}_g(f), h \rangle$$

hold for all $f \in H^{2,\lambda}_0$ and $h \in (H^{2,\lambda}_0)^*$. Owing to $H^{2,\lambda}_0$ is weak* dense in $H^{2,\lambda}$, we say that $V^{**}_g = V_g$ as operators on $H^{2,\lambda}$. Hence, $V_g$ is bounded on $H^{2,\lambda}$.

**Lemma 2.2.** Suppose $0 < \lambda < 1$ and $g \in \mathcal{H}(\mathbb{D})$. If $V_g$ is bounded on $H^{2,\lambda}$, then the following statements are equivalent.

(i) $V_g$ is weakly compact on $H^{2,\lambda}$.

(ii) $V_g$ is weakly compact on $H^{2,\lambda}_0$.

(iii) $V_g$ is compact on $H^{2,\lambda}_0$.

(iv) $V_g$ is compact on $H^{2,\lambda}$.

(v) $V_g(H^{2,\lambda}) \subset H^{2,\lambda}_0$.

**Proof.** By the proof of Lemma 2.1, we conclude that $V^{**}_g = V_g$. According to [4], the equivalence of (i), (ii) and (v) can be easily obtained.

Next, we show that (iii) and (iv) are equivalent. Because $H^{2,\lambda}_0$ is a subspace of $H^{2,\lambda}$ and they share the same norm, (iii) implies (iv). Conversely, let $V_g$ be compact on $H^{2,\lambda}_0$. Using $V^{**}_g = V_g$ again, and together with [4, Theorem VI.5.2], we get that (iv) and (iii) are equivalent.

Finally, we verify that (i) and (iii) are equivalent. (iii) $\Rightarrow$ (i) is obvious. To finish the proof, for a given subarc $I \subset \partial \mathbb{D}$, we consider the functions

$$f_w(z) = \frac{1}{(1 - wz)^{1-\lambda}}, \quad z \in \mathbb{D},$$
where \( w = (1 - |I|)z \) and \( z \) is the center of \( I \). Note that \( f_w \in H^{2,\lambda} \) and
\[
\sup_{w \in \mathbb{D}} \| f_w \|_{H^{2,\lambda}} < \infty.
\]
If (i) is true, then the equivalence of (i) and (v) gives that \( Vg(H^{2,\lambda}) \subseteq H_0^{2,\lambda} \). It follows that
\[
Vg(f_w)(z) = \int_0^z f_w(\xi)g'(\xi) d\xi, \quad w \in \mathbb{D},
\]
belong to \( H_0^{2,\lambda} \). Similar to (2), we have
\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} |f_w(z)|^2 |g'(z)|^2 (1 - |z|^2) dm(z) = 0.
\]
Hence,
\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} |g'(z)|^2 (1 - |z|^2) dm(z) = 0,
\]
which means that \( g \in VMOA \) by [7]. Combining this with [8] implies that \( Vg \) is compact on \( H^{2,\lambda} \).

Suppose now that \( (\varphi_t)_{t \geq 0} \) is a semigroup of self-maps of \( \mathbb{D} \) and \( (C_t)_{t \geq 0} \) is the corresponding composition semigroup on \( H^{2,\lambda} \). Since each \( \varphi_t \) is univalent, we know that \( C_t \) is bounded on \( H^{2,\lambda} \) ([16, Corollary 1]), and \( \sup_{t \in [0,1]} \| C_t \| < \infty \). If \( f \in H_0^{2,\lambda} \) and \( \varepsilon > 0 \), then there exists a polynomial \( P \) such that \( \|f - P\|_{H^{2,\lambda}} < \varepsilon \) ([13, Lemma 2.8]). Hence,
\[
\| C_t(f) - C_t(P) \|_{H^{2,\lambda}} < \varepsilon \left( \frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \right)^{1/2}.
\]
Since \( C_t(P) \in H_0^{2,\lambda} \), it follows that \( C_t(f) \in H_0^{2,\lambda} \). Therefore \( C_t : H_0^{2,\lambda} \to H_0^{2,\lambda} \) exists as a bounded operator with \( \| C_t \| \leq \left( \frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \right)^{1/2} \). Thus, we can define the composition operator \( S_t = C_t \) on \( H_0^{2,\lambda} \). It is clear that \( (S_t)_{t \geq 0} \) is strongly continuous on \( H_0^{2,\lambda}, 0 < \lambda < 1 \), by [1, Corollary 1.3].

**Lemma 2.3.** Let \( (\varphi_t)_{t \geq 0} \) be a semigroup of self-maps of \( \mathbb{D} \), \( (C_t)_{t \geq 0} \) be the corresponding composition semigroup on \( H^{2,\lambda} \), and \( S_t = C_t |_{H_0^{2,\lambda}} \) for \( 0 < \lambda < 1 \). Then \( S_t^{**} = C_t \) for all \( t \geq 0 \), where \( S_t^{**} \) means the second adjoint operator of \( S_t \) under the pairing (7).

**Proof.** For \( f \in H_0^{2,\lambda} \) and \( h \in (H_0^{2,\lambda})^* \), we have
\[
\langle S_t(f), h \rangle = \langle f, S_t^*(h) \rangle = \langle S_t^*(h), f \rangle = \langle h, S_t^{**}(f) \rangle = \langle S_t^{**}(f), h \rangle,
\]
which gives
\[
S_t^{**}(f) = S_t(f) \quad \text{for all } f \in H_0^{2,\lambda}.
\]
Therefore,
\[
C_t |_{H_0^{2,\lambda}} = S_t = S_t^{**} |_{H_0^{2,\lambda}}.
\]
Since $H^2_{0}$ is weak dense in $H^2$, the conclusion follows.

**Lemma C** ([5]). Let $(T_t)_{t \geq 0}$ be a strongly continuous composition semigroup on a Banach space $X$ with the infinitesimal generator $A$ and let $\omega_0$ be the growth bound of $(T_t)_{t \geq 0}$, i.e.

\[
\omega_0 = \lim_{t \to \infty} \frac{\log \|T_t\|}{t}.
\]

(i) If $\delta > \omega_0$, then there is a constant $M_\delta$ such that $\|T_t\| \leq M_\delta e^{\delta t}$, $t \geq 0$;

(ii) If $\Re(\sigma) > \omega_0$, then $\sigma \in \rho(A)$ and

\[
R(\sigma, A)(f) = \int_0^\infty e^{-\sigma t} T_t(f) dt, \quad f \in X.
\]

**Lemma 2.4.** Let $(\varphi_t)_{t \geq 0}$ be a non-trivial semigroup of self-maps of $\mathbb{D}$ with the DW point $b = 0$, the infinitesimal generator $G$ and Koenigs function $h$. Suppose $S_t$ is the corresponding composition semigroup on $H^2_{+0}$, $0 < \lambda < 1$, with the infinitesimal generator $\Gamma$. Then for $\sigma \in \rho(\Gamma)$, the resolvent operator of $\Gamma$ has the following representation:

\[
R(\sigma, \Gamma)f(z) = -\frac{1}{G'(0)} \frac{1}{(h(z))^{-\frac{\sigma}{\alpha(0)}}} \int_0^z f(\zeta)(h(\zeta))^{-\frac{\sigma}{\alpha(0)}} h'(\zeta) d\zeta.
\]  

(8)

In particular, $-G'(0)$ belongs to $\rho(\Gamma)$ and hence

\[
R(-G'(0), \Gamma)f(z) = -\frac{1}{G'(0) h(z)} \int_0^z f(\zeta) h'(\zeta) d\zeta.
\]  

(9)

**Proof.** Write

\[
R := -\frac{1}{G'(0)} \frac{1}{(h(z))^{-\frac{\sigma}{\alpha(0)}}} \int_0^z f(\zeta)(h(\zeta))^{-\frac{\sigma}{\alpha(0)}} h'(\zeta) d\zeta.
\]

It is easy to check that

\[(\sigma I - \Gamma)R = R(\sigma I - \Gamma) = I,
\]

which shows that $R$ is the resolvent operator of $\Gamma$ and (8) holds. Since each $\varphi_t$ is univalent, we immediately get that each $S_t$ maps $H^2_{+0}$ into itself and so

\[
\omega_0 := \lim_{t \to \infty} \frac{\log \|S_t\|}{t} = 0.
\]

By (1), we have

\[
G(z) = -zP(z), \quad \Re(P(z)) \geq 0, \quad z \in \mathbb{D},
\]

and

\[
\Re(-G'(0)) = \Re(P(0)) \geq 0.
\]

If $\Re(-G'(0)) > 0$, by (ii) of Lemma C, $-G'(0) \in \rho(\Gamma)$. If $\Re(-G'(0)) = 0$, write $G(z) = -iz$, where $\alpha \in \mathbb{R} \setminus \{0\}$. By [3, Theorem 2],

\[
\Gamma(f)(z) = G(z)f'(z) = -izf'(z).
\]
Thus, \((iaI - \Gamma)(f) = g\) has the unique analytic solution

\[
f(z) = \frac{1}{i\alpha z} \int_0^z g(\xi) d\xi.
\]

It is not difficult to see that the operator

\[
g \to \frac{1}{i\alpha z} \int_0^z g(\xi) d\xi
\]

is bounded on \(H^{2,\lambda}\). Hence, it is bounded on \(H^{2,\lambda}_0\). Therefore, \(-G'(0) \in \rho(\Gamma)\). Choosing \(\sigma = -G'(0)\) in (8), we obtain (9). \(\square\)

### 3. The proof of Theorem 1.1

Now we are going to prove Theorem 1.1. Suppose \(H^{2,\lambda}_0 = [\varphi_1, H^{2,\lambda}]\). By (i) of Lemma C, there are two positive constants \(\delta\) and \(M_\delta\) such that \(\|S_u\| \leq M_\delta e^{\delta u}\) for \(u \geq 0\). By (ii) of Lemma C, we choose a large enough real number \(\sigma > \delta\) such that \(\sigma \in \rho(\Gamma)\) and we have

\[
R(\sigma, \Gamma)(f) = \int_0^\infty e^{-\sigma u} S_u(f) du, \quad f \in H^{2,\lambda}_0.
\]

Thus,

\[
S_t \circ R(\sigma, \Gamma)(f) = \int_0^\infty e^{-\sigma u} S_{t+u}(f) du = e^{\sigma t} \int_0^\infty e^{-\sigma u} S_u(f) du.
\]

Accordingly,

\[
S_t \circ R(\sigma, \Gamma)(f) - R(\sigma, \Gamma)(f) = (e^{\sigma t} - 1) \int_t^\infty e^{-\sigma u} S_u(f) du - \int_0^t e^{-\sigma u} S_u(f) du.
\]

Therefore,

\[
\|S_t \circ R(\sigma, \Gamma) - R(\sigma, \Gamma)(f)\|_{H^{2,\lambda}_0}
\]

\[
\leq \left( |e^{\sigma t} - 1| \int_0^\infty e^{-\sigma u} \|S_u\| du + \int_0^t e^{-\sigma u} \|S_u\| du \right) \|f\|_{H^{2,\lambda}_0}.
\]

Thus,

\[
\|S_t \circ R(\sigma, \Gamma) - R(\sigma, \Gamma)\| \leq M_\delta \left( |e^{\sigma t} - 1| \int_t^\infty e^{-(\sigma-\delta)u} du + \int_0^t e^{-(\sigma-\delta)u} du \right),
\]

and so

\[
\lim_{t \to 0} \|S_t \circ R(\sigma, \Gamma) - R(\sigma, \Gamma)\| = 0.
\]

By Lemma 2.3, \(S_t^{**} = C_t\). Recalling that \(S_t\) commutes with \(R(\sigma, \Gamma)\), we have

\[
\lim_{t \to 0} \|C_t \circ R(\sigma, \Gamma)^{**} - R(\sigma, \Gamma)^{**}\| = 0.
\]
This implies

\[ \lim_{t \to 0} \|C_t \sigma R(\sigma, \Gamma)^{**}(f) - R(\sigma, \Gamma)^{**}(f)\|_{H^{2,\lambda}} = 0, \quad f \in H^{2,\lambda}, \]

which yields that \(R(\sigma, \Gamma)^{**}(H^{2,\lambda}) \subset [\varphi_1, H^{2,\lambda}] = H^{2,\lambda}_0\). According to [4, Theorem VI.4.2], we know that \(R(\sigma, \Gamma)\) is weakly compact on \(H^{2,\lambda}_0\) for a large enough real number \(\sigma\). For a general \(\sigma \in \rho(\Gamma)\), using the resolvent equation

\[ R(\sigma, \Gamma) - R(\mu, \Gamma) = (\mu - \sigma)R(\sigma, \Gamma)R(\mu, \Gamma), \quad \sigma, \mu \in \rho(\Gamma), \]

we obtain that \(R(\sigma, \Gamma)\) is weakly compact for some \(\sigma \in \rho(\Gamma)\) if and only if it is weakly compact for every \(\sigma \in \rho(\Gamma)\).

Conversely, write \(Y = [\varphi_1, H^{2,\lambda}]\) and then \(H^{2,\lambda}_0 \subset Y \subsetneq H^{2,\lambda}\) by [6]. By [3, Theorem 2], the restriction of \((C_t)_{t \geq 0}\) on \(Y\) is a strongly continuous semigroup with the infinitesimal generator \(\Delta(f) = Gf'\). It is clear that the domain of \(\Gamma\)

\[ D(\Gamma) = \{ f \in H^{2,\lambda}_0 : Gf' \in H^{2,\lambda}_0 \} \subset D(\Delta) = \{ f \in Y : Gf' \in Y \}, \]

and that \(\Delta\) is an extension of \(\Gamma\). Let \(\sigma\) be a large enough real number such that \(\sigma \in \rho(\Gamma) \cap \rho(\Delta)\). An argument similar to that in the proof of Lemma 2.3 shows that

\[ R(\sigma, \Gamma)^{**}|_{H^{2,\lambda}_0} = R(\sigma, \Gamma), \quad R(\sigma, \Gamma)^{**}|_Y = R(\sigma, \Delta). \]

On the other hand,

\[ D(\Delta) = \{ f \in Y : Gf' \in Y \} = \{ f \in Y : g = Gf' - \sigma f \in Y \} = \{ f \in Y : f = R(\sigma, \Delta)(g), g \in Y \} = R(\sigma, \Delta)(Y). \]

Thus,

\[ D(\Delta) = R(\sigma, \Gamma)^{**}|_Y(\varphi_1, H^{2,\lambda}) \subset R(\sigma, \Gamma)^{**}(H^{2,\lambda}) \subset H^{2,\lambda}_0. \]

By [3, Theorem 1], we have

\[ Y = [\varphi_1, H^{2,\lambda}] = D(\Delta) \subset H^{2,\lambda}_0, \]

which means that

\[ H^{2,\lambda}_0 = [\varphi_1, H^{2,\lambda}]. \]

Next, we are going to prove the second part of Theorem 1.1. By Lemma 2.4, we know that \(-G'(0) \in \rho(\Gamma)\) and

\[ R_h(f) = R(-G'(0), \Gamma)f(z) = -\frac{1}{G'(0)h(z)} \int_0^z f(\zeta)h'(\zeta)\,d\zeta. \]

By using the techniques mentioned in [12], the operator \(R_h\) and the multiplier operator

\[ M_1(f)(z) = I(z)f(z) = zf(z) \]

satisfy the following identities:

\[ M_1P_h = -G'(0)R_hM_1, \quad Q_h = P_h + Q_hP_h, \quad (10) \]
where
\[ P_h f(z) = \frac{1}{zh(z)} \int_0^z f(\xi)\xi h'(\xi) \, d\xi \]
and
\[ Q_h f(z) = \frac{1}{z} \int_0^z f(\xi)\frac{\xi h'(\xi)}{h(\xi)} \, d\xi. \]

To finish our proof, by the first part of the theorem, it suffices to show that \( R_h \) is weakly compact on \( H^{2,\lambda}_0 \) if and only if (6) holds. A simple computation shows that
\[ Q_h(f)(z) = J(f)(z) + L_h M_f(f)(z), \]
where
\[ J(f)(z) = \frac{1}{z} \int_0^z f(\xi) \, d\xi \]
and
\[ L_h f(z) = \frac{1}{z} \int_0^z f(\xi) \left( \log \frac{h(\xi)}{\xi} \right) \, d\xi. \]

Since the DW point \( b = 0 \), we have
\[ h'(z)G(z) = G'(0)h(z), \quad z \in \mathbb{D}. \]

Thus, (5) gives
\[ \sup_{\xi < z} \frac{1}{|I|} \int_{S(I)} \left| \frac{zh'(z)}{h(z)} \right|^2 (1 - |z|) \, dm(z) < \infty, \]
which shows that \( \log \frac{h(z)}{z} \in BMOA \). By [8] and Lemma 2.1, \( L_h \) is bounded on \( H^{2,\lambda}_0 \), and so \( Q_h \) is bounded on \( H^{2,\lambda}_0 \). By (10), \( R_h \) is bounded on \( H^{2,\lambda}_0 \) and therefore, \( P_h \) is bounded on \( H^{2,\lambda}_0 \). Meanwhile, (6) is equivalent to
\[ \lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} \left| \frac{zh'(z)}{h(z)} \right|^2 (1 - |z|^2) \, dm(z) = 0, \]
which shows that \( \log \frac{h(z)}{z} \in VMOA \). Similarly, we obtain that (6) is equivalent to that \( R_h \) is weakly compact on \( H^{2,\lambda}_0 \) see [4, Theorem VI.4.5]. The proof is complete.

The following corollary is closely related to Theorem B.

**Corollary 3.1.** Suppose \( 0 < \lambda < 1 \) and \( (\varphi_t)_{t \geq 0} \) is a non-trivial semigroup of analytic self-maps of \( \mathbb{D} \) with the DW point in \( \mathbb{D} \) and infinitesimal generator \( G \). If condition (5) holds, then \( H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}] \) implies that
\[ \lim_{|z| \to 1} \frac{1 - |z|}{G(z)} = 0. \]
Proof. Suppose $H_0^{2,1} = [\varphi, H^{2,1}]$. By Theorem 1.1, we have that (6) holds. A standard argument (cf. [7]) gives

$$\lim_{|a| \to 1} \int_{|z| < r_0} \frac{1}{|G(z)|^2} (1 - |\sigma_a(z)|^2) dm(z) = 0,$$

where $\sigma_a(z) = \frac{a - z}{1 - \overline{a} z}$, $a \in \mathbb{D}$, is the Möbius transformation of $\mathbb{D}$. For $0 < r < 1$, let $\mathbb{D}(a, r) = \{z \in \mathbb{D} : |\sigma_a(z)| < r\}$ be the pseudohyperbolic disk with center $a \in \mathbb{D}$ and radius $r$. By [17], we see that

$$|1 - \bar{a} z|^2 \approx |1 - z|^2 \approx (1 - |a|^2)^2 \approx m(\mathbb{D}(a, r)), \quad z \in \mathbb{D}(a, r).$$

Choose an $r_0 \in (0, 1)$. By the subharmonicity, we obtain

$$\int_{\mathbb{D}(a, r_0)} \frac{1}{|G(z)|^2} (1 - |\sigma_a(z)|^2) dm(z) \geq (1 - r_0^2) \int_{\mathbb{D}(a, r_0)} \frac{1}{|G(z)|^2} dm(z) \geq (1 - r_0^2) \frac{(1 - |a|^2)^2}{|G(a)|^2}.$$

Letting $|a| \to 1$, by (11) we obtain

$$\lim_{|a| \to 1} \frac{1 - |a|}{G(a)} = 0.$$

Thus, Corollary 3.1 is proved. \qed

References


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