Strongly continuous composition semigroups on analytic Morrey spaces

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ABSTRACT. For a semigroup \( (\varphi_t)_{t \geq 0} \) consisting of analytic self-maps from the unit disk \( \mathbb{D} \) to itself, a strongly continuous composition semi-group \( (C_t)_{t \geq 0} \) induced by \( (\varphi_t)_{t \geq 0} \) on analytic Morrey spaces \( H^{2,\lambda}, 0 < \lambda < 1 \), is investigated. By the weak compactness of resolvent operator, we give a complete characterization of \( H^{2,\lambda} = [\varphi, H^{2,\lambda}] \) for \( 0 < \lambda < 1 \) in terms of the infinitesimal generator if the Denjoy-Wolff point of \( (\varphi_t)_{t \geq 0} \) is in \( \mathbb{D} \).

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1. Introduction

Recall that a family \( (\varphi_t)_{t \geq 0} \) of analytic self-maps of the unit disk \( \mathbb{D} \) in the complex plane \( \mathbb{C} \) is said to be a semigroup if:

(i) \( \varphi_0 \) is the identity map \( I \), i.e. \( \varphi_0(z) = z, z \in \mathbb{D} \);
(ii) \( \varphi_{t+s} = \varphi_t \circ \varphi_s \) for all \( t, s \geq 0 \);
(iii) for each \( z \in \mathbb{D} \), \( \varphi_t(z) \to z \) as \( t \to 0^+ \).

A semigroup \( (\varphi_t)_{t \geq 0} \) is said to be trivial if each \( \varphi_t \) is the identity of \( \mathbb{D} \). By [12], every non-trivial semigroup \( (\varphi_t)_{t \geq 0} \) has a unique common fixed point \( b \in \mathbb{D} \) with \( |\varphi_t'(b)| \leq 1 \) for all \( t \geq 0 \), called the Denjoy-Wolff point (DW point) of \( (\varphi_t)_{t \geq 0} \). The infinitesimal generator of \( (\varphi_t)_{t \geq 0} \) is the function

\[
G(z) = \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t} = \frac{\partial \varphi_t(z)}{\partial t}
\]

Received May 31, 2022.
2010 Mathematics Subject Classification. 30D45, 30D99, 30H25, 47B38.
Key words and phrases. composition operator semigroup; strongly continuous; maximal closed subspace; analytic Morrey space; Denjoy-Wolff point.

This research is supported by NNSF of China (No.11720101003, 12271328) and Guangdong Basic and Applied-basic Research Foundation (No. 2022A1515012117).
This convergence holds uniformly on compact subsets of \( \mathbb{D} \), so \( G \in \mathcal{H}(\mathbb{D}) \), the set of all analytic functions on \( \mathbb{D} \). Moreover, \( G \) has a unique representation

\[
G(z) = (bz - 1)(z - b)P(z), \quad z \in \mathbb{D},
\]

where \( b \) is the DW point of \((\varphi_t)_{t \geq 0}\) and \( P \in \mathcal{H}(\mathbb{D}) \) with \( \text{Re}(P(z)) \geq 0 \) for \( z \in \mathbb{D} \). For every non-trivial semigroup \((\varphi_t)_{t \geq 0}\) with the infinitesimal generator \( G \), there exists a unique univalent function \( h \), the Koenigs function of \((\varphi_t)_{t \geq 0}\) on \( \mathbb{D} \), corresponding to \((\varphi_t)_{t \geq 0}\). If the DW point \( b \in \mathbb{D} \), then \( h(b) = 0 \), \( h'(b) = 1 \) and

\[
h(\varphi_t(z)) = e^{G(t)}h(z), \quad z \in \mathbb{D}, \quad t \geq 0.
\]

If the DW point \( b \in \partial \mathbb{D} = \{z : |z| = 1\} \), then \( h(0) = 0 \) and

\[
h(\varphi_t(z)) = h(z) + it, \quad z \in \mathbb{D}, \quad t \geq 0.
\]

Without loss of generality, the DW point \( b \in \mathbb{D} \) or \( b \in \partial \mathbb{D} \) can be written as \( b = 0 \) or \( b = 1 \). See [5] and [12] for more results about the composition semigroups.

For a given semigroup \((\varphi_t)_{t \geq 0}\) and a Banach space \( X \) consisting of analytic functions on \( \mathbb{D} \), we say that \((\varphi_t)_{t \geq 0}\) generates a strongly continuous composition semigroup \((C_t)_{t \geq 0}\) on \( X \) if \( C_t \) is bounded on \( X \) for \( t \geq 0 \) and

\[
\lim_{t \to 0^+} \|C_t(f) - f\|_X = 0 \quad \text{for all } f \in X,
\]

where \( C_t(f) = f \circ \varphi_t \) for \( f \in \mathcal{H}(\mathbb{D}) \). Here \( C_0 \) is the identity operator and \( C_{t+s} = C_t \circ C_s \) for \( t, s \geq 0 \). Denote by \([\varphi_t, X]\) the maximal subspace of \( X \) on which \((\varphi_t)_{t \geq 0}\) generates a strongly continuous composition semigroup \((C_t)_{t \geq 0}\). Note that \([\varphi_t, X]\) \( \subset X \) is obvious. By [2, 10, 11], we know that every semigroup \((\varphi_t)_{t \geq 0}\) generates a strongly continuous composition semigroup \((C_t)_{t \geq 0}\) on the Hardy space \( H^p, 1 \leq p < \infty \), the Bergman space \( A^p, 1 \leq p < \infty \), and the Dirichlet space \( \mathcal{D} \), respectively. In our notation, \([\varphi_t, H^p] = H^p, [\varphi_t, A^p] = A^p, [\varphi_t, \mathcal{D}] = \mathcal{D} \). However, not all analytic function spaces admit the property that the corresponding composition semigroups are strongly continuous on them. For this situation, we choose \( X = H^\infty \), the Bloch space \( B \), the spaces \( \Omega_p \) and \( \Omega_K \), for examples. See [3, 9, 15] for the details.

The authors of [6] considered the same problems for the analytic Morrey spaces \( H^{2, \lambda}, 0 \leq \lambda \leq 1 \). Let \( H^2 \) be the Hardy space of all analytic functions \( f \) on \( \mathbb{D} \) for which

\[
\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.
\]

Note that for \( f \in H^2 \), the function \( f(z) \) converges nontangentially to an \( L^2 \) function \( f(t) \) almost everywhere on \( \partial \mathbb{D} \). For \( 0 \leq \lambda \leq 1 \), the analytic Morrey space \( H^{2, \lambda} \) consisting of those functions \( f \in H^2 \) such that

\[
\|f\|_{H^{2, \lambda}} := \sup_{I \in \partial \mathbb{D}} \left( \frac{1}{|I|^\lambda} \int_I |f(t) - \varphi_f(t)|^2 \frac{dt}{2\pi} \right)^{1/2} < \infty,
\]
where \( f_t \) denotes the average of \( f \) over the arc \( I \subset \partial \mathbb{D} \) and \(|I|\) denotes the arc length of \( I \subset \partial \mathbb{D} \). It is clear that for \( \lambda = 0 \) or \( \lambda = 1 \), \( H^{2,\lambda} \) reduces to \( H^2 \) or \( BMOA \), the set of analytic functions in \( \mathbb{D} \) with boundary values of bounded mean oscillation. It is known (cf.\([14]\)), that \( \|f\|_{H^{2,\lambda}}^2 \) is equivalent to

\[
\sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) \mathrm{d}m(z),
\]

(2)

where \( S(I) \) is the Carleson box and \( \mathrm{d}m(z) \) is the normalized Lebesgue area measure on \( \mathbb{D} \).

It was shown in \([6]\) that for every non-trivial semigroup \((\varphi_t)_{t \geq 0}\),

\[
BMOA \subset H^{2,\lambda}_0 \subset [\varphi_t, H^{2,\lambda}] \subset H^{2,\lambda}, \quad 0 < \lambda < 1.
\]

(3)

Here, \( H^{2,\lambda}_0 \) is the closure of all polynomials in \( H^{2,\lambda} \). \([6, \text{Theorem} 3.1]\), the analogue of Sarason’s characterization of a function in \( VMOA \), showed that \( H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}] \) for \( \varphi_t(z) = e^{-t}z \) with the DW point \( b = 0 \). However, by choosing

\[
\varphi_t(z) = \frac{(e^{-t}((1+z^2)^{1/2} - 1) + 1)^{1/2} - 1}{(e^{-t}((1-z^2)^{1/2} - 1) + 1)^{1/2} + 1}, \quad 0 < \lambda < 1,
\]

with the DW point \( b = 0 \), we find that the function

\[
f_\lambda(z) = (\frac{1 + z}{1 - z})^{1/2} - 1 \in H^{2,\lambda} \setminus H^{2,\lambda}_0, \quad 0 < \lambda < 1.
\]

Since

\[
\|f_\lambda \circ \varphi_t - f_\lambda\|_{H^{2,\lambda}} = (1 - e^{-t})\|f_\lambda\|_{H^{2,\lambda}} \to 0
\]

as \( t \to 0 \), \( f_\lambda \in [\varphi_t, H^{2,\lambda}] \). It means that \( H^{2,\lambda}_0 \neq [\varphi_t, H^{2,\lambda}] \) holds for the semigroup \((\varphi_t)_{t \geq 0}\). In addition, we are able to find a semigroup \((\varphi_t)_{t \geq 0} = (e^{-t}z + 1 - e^{-t})_{t \geq 0} \) with the DW point \( b = 1 \), for example, such that \( H^{2,\lambda}_0 \neq [\varphi_t, H^{2,\lambda}] \).

A natural problem is to characterize the semigroup \((\varphi_t)_{t \geq 0}\) such that \( H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}] \) holds. The authors of \([6]\) obtained a sufficient condition for \( H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}] \) in terms of the infinitesimal generator of \((\varphi_t)_{t \geq 0}\) as follows.

**Theorem A** ([6]). Let \((\varphi_t)_{t \geq 0}\) be a semigroup of analytic self-maps of \( \mathbb{D} \) with the infinitesimal generator \( G \) and \( 0 < \lambda < 1 \). If

\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} \frac{1 - |z|}{|G(z)|^2} \mathrm{d}m(z) = 0,
\]

(4)

then \( H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}] \).

They also gave a necessary condition on the infinitesimal generator of a semigroup with the DW point \( b \in \mathbb{D} \) such that \( H^{2,\lambda}_0 = [\varphi_t, H^{2,\lambda}] \).
Theorem B ([6]). Let \((\varphi_t)_{t \geq 0}\) be a semigroup of analytic self-maps of \(\mathbb{D}\) with the DW point \(b \in \mathbb{D}\) and the infinitesimal generator \(G\). If for some \(\lambda \in (0, 1)\) we have \(H_0^{2, \lambda} = [\varphi_t, H^{2, \lambda}]\), then
\[
\lim_{|z| \to 1} \frac{(1 - |z|)^{1-\lambda}}{G(z)} = 0.
\]

The following result, Theorem 1.1, is our main result in this paper which gives a sufficient and necessary condition for \(H_0^{2, \lambda} = [\varphi_t, H^{2, \lambda}]\) in terms of the weakly compactness of the resolvent operator when the semigroup \((\varphi_t)_{t \geq 0}\) has a DW point in \(\mathbb{D}\). Moreover, this shows that when (5) holds, condition (4) in Theorem A is also necessary for \(H_0^{2, \lambda} = [\varphi_t, H^{2, \lambda}]\).

Theorem 1.1. Suppose \(0 < \lambda < 1\) and \((\varphi_t)_{t \geq 0}\) is a non-trivial semigroup of analytic self-maps of \(\mathbb{D}\) with the DW point \(b = 0\) and the infinitesimal generator \(G\). Denote by \(\Gamma\) the infinitesimal generator of the corresponding composition semigroup \((S_t)_{t \geq 0}\) on \(H_0^{2, \lambda}\) and denote by \(R(\sigma, \Gamma) = (\sigma - \Gamma)^{-1}\) the resolvent operator for \(\sigma \in \rho(\Gamma)\), the resolvent set of \(\Gamma\). Then \(H_0^{2, \lambda} = [\varphi_t, H^{2, \lambda}]\) if and only if the resolvent operator \(R(\sigma, \Gamma)\) is weakly compact on \(H_0^{2, \lambda}\). Moreover, if
\[
\sup_{t \in \mathbb{D}} \frac{1}{|I|} \int_{S(I)} \frac{1 - |z|}{|G(z)|^2} \, dm(z) < \infty, \tag{5}
\]
then \(H_0^{2, \lambda} = [\varphi_t, H^{2, \lambda}]\) if and only if
\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} \frac{1 - |z|}{|G(z)|^2} \, dm(z) = 0. \tag{6}
\]

Throughout the paper, the symbol \(A \approx B\) means that \(A \lesssim B \lesssim A\). We say that \(A \lesssim B\) if there exists a constant \(C > 0\) such that \(A \leq CB\).

2. Lemmas

For \(g \in \mathcal{H}(\mathbb{D})\), the Volterra type operator \(V_g\) on \(H^{2, \lambda}\) is defined by
\[
V_g(f)(z) = \int_0^z f(\xi)g'(\xi) \, d\xi, \quad f \in H^{2, \lambda}.
\]

The following Lemma 2.1 and Lemma 2.2 are extensions of the related results in [3].

Lemma 2.1. Let \(0 < \lambda < 1\) and \(g \in \mathcal{H}(\mathbb{D})\). Then the following are equivalent:

(i) \(V_g\) is bounded on \(H^{2, \lambda}\).
(ii) \(V_g\) is bounded on \(H_0^{2, \lambda}\).
The assertion follows.

\[ V_g(z^n) = \int_{0}^{z} \xi^n g'(\xi) d\xi \]

belong to \( H^{2,\lambda}_0 \) for all integers \( n \geq 1 \), and then \( V_g(P) \in H^{2,\lambda}_0 \) for all polynomials \( P \). Thus, for \( f \in H^{2,\lambda}_0 \), \( V_g(f) \) can be approximated by \( H^{2,\lambda}_0 \) functions since \( H^{2,\lambda}_0 \) is the closure of all polynomials in \( H^{2,\lambda} \). Bearing in mind that \( H^{2,\lambda}_0 \) is closed and the assertion follows.

(ii) \( \Rightarrow \) (i). Suppose \( V_g \) is bounded on \( H^{2,\lambda}_0 \). From [13], we know that the second dual of \( H^{2,\lambda}_0 \) is isomorphic to \( H^{2,\lambda} \) under the pairing:

\[ \langle f, h \rangle = \frac{1}{2\pi} \int_{\partial D} f(\xi) \overline{h(\xi)} |d\xi| \]  

(7)

for \( f \in H^{2,\lambda}_0 \) and \( h \in (H^{2,\lambda}_0)^* \). Let \( V_g^* \) be the adjoint of \( V_g \) acting on the dual space \((H^{2,\lambda}_0)^*\) under (2.1), and let \( V_g^{**} \) be the adjoint of \( V_g^* \) acting on \( H^{2,\lambda} \). Thus, by the definition of the adjoint operator,

\[ \langle V_g(f), h \rangle = \langle f, V_g^*(h) \rangle = \langle V_g^*(h), f \rangle = \langle h, V_g^{**}(f) \rangle = \langle V_g^{**}(f), h \rangle \]

hold for all \( f \in H^{2,\lambda}_0 \) and \( h \in (H^{2,\lambda}_0)^* \). Owing to \( H^{2,\lambda}_0 \) is weak* dense in \( H^{2,\lambda} \), we say that \( V_g^{**} = V_g \) as operators on \( H^{2,\lambda} \). Hence, \( V_g \) is bounded on \( H^{2,\lambda} \). □

Lemma 2.2. Suppose \( 0 < \lambda < 1 \) and \( g \in \mathcal{H}(\mathbb{D}) \). If \( V_g \) is bounded on \( H^{2,\lambda} \), then the following statements are equivalent.

(i) \( V_g \) is weakly compact on \( H^{2,\lambda} \);
(ii) \( V_g \) is weakly compact on \( H^{2,\lambda}_0 \);
(iii) \( V_g \) is compact on \( H^{2,\lambda}_0 \);
(iv) \( V_g \) is compact on \( H^{2,\lambda} \);
(v) \( V_g(H^{2,\lambda}_0) \subset H^{2,\lambda}_0 \).

Proof. By the proof of Lemma 2.1, we conclude that \( V_g^{**} = V_g \). According to [4], the equivalence of (i), (ii) and (v) can be easily obtained.

Next, we show that (iii) and (iv) are equivalent. Because \( H^{2,\lambda}_0 \) is a subspace of \( H^{2,\lambda} \) and they share the same norm, (iii) implies (iv). Conversely, let \( V_g \) be compact on \( H^{2,\lambda}_0 \). Using \( V_g^{**} = V_g \) again, and together with [4, Theorem VI.5.2], we get that (iv) and (iii) are equivalent.

Finally, we verify that (i) and (iii) are equivalent. (iii) \( \Rightarrow \) (i) is obvious. To finish the proof, for a given subarc \( I \subset \partial \mathbb{D} \), we consider the functions

\[ f_w(z) = \frac{1}{(1 - w\overline{z})^{1-\lambda}}, \quad z \in \mathbb{D}, \]
where \( w = (1 - |I|)\zeta \) and \( \zeta \) is the center of \( I \). Note that \( f_w \in H^{2, \lambda} \) and 
\[
\sup_{w \in \mathbb{D}} \|f_w\|_{H^{2, \lambda}} < \infty.
\]
If (i) is true, then the equivalence of (i) and (v) gives that \( V_g(H^{2, \lambda}) \subset H^{2, \lambda}_0 \). It follows that 
\[
V_g(f_w)(z) = \int_0^z f_w(\xi)g'(\xi)d\xi, \quad w \in \mathbb{D},
\]
belong to \( H^{2, \lambda}_0 \). Similar to (2), we have 
\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} |f_w(z)|^2 |g'(z)|^2 (1 - |z|^2)dm(z) = 0.
\]
Hence, 
\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} |g'(z)|^2 (1 - |z|^2)dm(z) = 0,
\]
which means that \( g \in VMOA \) by [7]. Combining this with [8] implies that \( V_g \) is compact on \( H^{2, \lambda} \). □

Suppose now that \( (\varphi_t)_{t \geq 0} \) is a semigroup of self-maps of \( \mathbb{D} \) and \( (C_t)_{t \geq 0} \) is the corresponding composition semigroup on \( H^{2, \lambda} \). Since each \( \varphi_t \) is univalent, we know that \( C_t \) is bounded on \( H^{2, \lambda} \) ([16, Corollary 1]), and \( \sup_{t \in [0,1]} \|C_t\| < \infty \). If \( f \in H^{2, \lambda}_0 \) and \( \varepsilon > 0 \), then there exists a polynomial \( P \) such that \( \|f - P\|_{H^{2, \lambda}} < \varepsilon \) ([13, Lemma 2.8]). Hence, 
\[
\|C_t(f) - C_t(P)\|_{H^{2, \lambda}} < \varepsilon \left(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|}\right)^{1-\lambda}.\]

Since \( C_t(P) \in H^{2, \lambda}_0 \), it follows that \( C_t(f) \in H^{2, \lambda}_0 \). Therefore \( C_t : H^{2, \lambda}_0 \to H^{2, \lambda}_0 \) exists as a bounded operator with \( \|C_t\| \leq \left(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|}\right)^{1-\lambda} \). Thus, we can define the composition operator \( S_t = C_t |_{H^{2, \lambda}_0} \) on \( H^{2, \lambda}_0 \). It is clear that \( (S_t)_{t \geq 0} \) is strongly continuous on \( H^{2, \lambda}_0 \), \( 0 < \lambda < 1 \), by [1, Corollary 1.3].

**Lemma 2.3.** Let \( (\varphi_t)_{t \geq 0} \) be a semigroup of self-maps of \( \mathbb{D} \), \( (C_t)_{t \geq 0} \) be the corresponding composition semigroup on \( H^{2, \lambda} \), and \( S_t = C_t |_{H^{2, \lambda}_0} \) for \( 0 < \lambda < 1 \). Then \( S_t^{**} = C_t \) for all \( t \geq 0 \), where \( S_t^{**} \) means the second adjoint operator of \( S_t \) under the pairing (7).

**Proof.** For \( f \in H^{2, \lambda}_0 \) and \( h \in (H^{2, \lambda}_0)^* \), we have 
\[
\langle S_t(f), h \rangle = \langle f, S_t^*(h) \rangle = \langle S_t^*(h), f \rangle = \langle h, S_t^{**}(f) \rangle = \langle S_t^{**}(f), h \rangle,
\]
which gives 
\[
S_t^{**}(f) = S_t(f) \quad \text{for all } f \in H^{2, \lambda}_0.
\]
Therefore, 
\[
C_t |_{H^{2, \lambda}_0} = S_t = S_t^{**} |_{H^{2, \lambda}_0}.
\]
Since \( H^{2,\lambda}_0 \) is weak* dense in \( H^{2,\lambda} \), the conclusion follows. \( \square \)

**Lemma C** ([1]). Let \((T_t)_{t \geq 0}\) be a strongly continuous composition semigroup on a Banach space \( X \) with the infinitesimal generator \( A \) and let \( \omega_0 \) be the growth bound of \((T_t)_{t \geq 0} \), i.e.

\[
\omega_0 = \lim_{t \to \infty} \frac{\log \|T_t\|}{t}.
\]

(i) If \( \delta > \omega_0 \), then there is a constant \( M_\delta \) such that \( \|T_t\| \leq M_\delta e^{\delta t} \), \( t \geq 0 \);

(ii) If \( \text{Re}(\sigma) > \omega_0 \), then \( \sigma \in \rho(A) \) and

\[
R(\sigma, A)(f) = \int_0^\infty e^{-\sigma t} T_t(f) dt, \quad f \in X.
\]

**Lemma 2.4.** Let \((\varphi_t)_{t \geq 0}\) be a non-trivial semigroup of self-maps of \( \mathbb{D} \) with the DW point \( b = 0 \), the infinitesimal generator \( G \) and Koenigs function \( h \). Suppose \( S_t \) is the corresponding composition semigroup on \( H^{2,\lambda}_0 \), \( 0 < \lambda < 1 \), with the infinitesimal generator \( \Gamma \). Then for \( \sigma \in \rho(\Gamma) \), the resolvent operator of \( \Gamma \) has the following representation:

\[
R(\sigma, \Gamma)f(z) = -\frac{1}{G'(0)} \frac{1}{(h(z))^{\frac{\lambda}{2}}} \int_0^z f(\zeta)(h(\zeta))^{-\frac{\lambda}{2}+1} h'(\zeta) d\zeta. \tag{8}
\]

In particular, \(-G'(0)\) belongs to \( \rho(\Gamma) \) and hence

\[
R(-G'(0), \Gamma)f(z) = -\frac{1}{G'(0)h(z)} \int_0^z f(\zeta)h'(\zeta) d\zeta. \tag{9}
\]

**Proof.** Write

\[
R := -\frac{1}{G'(0)} \frac{1}{(h(z))^{\frac{\lambda}{2}}} \int_0^z f(\zeta)(h(\zeta))^{-\frac{\lambda}{2}+1} h'(\zeta) d\zeta.
\]

It is easy to check that

\[
(\sigma I - \Gamma)R = R(\sigma I - \Gamma) = I,
\]

which shows that \( R \) is the resolvent operator of \( \Gamma \) and (8) holds. Since each \( \varphi_t \) is univalent, we immediately get that each \( S_t \) maps \( H^{2,\lambda}_0 \) into itself and so

\[
\omega_0 := \lim_{t \to \infty} \frac{\log \|S_t\|}{t} = 0.
\]

By (1), we have

\[
G(z) = -zP(z), \quad \text{Re}(P(z)) \geq 0, \ z \in \mathbb{D},
\]

and

\[
\text{Re}(-G'(0)) = \text{Re}(P(0)) \geq 0.
\]

If \( \text{Re}(-G'(0)) > 0 \), by (ii) of Lemma C, \(-G'(0)\) \( \in \rho(\Gamma) \). If \( \text{Re}(-G'(0)) = 0 \), write \( G(z) = -i\alpha z \), where \( \alpha \in \mathbb{R} \setminus \{0\} \). By [3, Theorem 2],

\[
\Gamma(f)(z) = G(z)f'(z) = -i\alpha f'(z).
\]
Thus, \((iaI - \Gamma)(f) = g\) has the unique analytic solution
\[ f(z) = \frac{1}{iaz} \int_{0}^{z} g(\xi) d\xi. \]

It is not difficult to see that the operator
\[ g \rightarrow \frac{1}{iaz} \int_{0}^{z} g(\xi) d\xi \]

is bounded on \(H^{2,\lambda}\). Hence, it is bounded on \(H^{2,\lambda}_{0}\). Therefore, \(-G'(0) \in \rho(\Gamma)\).

Choosing \(\sigma = -G'(0)\) in (8), we obtain (9). \(\square\)

3. The proof of Theorem 1.1

Now we are going to prove Theorem 1.1. Suppose \(H^{2,\lambda}_{0} = [\varphi_t, H^{2,\lambda}]\). By (i) of Lemma C, there are two positive constants \(\delta\) and \(M_{\delta}\) such that \(\|S_{u}\| \leq M_{\delta} e^{\delta u}\) for \(u \geq 0\). By (ii) of Lemma C, we choose a large enough real number \(\sigma > \delta\) such that \(\sigma \in \rho(\Gamma)\) and we have
\[ R(\sigma, \Gamma)(f) = \int_{0}^{\infty} e^{-\sigma u} S_{u}(f) du, \quad f \in H^{2,\lambda}_{0}. \]

Thus,
\[ S_{t} \circ R(\sigma, \Gamma)(f) = \int_{0}^{\infty} e^{-\sigma u} S_{t+u}(f) du = e^{\sigma t} \int_{t}^{\infty} e^{-\sigma u} S_{u}(f) du. \]

Accordingly,
\[ S_{t} \circ R(\sigma, \Gamma)(f) - R(\sigma, \Gamma)(f) = (e^{\sigma t} - 1) \int_{t}^{\infty} e^{-\sigma u} S_{u}(f) du - \int_{0}^{t} e^{-\sigma u} S_{u}(f) du. \]

Therefore,
\[ \|S_{t} \circ R(\sigma, \Gamma)(f) - R(\sigma, \Gamma)(f)\|_{H^{2,\lambda}} \leq \left( |e^{\sigma t} - 1| \int_{t}^{\infty} e^{-\sigma u} \|S_{u}\| du + \int_{0}^{t} e^{-\sigma u} \|S_{u}\| du \right) \|f\|_{H^{2,\lambda}}. \]

Thus,
\[ \|S_{t} \circ R(\sigma, \Gamma) - R(\sigma, \Gamma)\| \leq M_{\delta} \left( |e^{\sigma t} - 1| \int_{t}^{\infty} e^{-(\sigma-\delta) u} du + \int_{0}^{t} e^{-(\sigma-\delta) u} du \right), \]

and so
\[ \lim_{t \to 0} \|S_{t} \circ R(\sigma, \Gamma) - R(\sigma, \Gamma)\| = 0. \]

By Lemma 2.3, \(S_{t}^{**} = C_{t}\). Recalling that \(S_{t}\) commutes with \(R(\sigma, \Gamma)\), we have
\[ \lim_{t \to 0} \|C_{t} \circ R(\sigma, \Gamma)^{**} - R(\sigma, \Gamma)^{**}\| = 0. \]
This implies
\[
\lim_{t \to 0} ||C_t \sigma R(\sigma, \Gamma)^{**}(f) - R(\sigma, \Gamma)^{**}(f)||_{H^{2,\lambda}} = 0, \quad f \in H^{2,\lambda},
\]
which yields that \( R(\sigma, \Gamma)^{**}(H^{2,\lambda}) \subset [\varphi, H^{2,\lambda}] = H_0^{2,\lambda} \). According to [4, Theorem VI.4.2], we know that \( R(\sigma, \Gamma) \) is weakly compact on \( H_0^{2,\lambda} \) for a large enough real number \( \sigma \). For a general \( \sigma \in \rho(\Gamma) \), using the resolvent equation
\[
R(\sigma, \Gamma) - R(\mu, \Gamma) = (\mu - \sigma)R(\sigma, \Gamma)R(\mu, \Gamma), \quad \sigma, \mu \in \rho(\Gamma),
\]
we obtain that \( R(\sigma, \Gamma) \) is weakly compact for some \( \sigma \in \rho(\Gamma) \) if and only if it is weakly compact for every \( \sigma \in \rho(\Gamma) \).

Conversely, write \( Y = [\varphi, H^{2,\lambda}] \) and then \( H_0^{2,\lambda} \subset Y \subset H^{2,\lambda} \) by [6]. By [3, Theorem 2], the restriction of \((C_t)_{t \geq 0}\) on \( Y \) is a strongly continuous semigroup with the infinitesimal generator \( \Delta(f) = Gf' \). It is clear that the domain of \( \Gamma \)
\[
D(\Gamma) = \{ f \in H_0^{2,\lambda} : G f' \in H_0^{2,\lambda} \} \subset D(\Delta) = \{ f \in Y : G f' \in Y \},
\]
and that \( \Delta \) is an extension of \( \Gamma \). Let \( \sigma \) be a large enough real number such that \( \sigma \in \rho(\Gamma) \cap \rho(\Delta) \). An argument similar to that in the proof of Lemma 2.3 shows that
\[
R(\sigma, \Gamma)^{**}|_{H_0^{2,\lambda}} = R(\sigma, \Gamma), \quad R(\sigma, \Gamma)|_{Y} = R(\sigma, \Delta).
\]
On the other hand,
\[
D(\Delta) = \{ f \in Y : G f' \in Y \}
= \{ f \in Y : g = G f' - \sigma f \in Y \}
= \{ f \in Y : f = R(\sigma, \Delta)(g), g \in Y \}
= R(\sigma, \Delta)(Y).
\]
Thus,
\[
D(\Delta) = R(\sigma, \Gamma)^{**}|_{Y}(Y) \subset R(\sigma, \Gamma)^{**}(H^{2,\lambda}) \subset H_0^{2,\lambda}.
\]
By [3, Theorem 1], we have
\[
Y = [\varphi, H^{2,\lambda}] = D(\Delta) \subset H_0^{2,\lambda},
\]
which means that
\[
H_0^{2,\lambda} = [\varphi, H^{2,\lambda}].
\]

Next, we are going to prove the second part of Theorem 1.1. By Lemma 2.4, we know that \(-G'(0) \in \rho(\Gamma)\) and
\[
R_h(f) : = R(-G'(0), \Gamma)f(z) = -\frac{1}{G'(0)h(z)} \int_0^z f(\zeta)h'(\zeta)d\zeta.
\]
By using the techniques mentioned in [12], the operator \( R_h \) and the multiplier operator
\[
M_h(f)(z) = I(z)f(z) = zf(z)
\]
satisfy the following identities:
\[
M_hP_h = -G'(0)R_hM_h, \quad Q_h = P_h + Q_hP_h, \quad (10)
\]
where
\[ P_h f(z) = \frac{1}{zh(z)} \int_0^z f(\xi)\xi h'(\xi) \, d\xi \]
and
\[ Q_h f(z) = \frac{1}{z} \int_0^z f(\xi) \frac{\xi h'(\xi)}{h(\xi)} \, d\xi. \]

To finish our proof, by the first part of the theorem, it suffices to show that \( R_h \) is weakly compact on \( H_0^{2,\lambda} \) if and only if (6) holds. A simple computation shows that
\[ Q_h(f)(z) = J(f)(z) + L_h M_f(f)(z), \]
where
\[ J(f)(z) = \frac{1}{z} \int_0^z f(\xi) \, d\xi \]
and
\[ L_h f(z) = \frac{1}{z} \int_0^z f(\xi) \left( \log \frac{h(\xi)}{\xi} \right)' \, d\xi. \]

Since the DW point \( b = 0 \), we have
\[ h'(z)G(z) = G'(0)h(z), \quad z \in \mathbb{D}. \]
Thus, (5) gives
\[ \sup_{l \in \mathbb{D}} \frac{1}{|l|} \int_{S(l)} \left| \frac{zh'(z)}{h(z)} \right|^2 (1 - |z|) \, dm(z) < \infty, \]
which shows that \( \log \frac{h(z)}{z} \in \text{BMOA} \). By [8] and Lemma 2.1, \( L_h \) is bounded on \( H_0^{2,\lambda} \), and so \( Q_h \) is bounded on \( H_0^{2,\lambda} \). By (10), \( R_h \) is bounded on \( H_0^{2,\lambda} \) and therefore, \( P_h \) is bounded on \( H_0^{2,\lambda} \). Meanwhile, (6) is equivalent to
\[ \lim_{|l| \to 0} \frac{1}{|l|} \int_{S(l)} \left| \frac{zh'(z)}{h(z)} \right|^2 (1 - |z|^2) \, dm(z) = 0, \]
which shows that \( \log \frac{h(z)}{z} \in \text{VMOA} \). Similarly, we obtain that (6) is equivalent to that \( R_h \) is weakly compact on \( H_0^{2,\lambda} \) see [4, Theorem VI.4.5]. The proof is complete.

The following corollary is closely related to Theorem B.

**Corollary 3.1.** Suppose \( 0 < \lambda < 1 \) and \( (\varphi_t)_{t \geq 0} \) is a non-trivial semigroup of analytic self-maps of \( \mathbb{D} \) with the DW point in \( \mathbb{D} \) and infinitesimal generator \( G \). If condition (5) holds, then \( H_0^{2,\lambda} = [\varphi_t, H^{2,\lambda}] \) implies that
\[ \lim_{|z| \to 1} \frac{1 - |z|}{G(z)} = 0. \]
Proof. Suppose $H_0^{2,1} = [\varphi_1, H^{2,1}]$. By Theorem 1.1, we have that (6) holds. A standard argument (cf. [7]) gives

$$\lim_{|a| \to 1} \int_{\mathbb{D}} \frac{1}{|G(z)|^2} (1 - |\sigma_a(z)|^2) dm(z) = 0,$$

(11)

where $\sigma_a(z) = \frac{az - z}{1 - az}$, $a \in \mathbb{D}$, is the Möbius transformation of $\mathbb{D}$. For $0 < r < 1$, let $\mathbb{D}(a, r) = \{ z \in \mathbb{D} : |\sigma_a(z)| < r \}$ be the pseudohyperbolic disk with center $a \in \mathbb{D}$ and radius $r$. By [17], we see that

$$|1 - az|^2 \approx (1 - |z|^2)^2 \approx (1 - |a|^2)^2 \approx m(\mathbb{D}(a, r)), \quad z \in \mathbb{D}(a, r).$$

Choose an $r_0 \in (0, 1)$. By the subharmonicity, we obtain

$$\int_{\mathbb{D}} \frac{1}{|G(z)|^2} (1 - |\sigma_a(z)|^2) dm(z) \geq (1 - r_0^2) \int_{\mathbb{D}(a, r)} \frac{1}{|G(z)|^2} dm(z) \geq (1 - r_0^2) \frac{(1 - |a|^2)^2}{|G(a)|^2}.$$  

Letting $|a| \to 1$, by (11) we obtain

$$\lim_{|a| \to 1} \frac{1 - |a|}{G(a)} = 0.$$

Thus, Corollary 3.1 is proved. □

References


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This paper is available via http://nyjm.albany.edu/j/2022/28-60.html.