Sub-Hilbert relation for Fock–Sobolev type spaces

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Abstract. In this paper, two specific sub-Hilbert spaces are studied. They arise from the action of a Toeplitz operator on Fock–Sobolev type spaces, induced by a general Gaussian type weight. The argument is based on analysing the reproducing kernel of the corresponding sub-Hilbert space.

1. Introduction

This paper is concerned with sub-Hilbert functional spaces of analytic functions on planar domains. Suppose \( T \) is a bounded operator on a given Hilbert space \( \mathcal{H} \). We denote by \( \mathcal{M}(T) \) the range of \( T \), which is equipped with the following inner product:

\[
\langle Tx, Ty \rangle_{\mathcal{M}(T)} = \langle x, y \rangle_{\mathcal{H}} \quad x, y \in H \ominus \ker T.
\]

Then \( \mathcal{M}(T) \) is a Hilbert space. If, in addition, \( T \) is a contraction operator, the Hilbert space

\[
\mathcal{M}((I - TT^*)^{1/2})
\]

is called the complemented space to \( \mathcal{M}(T) \) is denoted by \( \mathcal{H}(T) \) and is called a sub-Hilbert space.

1. Introduction

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The pioneering work on sub-Hilbert spaces was done by L. de Branges, J. Rovnyak and D. Sarason [8, 9, 10, 18]. For further reading on the spaces introduced by de Branges and Rovnyak, their equivalent formulations, and their applications in function theory and operator theory, see [3]. Sarason’s monograph [18] contains extensive investigation of sub-Hilbert spaces arising from Toeplitz operators $T_f$ acting on the Hardy space on the unit circle; in this context, it is customary to agree on the notation $M(T_f) = M(f)$ and $\mathcal{H}(T_f) = \mathcal{H}(f)$.

Later, continuing Sarason’s work, Kehe Zhu introduced sub-Bergman Hilbert spaces on the unit disk [21, 22]. To provide a brief account on this issue, we recall that the standard weighted Bergman space $A^2_{\alpha}$, for $\alpha > -1$, consists of all analytic functions on the unit disk for which the integral

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \, dx \, dy$$

is finite. The norm of a function in the weighted Bergman space is given by

$$\|f\|^2 = \frac{\alpha + 1}{\pi} \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \, dx \, dy.$$ 

We shall at times write

$$dA_{\alpha}(z) = \frac{\alpha + 1}{\pi} (1 - |z|^2)^\alpha \, dx \, dy,$$

for normalized weighted area measure in the unit disk. Note that $A^2_{\alpha}$ is a reproducing kernel functional Hilbert space whose kernel is given by

$$K^2_{\alpha}(w) = \sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha + 2)}{n! \Gamma(\alpha + 2)} (\overline{z}w)^n = \frac{1}{(1 - w\overline{z})^{\alpha + 2}}, \quad (z, w) \in \mathbb{D} \times \mathbb{D}.$$

The Bergman projection

$$P_{\alpha} : L^2(\mathbb{D}, dA_{\alpha}) \to A^2_{\alpha}(\mathbb{D})$$

is defined by

$$P_{\alpha}f(z) = \int_{\mathbb{D}} f(w)K^2_{\alpha}(w) dA_{\alpha}(w).$$

Now, let $\varphi$ be an analytic function in the unit disk satisfying $\|f\|_{\infty} \leq 1$. For $\alpha \geq 0$, we consider the Toeplitz operator

$$T^2_{\varphi}(f) = P_{\alpha}(\varphi f), \quad f \in A^2_{\alpha}.$$

For $\alpha = 0$, the unweighted Bergman space, Kehe Zhu [21, 22] studied the sub-Bergman Hilbert spaces $\mathcal{H}_{\alpha}(\varphi) := \mathcal{H}(T^2_{\varphi})$ and $\mathcal{H}(\overline{\varphi}) := \mathcal{H}(T^2_{\overline{\varphi}})$. He proved that these sub-Bergman Hilbert spaces coincide as sets, moreover, both spaces contain the Banach space of all bounded analytic functions on the unit disk. Zhu further showed that for the symbol $z^m$, and more generally, for a finite Blaschke product $B$, we have

$$\mathcal{H}(B) = \mathcal{H}(\overline{B}) = H^2,$$

where $H^2$ denotes the Hardy space of analytic functions on the unit disk.
Later, in 2010, Abkar and Jafarzade [1] extended Zhu’s results to the standard weighted Bergman spaces $A^2_\alpha$ where $\alpha \geq 0$. They proved that $H^\infty \subset \mathcal{H}_\alpha(\varphi) = \mathcal{H}_\alpha(\varphi)$, and for a finite Blaschke product $B$,

$$H^\infty \subset \mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = A^2_{\alpha-1}.$$  

In 2014, this line of investigation was adapted by Nowak and Rososczuk in [15] where the authors extended the latter result for $-1 < \alpha < 0$. They proved that

$$\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B}) = D_{\alpha+1},$$

where the Dirichlet space $D_{\alpha}$ consists of all analytic functions $f$ in the unit disk such that $f' \in L^2(\mathbb{D}, dA_\alpha)$. See also [20], and [16] where in the latter the authors studied similar problems in the unit ball of $n$-dimensional complex space $\mathbb{C}^n$.

Inspired by the aforementioned works, we will study the concept of a sub-Hilbert space in the context of Fock-type spaces $F^2_{\alpha,\beta,s}$, where the indices $\alpha$ and $\beta$ appear in the exponential part of the weight, and $s$ can be thought of as the order of the fractional derivative; see the next section. However, on these spaces, multiplication by an entire non-constant function is never bounded, let alone contractive. We will therefore focus our attention to the symbols of the type $f(z) = (z/|z|)^m$. We prove

**Theorem 1.** Let $\alpha, \beta > 0$, $s \in \mathbb{R}$ and $m \in \mathbb{N}$, and let $T^\alpha_\beta,f$ be the Toeplitz operator on $F^2_{\alpha,\beta,s}$ induced by the symbol $f(z) = (z/|z|)^m$. We then have

$$\mathcal{H}(f) = \mathcal{H}(\overline{f}) = F^2_{\alpha,\beta,s+\beta/2}.$$

### 2. Fock-Sobolev type spaces

Let $\mathbb{C}$ denote the complex plane, $H(\mathbb{C})$ the space of entire functions, and $dA(z)$ the Lebesgue area measure on $\mathbb{C};$

$$dA(z) = \frac{1}{\pi}dxdy, \quad z = x + iy.$$  

For $\alpha, \beta > 0$ and $s \in \mathbb{R}$, we consider the weight

$$d\lambda_{\alpha,\beta,s}(z) = |z|^{2s}e^{-\alpha|z|^\beta}dA(z).$$

In the literature, it is common to normalize $d\lambda_{\alpha,\beta,s}$ into a probability measure. However, when $s \leq -1$, this weight is no longer integrable, and cannot be normalized in an obvious way. We refrain from normalizing the weight altogether because of this.
2.1. Case $s > -1$. We define the generalized Fock-Sobolev type space $F^2_{\alpha, \beta, s}$ as those elements in $H(C)$ that are square integrable over $C$ with respect to $d\lambda_{\alpha, \beta, s}$. That is,

$$F^2_{\alpha, \beta, s} = L^2_{\alpha, \beta, s} \cap H(C).$$

It is easy to see that $F^2_{\alpha, \beta, s}$ is a closed subspace of $L^2_{\alpha, \beta, s} = L^2(C, d\lambda_{\alpha, \beta, s})$, and a Hilbert space with the inner product

$$\langle f, g \rangle_{\alpha, \beta, s} = \int_C f(z)\overline{g(z)}d\lambda_{\alpha, \beta, s}(z).$$

2.2. Case $s \leq -1$. The spaces $F^2_{\alpha, \beta, s}$ also make sense for $s \leq -1$, but in that case, following the definition above would require the members $F^2_{\alpha, \beta, s}$ to have a deep enough zero at the origin. In [6] two ways to overcome this are presented. First, one could replace the term $|z|^{2s}$ in $d\lambda_{\alpha, \beta, s}$ by $(1 + |z|)^{2s}$. However, the other approach from [6] fits our calculations better. Given

$$f(z) = \sum_{k=0}^{\infty} f_k z^k,$$

let us denote by $p_N(f)$ the degree $N$ Maclaurin polynomial of $f$;

$$p_N(f)(z) = \sum_{k=0}^{N} f_k z^k.$$

Then, denote by $R_N(f) = f - p_N(f)$ the remainder, which in our case is going to determine the membership in $F^2_{\alpha, \beta, s}$.

By using the ceiling function, we define $N = -[s] - 1$ and introduce the inner product

$$\langle f, g \rangle_{\alpha, \beta, s} = \int_C R_N(f)(z)\overline{R_N(g)(z)}d\lambda_{\alpha, \beta, s}(z) + \sum_{k=0}^{N} f_k \overline{g_k}.$$

The space $F^2_{\alpha, \beta, s}$ consists of entire functions $f$ with

$$\|f\|_{\alpha, \beta, s}^2 := \langle f, f \rangle_{\alpha, \beta, s} < \infty,$$

and by the virtue of the above definition, always contains all polynomials. In practice, we will not need to worry about this definition, as we are only interested in $R_N(f)$ for large enough $N$.

2.3. Relation to other Fock spaces. For particular choice of parameters, the spaces $F^2_{\alpha, \beta, s}$ reduce to more well-known spaces. The choice $(\alpha, \beta, s) = (\alpha, 2, 0)$ gives rise to classical Fock spaces, where standard references include the book of Folland [11] and the more recent book of Zhu [23].

Adding the parameter $s$ is known to be equivalent to the membership of (fractional) derivatives in the standard Fock space. This motivates the terminology
Fock-Sobolev space, which corresponds to the choice \((\alpha, \beta, s) = (\alpha, 2, s)\) studied in [7, 6, 5]. These references do not always contain \(\alpha\) as a parameter, but passage to this more general case is easy for most purposes of this paper.

In [4], Bommier-Hato, Engliš and Youssfi studied the so-called Fock-type spaces. These correspond to changing the Gaussian in the weight: \((\alpha, \beta, s) = (1, \beta, 0)\). Here again, slightly more general parameters do not cause much of an obstacle.

Finally, there are several generalizations of the Fock-spaces to the case where the weight is non-radial; we mention [12], [14] and [19], but there are many more. These spaces are often called generalized Fock spaces, but we refrain from studying them, because having a radial weight is essential for our approach.

3. Gamma function and reproducing kernels

3.1. Gamma function. The Euler Gamma function (or simply the Gamma function) is a well-known special function that generalizes the concept of a factorial to non-integer values. As we have already seen, it appears naturally in the context of exponential weights.

The Gamma function can be defined by a convergent improper integral:

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad R(z) > 0.
\]

The Gamma function satisfies the crucial recurrence relation: \(\Gamma(z + 1) = z\Gamma(z)\), and the following standard estimate for fixed complex numbers \(a\) and \(b\)

\[
\frac{\Gamma(z + a)}{\Gamma(z + b)} \approx z^{a-b}, \quad z \to \infty.
\]

In this paper, we will need a more refined variant of the latter. The following formula can be found in [13].

\[
\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b}\left[1 + \frac{(a - b)(a + b - 1)}{2z} + \frac{(a - b)(a - b - 1)}{24z^2}ight]
\]

\[
\times \{3(a + b - 1)^2 - a + b - 1\} [1 + O(z^{-3})], \quad z \to \infty. \tag{1}
\]

Lemma 2. The Gamma function satisfies

\[
1 - \frac{(\Gamma(z + \frac{a+b}{2}))^2}{\Gamma(z+a)\Gamma(z+b)} = \frac{(a - b)^2}{4z} + O(z^{-2}), \quad z \to \infty.
\]

Proof. By using the equation (1) we can obtain estimates for \(\frac{\Gamma(z + \frac{a+b}{2})}{\Gamma(z+a)}\) and \(\frac{\Gamma(z + \frac{a+b}{2})}{\Gamma(z+b)}\), so that we have

\[
\frac{\Gamma(z + \frac{a+b}{2})}{\Gamma(z + a)} = z^{\frac{b-a}{2}}\left[1 + \frac{(b - a)(3a + b - 2)}{8z} + O(z^{-2})\right]
\]
and
\[ \frac{\Gamma(z + \frac{a+b}{2})}{\Gamma(z + b)} = z^{\frac{a-b}{2}} \left[ 1 + \frac{(a-b)(a+3b-2)}{8z} + O(z^{-2}) \right] \]
as \( z \to \infty \). Multiplying these completes the proof. \( \square \)

The equation (1) can also be used to partially refine the recurrence relation:

**Lemma 3.** For any complex number \( \delta \) the Gamma function satisfies
\[ \Gamma(z + \delta) = z^\delta \Gamma(z), \quad z \to \infty. \]

**Proof.** The proof is easy and we omit the details. \( \square \)

### 3.2. Reproducing kernels and projections.

The approach of this paper is based on identifying the sub-Hilbert space by calculating its reproducing kernel. This is a well-known approach, see [1, 18, 21, 22]. The theory of reproducing kernels is a fascinating field in its own right, extending far beyond what is needed here. Some classical references include [2] and [17].

Since the weight \( d\lambda_{\alpha,\beta,s} \) is radial, the Fock-type space \( F_{\alpha,\beta,s}^2 \) possesses a monomial Schauder basis. If \( s \leq -1 \) and \( n \leq -[s] - 1 \), we set \( e^{\alpha,\beta,s}_n(z) = z^n \) and observe that \( \|e^{\alpha,\beta,s}_n\|_{\alpha,\beta,s} = 1 \). Otherwise, we compute in polar coordinates and using the change of variables \( t = \alpha r^\beta \):

\[ \|z^n\|^2_{\alpha,\beta,s} = \int_C |z|^{2n} |z|^{2s} e^{-\alpha|z|^\beta} \, dA(z) \]
\[ = \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} r^{2n+2s+1} e^{-\alpha r^\beta} \, d\theta \, dr \]
\[ = \frac{2}{\beta \alpha} \int_0^\infty t^{\frac{2n+2s+2}{\beta} - 1} e^{-t} \, dt \]
\[ = \frac{2}{\beta \alpha} \Gamma\left( \frac{2n+2s+2}{\beta} \right). \]

So, for \( n > -[s] - 1 \), we observe that then the functions
\[ e^{\alpha,\beta,s}_n = \sqrt{\frac{\beta \alpha}{2\Gamma\left( \frac{2n+2s+2}{\beta} \right)}} z^n \]
are unit vectors, and \( (e^{\alpha,\beta,s}_n)_{n=0}^\infty \) forms the basis of \( F_{\alpha,\beta,s}^2 \).

Let \( K^{\alpha,\beta,s}_z \) denote the reproducing kernel of \( F_{\alpha,\beta,s}^2 \) – that is, the unique function in \( F_{\alpha,\beta,s}^2 \) with the property
\[ f(z) = \langle f, K^{\alpha,\beta,s}_z \rangle_{\alpha,\beta,s}, \quad f \in F_{\alpha,\beta,s}^2. \]
By a well-known identity, we obtain
\[ K_z^{\alpha,\beta,s}(\xi) = \sum_{n=0}^{\infty} \frac{\beta\alpha^{2n+2} \xi^n}{2^n \Gamma\left(\frac{2n+2+2\beta}{\beta}\right)} \]
when \( s > -1 \). When \( s \leq -1 \), we obtain
\[ K_z^{\alpha,\beta,s}(\xi) = \sum_{n=0}^{\infty} e_n^{\alpha,\beta,s}(\xi) e_n^{\alpha,\beta,s}(z) \]
\[ = \frac{1 - (\xi\xi)^{-|s|}}{1 - \xi\xi} + \sum_{n=-|s|}^{\infty} \frac{\beta\alpha^{2n+2} \xi^n}{2^n \Gamma\left(\frac{2n+2+2\beta}{\beta}\right)} \]
In either case, we are only interested the asymptotics of the general term in the sum as \( n \) is large; that is
\[ \frac{\beta\alpha^{2n+2} \xi^n}{2^n \Gamma\left(\frac{2n+2+2\beta}{\beta}\right)} \approx \frac{\alpha^{2n+2}}{\Gamma\left(\frac{2n+2+2\beta}{\beta}\right)}. \tag{2} \]
In general, these power series can be understood in terms of the generalized Mittag-Leffler functions; see [4]. Of course, it is well-known (there are many references, see for instance [23]) that
\[ K_z^{\alpha,2,0}(\xi) = \alpha e^{\alpha\xi\xi}. \]
Finally, we are now able to write the orthogonal projection (the Bergman projection) \( P^{\alpha,\beta,s} : L^2_{\alpha,\beta,s} \to F^2_{\alpha,\beta,s} \) as
\[ P^{\alpha,\beta,s} f(z) = \int_C \overline{f(\xi)} K_z^{\alpha,\beta,s}(\xi) d\lambda_{\alpha,\beta,s}(\xi). \]
By the standard theory of orthogonal projections, \( P^{\alpha,\beta,s} \) is bounded; in fact the norm of \( P^{\alpha,\beta,s} \) is one.

4. The main results

4.1. Toeplitz operators. Before proving the main result, a short discussion on Toeplitz operators in order. Given an essentially bounded function \( f \) on the complex plane, let \( M_f \) denote the multiplication induced by \( f \). It is clearly bounded from \( F^2_{\alpha,\beta,s} \to L^2_{\alpha,\beta,s} \). The Toeplitz operator
\[ T_f^{\alpha,\beta,s} : F^2_{\alpha,\beta,s} \to F^2_{\alpha,\beta,s} \]
is then defined as
\[ T_f^{\alpha,\beta,s} = P^{\alpha,\beta,s} M_f. \]
Observe that $T_{\alpha,\beta,s}^f$ is contractive, whenever $\|f\|_\infty \leq 1$. Since orthogonal projections are self-adjoint, it is easy to see that the adjoint of $T_{\alpha,\beta,s}^f$ equals $T_{\alpha,\beta,s}$.

In particular, if $\|f\|_\infty \leq 1$, the operators

$$I - T_{\alpha,\beta,s}^f T_{\alpha,\beta,s}^* \quad \text{and} \quad I - T_{\alpha,\beta,s}^* T_{\alpha,\beta,s}$$

are positive.

In [1, 18, 21, 22] the authors study function spaces on the unit disk, and problem of sub-Hilbert spaces induced by a Toeplitz operator (given by the orthogonal projection of the respective space). Special focus is given to symbols $\alpha, \beta$.

In [8] the authors study function spaces on the unit disk, and problem of sub-Hilbert spaces induced by a Toeplitz operator (given by the orthogonal projection of the respective space). Special focus is given to symbols $\alpha, \beta$.

Neither option seems to work directly for our setting. Instead we take:

$$f(z) = \left( \frac{z}{|z|} \right)^m$$

and

$$\overline{f}(z) = \left( \frac{\overline{z}}{|z|} \right)^m,$$

where the contractivity requirement is automatically satisfied.

### 4.2. Proof of the main result

We will make use of the following result, which can be found in [18] (it is proven for the unit disk, but the exact same argument works for any reproducing kernel Hilbert space).

**Lemma 4.** Let $H$ be a reproducing kernel Hilbert space over a domain $\Omega$, $K_z$ its reproducing kernel and $T^* : H \to H$ a contraction. Then the reproducing kernel of $\mathcal{K}(T)$ is given by $(I - TT^*)K_z$.

Note that every $F^2_{\alpha,\beta,s}$ is isometrically isomorphic to a weighted $\ell^2$ space, with the weight coming from the moments of the weight $d\lambda_{\alpha,\beta,s}(z)$. On the other hand, also the reproducing kernel is related to these moments. Therefore, in order to determine $\mathcal{K}(T)$ and $\mathcal{K}(T^*)$, it suffices to study the asymptotic of the power series expansion of the reproducing kernel.

We are now in position to prove the main theorem. Let $\alpha, \beta > 0$ and $s \in \mathbb{R}$, and let $T_{\alpha,\beta,s}$ be the Toeplitz operator on $F^2_{\alpha,\beta,s}$ induced by the symbol $f(z) = (z/|z|)^m$. We then have

$$\mathcal{K}(f) = \mathcal{K}(\overline{f}) = F^2_{\alpha,\beta,s+\beta/2}.$$

We now prove this.

**Proof.** Suppose $m$ is a natural number. We will calculate the reproducing kernels of sub-Fock-Sobolev Hilbert spaces. The formula

$$(I - T_{\alpha,\beta,s}^f T_{\alpha,\beta,s}^*)K_{\alpha,\beta,s}$$

gives the reproducing kernels of these spaces. So, we consider the Toeplitz operator induced by $(\frac{z}{|z|})^m$ in Fock-Sobolev spaces $F^2_{\alpha,\beta,s}$. By using the definition
of Toeplitz operator and the formula (2), we have

$$\frac{2}{\beta} \alpha^{-\frac{2s-2}{\beta}} T_{\langle |z| \rangle} z^n = \int_{\mathbb{C}} \left( \frac{\xi}{|\xi|} \right)^m |\xi| \sum_{k \geq 0} \frac{1}{\Gamma\left(\frac{2k+2s+2}{\beta}\right)} \left(\alpha^{2/\beta} z^{\xi} \right)^k |\xi|^{2s} e^{-\alpha|\xi|^\beta} dA(\xi)$$

By a similar calculation for $T_{\langle |z| \rangle^m} z^n$, using (2) we have

$$\frac{2}{\beta} \alpha^{-\frac{2s-2}{\beta}} T_{\langle |z| \rangle^m} z^n = \int_{\mathbb{C}} \left( \frac{\xi}{|\xi|} \right)^m |\xi| \sum_{k \geq 0} \frac{1}{\Gamma\left(\frac{2k}{\alpha} \right)} \left(\alpha^{2/\beta} z^{\xi} \right)^k |\xi|^{2s} e^{-\alpha|\xi|^\beta} dA(\xi)$$
\[
\Gamma \left( \frac{2}{\beta} (n + s + 1) - \frac{m}{\beta} \right) z^{n - m} \alpha^{\frac{1}{\beta} \left( -m - 2s - 2 \right)}.
\]

It follows that
\[
\left( I - T_{(z/|z|)^m} T_{(\bar{z}/|z|)^m} \right) z^n = 
\begin{align*}
&\left( 1 - \frac{\Gamma \left( \frac{2}{\beta} (n + s + 1) + \frac{m}{\beta} \right) \Gamma \left( \frac{2}{\beta} (n + s + 1) + \frac{m}{\beta} \right)}{\Gamma \left( \frac{2}{\beta} (s + m + n + 1) \right) \Gamma \left( \frac{2}{\beta} (s + n + 1) \right)} \right) z^n.
\end{align*}
\]

From Lemma (2), we conclude that
\[
1 - \frac{\Gamma \left( \frac{2}{\beta} (n + s + 1) + \frac{m}{\beta} \right) \Gamma \left( \frac{2}{\beta} (n + s + 1) + \frac{m}{\beta} \right)}{\Gamma \left( \frac{2}{\beta} (s + m + n + 1) \right) \Gamma \left( \frac{2}{\beta} (s + n + 1) \right)} \asymp \frac{1}{n},
\]
therefore
\[
\left( I - T_{(z/|z|)^m} T_{(\bar{z}/|z|)^m} \right) K_{z, \alpha, \beta}^s (\xi) = \sum_{n=0}^{\infty} \frac{1}{n} \frac{1}{\Gamma \left( \frac{2n + 2s + 2}{\beta} \right)} (\alpha^{2/\beta} \xi \bar{\xi})^n.
\]

For large enough \( n \), we have
\[
n \Gamma \left( \frac{2n + 2s + 2}{\beta} \right) \asymp \left( \frac{2n + 2s + 2}{\beta} \right) \Gamma \left( \frac{2n + 2s + 2}{\beta} \right) = \Gamma \left( \frac{2n + 2(s + \frac{\beta}{2}) + 2}{\beta} \right),
\]
Substituting (4) into (3), we get the reproducing kernel of Fock-Sobolev space of order \( s + \frac{\beta}{2} \), which completes the proof.

As a consequence of the main theorem, we obtain the following corollary for the Fock-Sobolev space \( F_{\alpha,2,s}^2 \).

**Corollary 5.** Let \( f(z) = (z/|z|)^m \), and let us consider Toeplitz operators acting on \( F_{\alpha,2,s}^2 \). Then
\[
H(f) = H(\overline{f}) = F_{\alpha,2,s+1}^2.
\]

Note that this is in line with the well-known Bergman space results of Zhu [21, 22] and Abkar-Jafarzadeh [1].
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