Potential density of projective varieties having an int-amplified endomorphism

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Abstract. We consider the potential density of rational points on an algebraic variety defined over a number field $K$, i.e., the property that the set of rational points of $X$ becomes Zariski dense after a finite field extension of $K$. For a non-uniruled projective variety with an int-amplified endomorphism, we show that it always satisfies potential density. When a rationally connected variety admits an int-amplified endomorphism, we prove that there exists some rational curve with a Zariski dense forward orbit, assuming the Zariski dense orbit conjecture in lower dimensions. As an application, we prove the potential density for projective varieties with int-amplified endomorphisms in dimension $\leq 3$. We also study the existence of densely many rational points with the maximal arithmetic degree over a sufficiently large number field.

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1. Introduction

Let $K$ be a number field with a fixed algebraic closure $\overline{K}$. Given a variety $X$ over $K$, we are interested in the set of $K$-rational points $X(K)$ of $X$. More specifically, we study the potential density of varieties over $K$.

Definition 1.1. A variety $X$ defined over a number field $K$ is said to satisfy potential density if there is a finite field extension $K \subseteq L$ such that $X_L(L)$ is Zariski dense in $X_L$, where $X_L := X \times_{\text{Spec } K} \text{Spec } L$. 

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The potential density of varieties over number fields has been investigated in several papers. The potential density problem is attractive because the potential density of a variety is pretty much governed by its geometry. See [Cam04] for a conjecture characterising varieties satisfying potential density. However, algebraic varieties for which the potential density is verified are very few. See [Has03] for a survey of studies on the potential density problem.

In this paper, we first study the potential density of varieties admitting int-amplified endomorphisms. For the definition of int-amplified endomorphisms, see 2.1(11). Recently, the equivariant minimal model program for varieties with int-amplified endomorphisms was established (cf. [MZ20]). It has been used to study arithmetic-dynamical problems (cf. [MY19], [MMSZ20]). It turns out that the equivariant minimal model program is also useful for the potential density problem.

Our main conjecture is the following.

**Conjecture 1.2 (Potential density under int-amplified endomorphisms).** Let $X$ be a projective variety defined over a number field $K$. Suppose that $X$ admits an int-amplified endomorphism. Then $X$ satisfies potential density.

The endomorphism being int-amplified is a crucial assumption in Conjecture 1.2 above. Indeed, consider $X = X_1 \times C$ where $X_1$ is any smooth projective variety and $C$ is any smooth projective curve of genus at least 2. Such $X$ does not satisfy potential density (cf. Remark 1.4(2)). It does not have any int-amplified endomorphisms either; this is because every surjective endomorphism $f$ of $X$, after iteration, has the form $(x_1, x_2) \mapsto (g(x_1, x_2), x_2)$ for some morphism $g : X_1 \times C \rightarrow X_1$ by [San20, Lemma 4.5], and hence descends to the identity map $id_C$ on $C$ via the natural projection $X \rightarrow C$; thus, the iteration and hence $f$ itself are not int-amplified (cf. [Men20, Lemma 3.7 and Theorem 1.1]).

One might think that Conjecture 1.2 is too strong. In fact, the following even stronger conjecture has already been long outstanding. We refer to Medvedev–Scanlon [MS14, Conjecture 7.13], and Amerik–Bogomolov–Rovinsky [ABR11] for the details.

**Conjecture 1.3 (Zariski dense orbit conjecture).** Let $X$ be a variety defined over an algebraically closed field $k$ of characteristic zero and $f : X \rightarrow X$ a dominant rational map. If the $f^*$-invariant function field $k(X)^f$ is trivial, that is, $k(X)^f = k$, then there exists some $x \in X(k)$ whose (forward) $f$-orbit $O_f(x) := \{f^n(x) \mid n \geq 0\}$ is well-defined (i.e., $f$ is defined at $f^n(x)$ for any $n \geq 0$) and Zariski dense in $X$.

Note that Conjecture 1.3 with $f$ being int-amplified implies Conjecture 1.2 (cf. Lemmas 2.2 and 2.3).

**Remark 1.4.** We recall some known cases of the potential density problem and Conjecture 1.3.

1. Unirational varieties and abelian varieties over number fields satisfy potential density (cf. [Has03, Corollary 3.3 and Proposition 4.2]).
(2) Let $X$ be a variety with a dominant rational map $X \to C$ to a curve of genus $\geq 2$ over a number field. Then $X$ does not satisfy potential density (cf. [Fal83] and [Has03, Proposition 3.1]).

(3) Conjecture 1.3 holds for any pair $(X, f)$ with $X$ being a curve (cf. [Ame11, Corollary 9]).

(4) Conjecture 1.3 holds for any pair $(X, f)$ with $X$ being a projective surface and $f$ a surjective endomorphism of $X$ (cf. [Xie19, JXZ20]).

We first prove Conjecture 1.2 for rationally connected varieties in dimension $\leq 3$.

**Proposition 1.5.** Let $X$ be a rationally connected projective variety over the number field $K$. Suppose that $\dim X \leq 3$ and $X$ admits an int-amplified endomorphism. Then $X$ satisfies potential density.

Conjecture 1.2 also has a positive answer for non-uniruled varieties in any dimension:

**Proposition 1.6.** Let $X$ be a non-uniruled projective variety over the number field $K$. Suppose that $X$ admits an int-amplified endomorphism. Then $X$ satisfies potential density.

With the help of Propositions 1.5 and 1.6, we are able to show:

**Theorem 1.7.** Let $X$ be a normal projective variety over the number field $K$ with at worst $\mathbb{Q}$-factorial klt singularities. Suppose that $\dim X \leq 3$ and $X$ admits an int-amplified endomorphism. Then $X$ satisfies potential density.

In the last section, we study Question 1.9 below, which is also arithmetic in nature, initiated in [KS14] and further studied in [SS20] and [SS21].

**Definition 1.8** (cf. [SS20, Definition 1.4]). Let $X$ be a projective variety over a number field $K$ and $f : X \to X$ a surjective morphism. We recall the inequality

$$\alpha_f(x) \leq d_1(f)$$

between the arithmetic degree $\alpha_f(x)$ at a point $x \in X(\overline{K})$ and the first dynamical degree $d_1(f)$ of $f$ (cf. 2.1(12) and (13)). Let $L$ be an intermediate field: $K \subseteq L \subseteq \overline{K}$. We say that $(X, f)$ has densely many $L$-rational points with the maximal arithmetic degree if there is a subset $S \subseteq X(L)$ satisfying the following conditions:

1. $S$ is Zariski dense in $X_L$;
2. the equality $\alpha_f(x) = d_1(f)$ holds for all $x \in S$; and
3. $O_f(x_1) \cap O_f(x_2) = \emptyset$ for any pair of distinct points $x_1, x_2 \in S$.

Following [SS21], we introduce the following notation. We say that $(X, f)$ satisfies $(DR)_L$ if $(X, f)$ has densely many $L$-rational points with the maximal arithmetic degree. We say that $(X, f)$ satisfies $(DR)$ if there is a finite field extension $K \subseteq L (\subseteq \overline{K})$ such that $(X, f)$ satisfies $(DR)_L$. 

**Question 1.9.** Let $X$ be a projective variety over $K$ and $f : X \to X$ a surjective endomorphism. Assume that $X$ satisfies potential density. Does $(X, f)$ satisfy $(DR)$?

Question 1.9 has a positive answer for smooth projective surfaces when $d_1(f) > 1$ (cf. [SS21, Theorem 1.5]). We generalise it to (possibly singular) projective surfaces:

**Theorem 1.10.** Let $X$ be a normal projective surface over the number field $K$ satisfying potential density, and $f : X \to X$ a surjective morphism with $d_1(f) > 1$. Then $(X, f)$ satisfies $(DR)$.

The following is an affirmative answer to Question 1.9 for int-amplified endomorphisms on rationally connected threefolds:

**Theorem 1.11.** Let $X$ be a rationally connected smooth projective threefold over the number field $K$ and $f : X \to X$ an int-amplified endomorphism. Then $(X, f)$ satisfies $(DR)$.

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**2. Preliminaries**

**2.1. Notation and terminology**

1. Let $K$ be a number field. We work over $K$ when considering the potential density. We fix an algebraic closure $\overline{K}$ of $K$.
2. Let $k$ be an algebraically closed field of characteristic zero. We work over $k$ when considering geometric properties.
3. A variety means a geometrically integral separated scheme of finite type over a field.
4. Let $X$ be a variety over $K$ and $f : X \to X$ a morphism (over $K$). We denote $X^\vee_K := X \times_{\text{Spec} K} \text{Spec} K$ and $f^\vee_K : X^\vee_K \to X^\vee_K$ the induced morphism (over $\overline{K}$).
5. The symbol $\sim_R$ denotes the $\mathbb{R}$-linear equivalence on Cartier divisors.
6. We refer to [KM98] for definitions of Q-factoriality and klt singularities.
7. A variety $X$ of dimension $n$ is **uniruled** if there is a variety $U$ of dimension $n - 1$ and a dominant rational map $\mathbb{P}^1 \times U \to X$.
8. Let $X$ be a proper variety over a field $k$. We say that $X$ is **rationally connected** if there is a family of proper algebraic curves $U \to Y$ whose geometric fibres are irreducible rational curves with cycle morphism $U \to X$ such that $U \times_Y U \to X \times X$ is dominant (cf. [Kol96, Chapter IV, Definition 3.2]). When $k$ is algebraically closed of characteristic zero, if $X$ is rationally connected, then any two closed points of $X$ are connected by an irreducible rational curve over $k$ (by applying [Kol96, Chapter IV, Theorem 3.9] to a
resolution of \( X \). The converse holds when \( k \) is also uncountable (cf. [Kol96, Chapter IV, Proposition 3.6.2]).

(9) A normal projective variety \( X \) is said to be \( Q \)-abelian if there is a finite surjective morphism \( \pi : A \to X \), which is étale in codimension 1, with \( A \) being an abelian variety.

(10) For a morphism \( f : X \to X \) and a point \( x \in X \), the forward \( f \)-orbit of \( x \) is the set \( O_f(x) := \{ x, f(x), f^2(x), \ldots \} \). We denote the Zariski closure of \( O_f(x) \) by \( Z_f(x) \). More generally, for a closed subset \( Y \subseteq X \), we denote \( O_f(Y) := \bigcup_{n=0}^{\infty} f^n(Y) \) and its Zariski-closure \( Z_f(Y) := \overline{O_f(Y)} \). We say that \( O_f(Y) \) is Zariski dense if \( Z_f(Y) = X \).

(11) A surjective morphism \( f : X \to X \) of a projective variety is called int-amplified if there exists an ample Cartier divisor \( H \) on \( X \) such that \( f^*H - H \) is ample. In particular, polarised endomorphisms are int-amplified.

(12) Let \( X \) be a projective variety and \( f : X \to X \) a surjective morphism. The first dynamical degree \( d_1(f) \) of \( f \) is the limit
\[
d_1(f) := \lim_{n \to \infty} ((f^n)^*H - H)^{\dim X - 1}/n,
\]
where \( H \) is an ample Cartier divisor on \( X \). This limit always converges and is independent of the choice of \( H \) (cf. [DS05]). Dynamical degrees are invariant under the conjugation by generically finite maps (cf. [Zha09, Lemma 2.6]).

(13) Let \( X \) be a projective variety over \( K \) and \( f : X \to X \) a surjective morphism. Fix a (logarithmic) height function \( h_H \geq 1 \) associated to an ample Cartier divisor \( H \) on \( X \). For \( x \in X(\overline{K}) \), the arithmetic degree \( \alpha_f(x) \) of \( f \) at \( x \) is the limit
\[
\alpha_f(x) := \lim_{n \to \infty} h_H(f^n(x))^{1/n}.
\]
This limit always converges and is independent of the choices of \( H \) and \( h_H \) (cf. [KS16]).

**Lemma 2.2.** Let \( X \) be a projective variety over \( k \) and \( f : X \to X \) an int-amplified endomorphism. Then \( k(X)^f = k \). In particular, if Conjecture 1.3 holds for \( (X, f) \), then there exists some \( x \in X(k) \) such that \( O_f(x) \) is Zariski dense in \( X \).

**Proof.** Assume to the contrary that there is a nonconstant rational function \( \phi : X \to \mathbb{P}^1 \) such that \( \phi \circ f = \phi \). Let \( \Gamma \) be the graph of the rational map \( \phi : X \to \mathbb{P}^1 \) with projections \( \pi_1 : \Gamma \to X \) being birational and \( \pi_2 : \Gamma \to \mathbb{P}^1 \) being surjective. Then \( f \) lifts to an endomorphism \( f|_\Gamma \) on \( \Gamma \) such that \( \pi_1 \circ f|_\Gamma = f \circ \pi_1 \) and \( \pi_2 \circ f|_\Gamma = \pi_2 \). It follows from [Men20, Lemmas 3.4 and 3.5] that \( \text{id} : \mathbb{P}^1 \to \mathbb{P}^1 \) is int-amplified, which is absurd.

**Lemma 2.3.** Let \( X \) be a projective variety over \( K \), \( f : X \to X \) a surjective morphism, and \( Z \subseteq X \) a subvariety which satisfies potential density (e.g., \( Z \) is an abelian variety or unirational; see Remark 1.4(1)). If \( O_f(Z) \) is Zariski dense, then \( X \) satisfies potential density.
Proof. Replacing $K$ with a finite extension, we may assume that $Z(K)$ is Zariski dense in $Z$. Then the union $\bigcup_{n=0}^{\infty} f^n(Z(K))$ is a Zariski dense set of $K$-rational points of $X$. \qed

3. Rationally connected varieties: Proof of Proposition 1.5

Lemma 3.1. Let $X$ be a rationally connected projective variety over $k$ and of dimension $d \geq 1$, and $f : X \to X$ an int-amplified endomorphism. Assume Conjecture 1.3 in dimension $\leq d - 1$. Then there is a rational curve $C \subseteq X$ such that $O_f(C)$ is Zariski dense.

Proof. If we have a Zariski dense $f$-orbit $O_f(x)$, take any rational curve $C$ passing through $x$. Clearly, $O_f(C)$ is Zariski dense. So we may assume that $f$ has no Zariski dense orbit.

Replacing $f$ by some positive power, we can take a point $x \in X(k)$ such that $Z_f(x)$ is irreducible with dimension $r < d$ (cf. e.g. [MMSZ20, Lemma 2.7]). By [Fak03, Theorem 5.1], the subset of $X(k)$ consisting of $f$-periodic points is Zariski dense in $X$. Pick an $f$-periodic point $y \in X(k) \setminus Z_f(x)$. After iterating $f$, we may assume that $y$ is an $f$-fixed point. Take a rational curve $C \subseteq X$ containing $x$ and $y$. Set $W := Z_f(C)$. If $W = X$, we are done. So we may assume that $W \subset X$. If $\dim W = r$, then $W$ has its irreducible decomposition as $W = Z_f(x) \cup W_1 \cup \cdots \cup W_m$. There is some $n \geq 0$ such that $f^n(x) \in Z_f(x) \setminus \bigcup_{i=1}^{m} W_i$. Then $f^n(C) \subseteq W$ but $f^n(C) \not\subseteq \bigcup_{i=1}^{m} W_i$. Hence $f^n(C) \not\subseteq Z_f(x)$. In particular, $y = f^n(y) \in f^n(C) \subseteq Z_f(x)$, a contradiction. Thus $r < \dim W (< \dim X = d)$.

Now there exists an $f$-periodic irreducible component $W' \subseteq W$ with $r < \dim W' < d$. Replacing $f$ by a positive power, we may assume that $W'$ is $f$-invariant. Then $f|_{W'}$ is an int-amplified endomorphism on $W'$ (cf. [Men20, Lemma 2.2]). By assumption, Conjecture 1.3 holds for $(W', f|_{W'})$. So there exists some $w \in W'(k)$ such that $Z_f(w) = Z_{f|_{W'}}(w) = W'$ (cf. Lemma 2.2). In particular, $Z_f(w)$ is irreducible with $\dim Z_f(w) > r$. Continuing this process, the lemma follows. \qed

Corollary 3.2. Let $X$ be a rationally connected projective variety over $k$ and of dimension $\leq 3$, and $f : X \to X$ an int-amplified endomorphism. Then there is a rational curve $C \subseteq X$ such that $O_f(C)$ is Zariski dense.

Proof. This follows from Remark 1.4(3), (4), and Lemma 3.1. \qed

Proof of Proposition 1.5. By applying Corollary 3.2 to $(X_{\overline{k}}, f|_{X_{\overline{k}}})$, we know that there is a rational curve $C \subseteq X_{\overline{k}}$ such that $O_f(C)$ is Zariski dense. Replacing $K$ with a finite extension, we may assume that $C$ is defined over $K$. Then $O_f(C)$ is Zariski dense in $X$. The theorem follows from Lemma 2.3. \qed
4. Int-amplified endomorphisms: Proofs of Proposition 1.6 and Theorem 1.7

Lemma 4.1. (cf. [Men20, Theorem 1.9]) Let $X$ be a normal projective variety over $k$ and $f : X \to X$ an int-amplified endomorphism. Assume one of the following conditions.

(i) $X$ is non-uniruled.
(ii) $X$ has at worst $Q$-factorial klt singularities, and $K_X$ is pseudo-effective, i.e.,
    
    $K_X$ is in the closure of the cone of effective $\mathbb{R}$-divisors.

Then $X$ is a $Q$-abelian variety. In particular, $f$ has a Zariski dense orbit.

Proof. The first claim is [Men20, Theorem 1.9]. Now there is a finite cover $\pi : A \to X$ (étale in codimension 1) from an abelian variety $A$ with $f$ lifted to an int-amplified endomorphism $g$ on $A$ (cf. [NZ10, Lemma 2.12] and [Men20, Lemma 3.5]). Since Conjecture 1.3 holds for endomorphisms on abelian varieties (cf. [GS16]), $g$ has a Zariski dense orbit $O_g(a)$ for some $a \in A(k)$ (cf. Lemma 2.2). Then $O_f(\pi(a))$ is a Zariski dense orbit of $f$. □

Lemma 4.2. Let $X$ be a normal projective variety over $k$ and of dimension $\leq 3$ with at worst $Q$-factorial klt singularities. Let $f : X \to X$ be an int-amplified endomorphism. Then there exists a rational subvariety $Z \subseteq X$ of dimension $\geq 0$, such that $O_f(Z)$ is Zariski dense.

Proof. By Remark 1.4(3), (4), and Lemma 2.2, the assertion holds when $\dim X \leq 2$. Then by Corollary 3.2 and Lemma 4.1, we may assume that $X$ is a threefold that is uniruled and not rationally connected, and $K_X$ is not pseudo-effective.

By [MZ20], replacing $f$ with an iteration, we can run an $f$-equivariant minimal model program:

$$X = X_0 \dasharrow \cdots \dasharrow X_1 \dasharrow \cdots \dasharrow X_m = X' \xrightarrow{\pi} Y,$$

where each $\mu_i$ is a birational map and $\pi$ is a Mori fibre space with $\dim Y < \dim X' = 3$. If $\dim Y = 0$, then $X'$ is klt Fano. Hence $X'$ and $X$ are rationally connected (cf. [Zha06, Theorem 1]), contradicting our extra assumption. Thus $\dim Y = 1, 2$. Since $\dim Y \leq 2$, the int-amplified endomorphism $g := f|_Y$ has a Zariski dense orbit $O_g(y)$ by Remark 1.4(3), (4) and Lemma 2.2 (cf. [Men20, Lemmas 3.4 and 3.5]). A general fibre of $\pi$ is a klt Fano variety with dimension $\dim X' - \dim Y$ (cf. [KM98, Lemma 5.17(1)]). So, replacing $y$ by $g^N(y)$ for a suitable $N \geq 0$, we may assume that $F := \pi^{-1}(y)$ is a klt Fano variety of dimension equal to $\dim X' - \dim Y \in \{1, 2\}$, and hence a rational variety. Clearly, $O_f(F)$ is Zariski dense in $X$ by construction, where $F \subseteq X$ is the strict transform of $F \subseteq X'$. □

Proof of Proposition 1.6. Since being uniruled and the potential density are birational properties (cf. 2.1(7) and [Has03, Proposition 3.1]), they are invariant under the normalisation map. Also, since an int-amplified endomorphism on the variety $X$ lifts to an int-amplified endomorphism on its normalisation
(cf. [Men20, Lemma 3.5]), we may assume that $X$ is normal. Then the proposition follows from Lemmas 2.3 and 4.1.

**Proof of Theorem 1.10.** This follows from Lemmas 2.3 and 4.2.

**5. The maximal arithmetic degree: Proofs of Theorems 1.10 and 1.11**

In this section, we study Question 1.9. We first prepare some results and then we prove Theorem 1.10. We begin with:

**Lemma 5.1.** Let $X, Y$ be normal projective varieties over $K$, and $f : X \to X$ and $g : Y \to Y$ surjective endomorphisms. Assume that there is a surjective morphism $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. Then we have:

1. If $\pi$ is generically finite and $(X, f)$ satisfies (DR), then $(Y, g)$ also satisfies (DR).
2. Suppose $\pi$ is birational. Then $(X, f)$ satisfies (DR) if and only if so does $(Y, g)$.

**Proof.** Assume first that $\pi$ is generically finite. Let $X \xrightarrow{\pi'} X' \xrightarrow{\phi} Y$ be the Stein factorisation of $\pi$, where $\pi'$ is a projective morphism with connected fibres (indeed, $\pi'_* \mathcal{O}_X \cong \mathcal{O}_{X'}$) to a normal variety $X'$, and $\phi$ is a finite morphism (cf. [Har77, Chapter III, Corollary 11.5]). Since $\pi \circ f = g \circ \pi$ and $\phi$ is finite, we see that $\pi' \circ f$ contracts every fibre of $\pi'$. By the rigidity lemma (cf. [Deb01, Lemma 1.15]), there is a morphism $f' : X' \to X'$ such that $\pi' \circ f = f' \circ \pi'$ and $\phi \circ f' = g \circ \pi$. By [SS21, Lemma 3.2], for (1), we only need to show that $(X', f')$ satisfies (DR), which can be deduced from (2); for (2), we only need to show that if $(X, f)$ satisfies (DR), then so does $(Y, g)$.

Let $\Sigma \subseteq Y$ be the subset consisting of points $y$ such that $\dim \pi^{-1}(y) > 0$, and $E := \pi^{-1}(\Sigma) \subseteq X$, which is a closed proper subset. Since $\pi$ has connected fibres by Zariski’s Main Theorem (cf. [Har77, Chapter III, Corollary 11.4]), $\pi|_{X \setminus E} : X \setminus E \to Y \setminus \Sigma$ is an isomorphism. Since $g$ is finite, both $\Sigma$ and $Y \setminus \Sigma$ are $g^{-1}$-invariant. There is an induced surjective morphism $f|_{X \setminus E} : X \setminus E \to X \setminus E$ such that $\pi|_{X \setminus E} \circ f|_{X \setminus E} = g|_{Y \setminus \Sigma} \circ \pi|_{X \setminus E}$. Let $L$ be a finite field extension of $K$ such that $(X, f)$ satisfies (DR)$_L$. Then there exists a sequence of $L$-rational points $S_X = \{x_i\}_{i=1}^{\infty} \subseteq X(L) \setminus E$ such that

- (i) $S_X$ is Zariski dense in $X_L$;
- (ii) $\alpha_f(x_i) = d_i(f)$ for all $i$; and
- (iii) $\mathcal{O}_f(x_i) \cap \mathcal{O}_f(x_j) = \emptyset$ for $i \neq j$.

Thus $y_i := \pi(x_i)$ is well-defined and $S_Y := \{y_i\}_{i=1}^{\infty}$ satisfies the conditions of (DR)$_L$ for $(Y, g)$; note that $d_i(f) = d_i(g)$ and $\alpha_f(x_i) = \alpha_g(y_i)$ (cf. [Sil17, Lemma 3.2] in the smooth case, or [MMSZ20, Lemma 2.8] in general).

We need the following from [SS20].
Lemma 5.2 (cf. [SS20, Theorem 4.1]). Let $X$ be a projective variety over $K$ and $f : X \rightarrow X$ a surjective morphism with $d_1(f) > 1$. Assume the following condition:

(†) There is a numerically non-zero nef $\mathbb{R}$-Cartier divisor $D$ on $X$ such that $f^*D \sim_{\mathbb{R}} d_1(f)D$, and for any proper closed subset $Y \subseteq X$, there exists a morphism $g : \mathbb{P}^1_k \rightarrow X$ such that $g(Z) \not\subseteq Y$ and $g^*D$ is ample.

Then $(X, f)$ satisfies $(DR)_K$.

We also need the following structure theorem of endomorphisms.

Proposition 5.3 (cf. [JXZ20, Theorem 1.1]). Let $f : X \rightarrow X$ be a non-isomorphic surjective endomorphism of a normal projective surface over $k$. Then, replacing $f$ with a positive power, one of the following holds.

(i) $\rho(X) = 2$; there is a $\mathbb{P}^1$-fibration $X \rightarrow C$ to a smooth projective curve of genus $\geq 1$, and $f$ descends to an automorphism of finite order on the curve $C$.

(ii) $f$ lifts to an endomorphism $f|_V$ on a smooth projective surface $V$ via a generically finite surjective morphism $V \rightarrow X$.

(iii) $X$ is a rational surface.

Proof. We use [JXZ20, Theorem 1.1]. Cases (1), (3) and (8) imply our (ii). Cases (4) $\sim$ (7) and (9) lead to our (iii). Case (2) implies our (i), noting that $f$ cannot be polarised since it descends to an automorphism and hence $\rho(X) = 2$ by [MZ19, Theorem 5.4].

Proof of Theorem 1.10. When $f$ is an automorphism, we may take an equivariant resolution of $(X, f)$ and assume that $X$ is smooth (cf. Lemma 5.1). In this case, the theorem follows from [SS21, Theorem 1.5].

Now we assume that $\deg(f) \geq 2$. We apply Proposition 5.3 to $(X, f)$ (cf. [SS21, Lemma 3.3]). In either case, we may replace $K$ with a finite field extension so that the varieties and morphisms are defined over $K$.

In Case 5.3(ii), the theorem follows from Lemma 5.1 and [SS21, Theorem 1.5]. In Case 5.3(iii), the theorem is a consequence of [SS20, Theorem 1.11].

In Case 5.3(i), we may assume $g(C) = 1$; otherwise, $X$ does not satisfy potential density (cf. Remark 1.4(2)). Note that $\pi : X \rightarrow C$ has a section $S$ over $\overline{K}$ (the classical Tsen’s theorem). After replacing $K$ by a finite extension, we may assume that $S$ is defined over $K$. Let $F$ be a general fibre of $\pi$, which is a rational curve over $\overline{K}$ since $\pi$ is a $\mathbb{P}^1$-fibration. Replacing $K$ by a finite extension, we may assume that $F$ is defined over $K$. Then $S$ intersects $F$ at a $K$-rational point. Hence $F \cong \mathbb{P}^1$ over $K$. By [MZ19, Theorem 6.4], there is a numerically non-zero nef $\mathbb{R}$-Cartier divisor $D$ on $X$ such that $f^*D \sim_{\mathbb{R}} d_1(f)D$, after possibly replacing $K$ with a finite field extension. The numerical equivalence class of $D$ is not a multiple of that of the fibre $F$ since $f^*F \sim_{\mathbb{R}} F$ and $d_1(f) > 1$. Then $(D \cdot F) > 0$, by the Hodge index theorem. Thus, $(X, f)$ satisfies the condition (†) in Lemma 5.2 and hence satisfies $(DR)$. 

□
Before proving Theorem 1.11, we need a stronger version of Corollary 3.2 in dimension 3.

**Lemma 5.4.** Let $X$ be a rationally connected smooth projective threefold over $k$ and $f : X \rightarrow X$ an int-amplified endomorphism. Let $D$ be a numerically non-zero nef $\mathbb{R}$-Cartier divisor on $X$. Then there is a rational curve $C \subseteq X$ such that $O_f(C)$ is Zariski dense and $(D \cdot C) > 0$.

**Proof.** By [Yos21, Corollary 1.4], $X$ is of Fan type. Then there is a surjective morphism $\phi : X \rightarrow Y$ to a projective variety $Y$ such that $D \sim_{\mathbb{R}} \phi^* H$ for some ample $\mathbb{R}$-divisor on $Y$ by [Bir10, Theorem 3.9.1].

If $f$ has a Zariski dense orbit $O_f(x)$, then there is a rational curve passing through $x$ (such a curve exists since $X$ is rationally connected) and satisfying the claims. So we may assume that $f$ has no Zariski dense orbit.

Since Conjecture 1.3 is known for surfaces (cf. [JXZ20, Theorem 1.9]), we can take a point $x_0 \in X$ such that $\dim Z_f(x_0) = 2$ (cf. Proof of Lemma 3.1). Replacing $f$ by a power and $x_0$ by $f^N(x_0)$ for some integer $N \geq 0$, we may assume that $Z_f(x_0)$ is irreducible. We can take an $f$-periodic point $x_1 \in X$ such that $x_1 \notin Z_f(x_0) \cup \phi^{-1}(\phi(x_0))$ since the set of $f$-periodic points is Zariski dense in $X$ (cf. [Fak03, Theorem 5.1]). Take a rational curve $C \subseteq X$ containing $x_0, x_1$. We see that $O_f(C)$ is Zariski dense as in the proof of Lemma 3.1. Now $\phi(x)$ is not a point by construction, so

$$(D \cdot C) = (\phi^* H \cdot C) = (H \cdot \phi_* C) > 0.$$ 

Thus $C$ satisfies the claims. \hfill \Box

**Proof of Theorem 1.11.** By [MMSZZ21, Theorem 6.4], replacing $K$ by a finite extension, there is a numerically non-zero nef $\mathbb{R}$-Cartier divisor $D$ on $X$ such that $f^* D \sim_{\mathbb{R}} d_1(f)D$. Lemma 5.4 implies that, replacing $K$ with a finite extension so that the curve $C$ there (and $f$) are defined over $K$, the pair $(X, f)$ satisfies (†) in Lemma 5.2. Hence $(X, f)$ satisfies (DR). \hfill \Box

**References**


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