New York Journal of Mathematics

New York J. Math. **26** (2020) 116–128.

Indefinite Schwarz-Pick inequalities on the bidisk

Michio Seto

Abstract. Indefinite Schwarz-Pick inequalities for analytic self-maps of the bidisk are given as an application of the spectral theory on analytic Hilbert modules.

Dedicated to Professor Keiji Izuchi and Professor Takahiko Nakazi.

Contents

1. Introduction

The classical Schwarz-Pick inequality is fundamental in complex analysis and hyperbolic geometry, and also its functional analysis aspect has attracted a lot of interest. For example, Banach space theory related to the geometry derived from Schwarz-Pick inequality can be seen in Dineen [5]. In connection with operator theory, Schwarz-Pick type inequalities for analytic functions of one and several variables were discussed by Anderson-Rovnyak [3], Anderson-Dritschel-Rovnyak [2], Knese [12] and MacCluer-Stroethoff-Zhao [13, 14] in the context of Pick interpolation, realization formula, de Branges-Rovnyak space and composition operator. Now, the purpose of this paper is to give some variants of Schwarz lemma and Schwarz-Pick inequality for the bidisk. Here the author would like to emphasize the following three points:

(1) we deal with analytic self-maps of the bidisk,
(2) our inequalities are indefinite in a certain sense,
(3) our method is based on the theory of analytic Hilbert modules.

Received October 9, 2019.
2010 *Mathematics Subject Classification.* 46E22, 47B32.
Key words and phrases. Schwarz-Pick inequality, Hilbert module, Hardy space.

ISSN 1076-9803/2020

We shall introduce the language of the theory of Hilbert modules in the Hardy space over the bidisk. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, H^2 be the Hardy space over the bidisk $\mathbb{D}^2$, and H^∞ be the Banach algebra consisting of all bounded analytic functions on $\mathbb{D}^2$. Then H^2 is a Hilbert module over H^∞, that is, H^2 is a Hilbert space invariant under multiplication of functions in H^∞. A closed subspace $\mathcal{M}$ of H^2 is called a submodule if $\mathcal{M}$ is invariant under the module action. Comparing with the theory of the Hardy space over the unit disk $\mathbb{D}$, structure of submodules in H^2 is very complicated. However, there are some well-behaved classes of submodules in H^2. One of those classes was introduced by Izuchi, Nakazi and the author in [9], and those members are called submodules of INS type. In this paper, as an application of spectral theory on submodules of INS type, the following Schwarz-Pick type inequalities will be given (Theorem 4.2 and Theorem 4.3): if $\psi = (\psi_1, \psi_2)$ is an analytic self-map on $\mathbb{D}^2$, then

$$0 \leq d(\psi(z), \psi(w)) \leq \sqrt{2} d(z, w) < \sqrt{2} \quad (z, w \in \mathbb{D}^2),$$

where we set

$$d(z,w) = \sqrt{\left|\frac{z_1 - w_1}{1 - \overline{w_1} z_1}\right|^2 + \left|\frac{z_2 - w_2}{1 - \overline{w_2} z_2}\right|^2 - \left|\frac{z_1 - w_1}{1 - \overline{w_1} z_1} \cdot \frac{z_2 - w_2}{1 - \overline{w_2} z_2}\right|^2}$$

for $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{D}^2$. Further, if ψ belongs to a certain class defined in Section 2, then

$$0 \leq d(\psi(z), \psi(w)) \leq d(z, w) < 1 \quad (z, w \in \mathbb{D}^2).$$

This paper contains four sections. Section 1 is this introduction. In Section 2, three classes of tuples of analytic functions on $\mathbb{D}^2$ are defined, and we show they are nontrivial. In Sections 3 and 4, as an application of the theory of analytic Hilbert modules, indefinite variants of Schwarz lemma and Schwarz-Pick inequality are given, respectively.

2. Schur-Drury-Agler class

Let k_λ denote the reproducing kernel of H^2 at λ in $\mathbb{D}^2$, that is,

$$k_\lambda(z) = \frac{1}{(1 - \overline{\lambda_1} z_1)(1 - \overline{\lambda_2} z_2)} \quad (z = (z_1, z_2), \lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2).$$

Then we set

$$\mathcal{D} = \left\{ \sum_\lambda c_\lambda k_\lambda \text{ (a finite sum)} : \lambda \in \mathbb{D}^2, c_\lambda \in \mathbb{C} \right\},$$

the linear space generated by all reproducing kernels of H^2. We shall consider unbounded Toeplitz operators with symbols in H^2. Let f be a function in H^2. Then T_f denotes the multiplication operator of f, where we fix $\mathcal{D}$ for the domain of T_f . Then, since

$$\langle k_\lambda, T_f k_\mu \rangle = \langle \overline{f(\lambda)} k_\lambda, k_\mu \rangle \quad (\lambda, \mu \in \mathbb{D}^2),$$

T_f^* is defined on $\mathcal{D}$ and we have

$$T_f^* k_\lambda = \overline{f(\lambda)} k_\lambda \quad (\lambda \in \mathbb{D}^2).$$

Definition 2.1. *Let m and n be non-negative integers. We consider a tuple*

$$\Phi_{m,n} = (\varphi_1, \dots, \varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n})$$

of $m+n$ analytic functions in H^2. Then $\mathcal{S}(\mathbb{D}; m, n)$ denotes the set of all $\Phi_{m,n}$ satisfying the following operator inequality on $\mathcal{D}$:

$$0 \leq \sum_{j=1}^{m} T_{\varphi_j} T_{\varphi_j}^* - \sum_{k=m+1}^{m+n} T_{\varphi_k} T_{\varphi_k}^* \leq I.$$

Equivalently, $\Phi_{m,n}$ belongs to $\mathcal{S}(\mathbb{D}; m, n)$ if and only if

$$0 \leq \frac{\sum_{j=1}^{m} \overline{\varphi_j(\lambda)} \varphi_j(z) - \sum_{k=m+1}^{m+n} \overline{\varphi_k(\lambda)} \varphi_k(z)}{(1 - \overline{\lambda_1} z_1)(1 - \overline{\lambda_2} z_2)} \leq \frac{1}{(1 - \overline{\lambda_1} z_1)(1 - \overline{\lambda_2} z_2)}$$

as kernel functions.

Since the author has been influenced by Drury [6], in our paper, we would like to call $\mathcal{S}(\mathbb{D}^2; m, n)$ a Schur-Drury-Agler calss of $\mathbb{D}^2$. Here two remarks are given. First, unbounded functions are not excluded from $\mathcal{S}(\mathbb{D}^2; m, n)$ (cf. Definition 1 in Jury [11] for the Drury-Arveson space). Throughout this paper, a triplet $(\varphi_1, \varphi_2, \varphi_3)$ consisting of functions in H^∞ will be said to be bounded. Second, $\mathcal{S}(\mathbb{D}^2; m, n)$ is more restricted than the class, which might be called a Schur-Agler class in some literatures, consisting of tuples of functions in H^2 satisfying the operator inequality

$$I - \sum_{j=1}^{m} T_{\varphi_j} T_{\varphi_j}^* + \sum_{k=m+1}^{m+n} T_{\varphi_k} T_{\varphi_k}^* \geq 0.$$

In this paper, we will focus on the case where $m = 2$ and $n = 1$, that is,

$$\mathcal{S}(\mathbb{D}^2; 2, 1) = \{(\varphi_1, \varphi_2, \varphi_3) \in (H^2)^3 : 0 \leq T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^* \leq I\}.$$

This class is closely related to submodules of rank 3 (see Wu-S-Yang [15] and Yang [16, 17]). Further, we define other two classes as follows:

$$\mathcal{P}(\mathbb{D}^2; 2, 1) = \{(\varphi_1, \varphi_2, \varphi_3) \in (H^2)^3 : T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^* \geq 0\},$$
$$\mathcal{Q}(\mathbb{D}^2; 2, 1) = \{(\varphi_1, \varphi_2, \varphi_3) \in (H^2)^3 : I - T_{\varphi_1} T_{\varphi_1}^* - T_{\varphi_2} T_{\varphi_2}^* + T_{\varphi_3} T_{\varphi_3}^* \geq 0\}.$$

Trivially, $\mathcal{P}(\mathbb{D}^2; 2, 1) \cap \mathcal{Q}(\mathbb{D}^2; 2, 1) = \mathcal{S}(\mathbb{D}^2; 2, 1)$. First, we shall give examples of elements of $\mathcal{S}(\mathbb{D}^2; 2, 1)$.

Example 2.2. Let $\varphi_1 = \varphi_1(z_1)$ and $\varphi_2 = \varphi_2(z_2)$ be analytic functions of single variable. If $\|\varphi_1\|_\infty \leq 1$ and $\|\varphi_2\|_\infty \leq 1$, then $(\varphi_1, \varphi_2, \varphi_1\varphi_2)$ belongs to

$\mathcal{S}(\mathbb{D}^2; 2, 1)$. Indeed, since T_{φ_1} and T_{φ_2} are doubly commuting contractions,

$$\begin{aligned} &I - T_{\varphi_1}T_{\varphi_1}^* - T_{\varphi_2}T_{\varphi_2}^* + T_{\varphi_1\varphi_2}T_{\varphi_1\varphi_2}^* \\ &= (I - T_{\varphi_1}T_{\varphi_1}^*)(I - T_{\varphi_2}T_{\varphi_2}^*) \\ &= (I - T_{\varphi_1}T_{\varphi_1}^*)^{1/2}(I - T_{\varphi_2}T_{\varphi_2}^*)(I - T_{\varphi_1}T_{\varphi_1}^*)^{1/2} \\ &\geq 0, \end{aligned}$$

and

$$T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_1\varphi_2}T_{\varphi_1\varphi_2}^* = T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}(I - T_{\varphi_1}T_{\varphi_1}^*)T_{\varphi_2}^* \geq 0.$$

In particular, (z_1, z_2, z_1z_2) belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$ and

$$T_{z_1}T_{z_1}^* + T_{z_2}T_{z_2}^* - T_{z_1z_2}T_{z_1z_2}^*$$

is the orthogonal projection of H^2 onto the submodule generated by z_1 and z_2.

Example 2.3. Let $\psi(z) = (\psi_1(z), \psi_2(z))$ be an analytic self-map of $\mathbb{D}^2$. Then, trivially, $\operatorname{ran} T_{\psi_1\psi_2/\sqrt{2}}$ is a subspace of $\operatorname{ran} T_{\psi_1}$. Hence, by the Douglas range inclusion theorem and $\|T_{\psi_j}\| \leq 1$, we have

$$0 \leq T_{\psi_1\psi_2/\sqrt{2}}T_{\psi_1\psi_2/\sqrt{2}}^* \leq \frac{1}{2}T_{\psi_1}T_{\psi_1}^* \leq T_{\psi_1}T_{\psi_1}^* + T_{\psi_2}T_{\psi_2}^* \leq 2I.$$

Therefore, we have

$$\begin{aligned} 0 &\leq \frac{1}{2}(T_{\psi_1}T_{\psi_1}^* + T_{\psi_2}T_{\psi_2}^* - T_{\psi_1\psi_2/\sqrt{2}}T_{\psi_1\psi_2/\sqrt{2}}^*) \\ &= T_{\psi_1/\sqrt{2}}T_{\psi_1/\sqrt{2}}^* + T_{\psi_2/\sqrt{2}}T_{\psi_2/\sqrt{2}}^* - T_{\psi_1\psi_2/2}T_{\psi_1\psi_2/2}^* \\ &\leq T_{\psi_1/\sqrt{2}}T_{\psi_1/\sqrt{2}}^* + T_{\psi_2/\sqrt{2}}T_{\psi_2/\sqrt{2}}^* \\ &\leq I. \end{aligned}$$

Thus $(\psi_1/\sqrt{2}, \psi_2/\sqrt{2}, \psi_1\psi_2/2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$ for any analytic self-map (ψ_1, ψ_2) of $\mathbb{D}^2$.

Example 2.4. Further non-trivial examples of elements in $\mathcal{S}(\mathbb{D}^2; 2, 1)$ related to the theory of Hilbert modules in H^2 can be obtained from Theorem 4.3 in Wu-S-Yang [15].

$\mathcal{P}(\mathbb{D}^2; 2, 1)$ and $\mathcal{Q}(\mathbb{D}^2; 2, 1)$ are closed under composition of elements in $\mathcal{Q}(\mathbb{D}^2; 2, 1)$ in the following sense (cf. Theorem 2 in Jury [11]).

Theorem 2.5. *Let $(\varphi_1, \varphi_2, \varphi_3)$ be a triplet in $\mathcal{P}(\mathbb{D}^2; 2, 1)$ (resp. $\mathcal{Q}(\mathbb{D}^2; 2, 1)$), and $\psi = (\psi_1, \psi_2)$ be an analytic self-map of $\mathbb{D}^2$. If $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{Q}(\mathbb{D}^2; 2, 1)$, then $(\varphi_1 \circ \psi, \varphi_2 \circ \psi, \varphi_3 \circ \psi)$ belongs to $\mathcal{P}(\mathbb{D}^2; 2, 1)$ (resp. $\mathcal{Q}(\mathbb{D}^2; 2, 1)$).*

Proof. We set

$$\Phi(z, \lambda) = \overline{\varphi_1(\lambda)}\varphi_1(z) + \overline{\varphi_2(\lambda)}\varphi_2(z) - \overline{\varphi_3(\lambda)}\varphi_3(z).$$

If $(\varphi_1, \varphi_2, \varphi_3)$ belongs to $\mathcal{P}(\mathbb{D}^2; 2, 1)$, then, for any $\lambda_1, \dots, \lambda_n$ in $\mathbb{D}^2$, we have

$$\begin{aligned}
&\langle (T_{\varphi_1\circ\psi}T^*_{\varphi_1\circ\psi} + T_{\varphi_2\circ\psi}T^*_{\varphi_2\circ\psi} - T_{\varphi_3\circ\psi}T^*_{\varphi_3\circ\psi}) \sum_{i=1}^n c_i k_{\lambda_i}, \sum_{j=1}^n c_j k_{\lambda_j} \rangle \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \Phi(\psi(\lambda_j), \psi(\lambda_i)) \langle k_{\lambda_i}, k_{\lambda_j} \rangle \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \Phi(\psi(\lambda_j), \psi(\lambda_i)) \langle k_{\psi(\lambda_i)}, k_{\psi(\lambda_j)} \rangle \frac{\langle k_{\lambda_i}, k_{\lambda_j} \rangle}{\langle k_{\psi(\lambda_i)}, k_{\psi(\lambda_j)} \rangle} \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \langle (T_{\varphi_1}T^*_{\varphi_1} + T_{\varphi_2}T^*_{\varphi_2} - T_{\varphi_3}T^*_{\varphi_3}) k_{\psi(\lambda_i)}, k_{\psi(\lambda_j)} \rangle \frac{\langle k_{\lambda_i}, k_{\lambda_j} \rangle}{\langle k_{\psi(\lambda_i)}, k_{\psi(\lambda_j)} \rangle}.
\end{aligned}$$

Hence, by the definition of $\mathcal{Q}(\mathbb{D}^2; 2, 1)$ and Schur's theorem, we have

$$T_{\varphi_1\circ\psi}T^*_{\varphi_1\circ\psi} + T_{\varphi_2\circ\psi}T^*_{\varphi_2\circ\psi} - T_{\varphi_3\circ\psi}T^*_{\varphi_3\circ\psi} \geq 0.$$

Therefore, $(\varphi_1\circ\psi, \varphi_2\circ\psi, \varphi_3\circ\psi)$ belongs to $\mathcal{P}(\mathbb{D}^2; 2, 1)$. Similarly, considering $1 - \Phi$, we have the statement on $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. □

Corollary 2.6. *Suppose that $\psi = (\psi_1, \psi_2)$ is an analytic self-map of $\mathbb{D}^2$ and $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$. Then $(\varphi_1 \circ \psi, \varphi_2 \circ \psi, \varphi_3 \circ \psi)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$ for any triplet $(\varphi_1, \varphi_2, \varphi_3)$ in $\mathcal{S}(\mathbb{D}^2; 2, 1)$.*

3. Indefinite Schwarz lemmas

In this section, we shall give inequalities which can be seen as variants of Schwarz lemma. We need several lemmas.

Lemma 3.1. *Let T be a non-negative bounded linear operator, and P be an orthogonal projection on a Hilbert space $\mathcal{H}$. If there exists some constant $c > 0$ such that $0 \leq T \leq cP$, then we may take $c = \|T\|$.*

Proof. By elementary theory of self-adjoint operators, we have the conclusion. □

Lemma 3.2. *Let $(\varphi_1, \varphi_2, \varphi_3)$ be a bounded triplet in $\mathcal{P}(\mathbb{D}^2; 2, 1)$. Then φ_3 belongs to $\varphi_1 H^2 + \varphi_2 H^2$.*

Proof. Applying the Douglas range inclusion theorem to the operator inequality

$$T_{\varphi_3}T^*_{\varphi_3} \leq T_{\varphi_1}T^*_{\varphi_1} + T_{\varphi_2}T^*_{\varphi_2},$$

we have

$$\operatorname{ran} T_{\varphi_3} \subseteq \operatorname{ran} \sqrt{T_{\varphi_1}T^*_{\varphi_1} + T_{\varphi_2}T^*_{\varphi_2}} = \operatorname{ran} T_{\varphi_1} + \operatorname{ran} T_{\varphi_2}$$

(see Theorem 2.2 attributed to Crimmins in Fillmore-Williams [7] or Theorem 3.6 in Ando [4]). This concludes the proof. □

Lemma 3.3. *Let* $(\varphi_1, \varphi_2, \varphi_3)$ *be a bounded triplet in* $\mathcal{P}(\mathbb{D}^2; 2, 1)$*. If*

$$\varphi_1(0,0) = \varphi_2(0,0) = 0,$$

then

$$0 \leq |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_3(z)|^2 \leq \|T\|(|z_1|^2 + |z_2|^2 - |z_1 z_2|^2)$$

for any $z = (z_1, z_2)$ *in* $\mathbb{D}^2$*, where we set*

$$T = T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*.$$

Proof. Suppose that φ_1, φ_2 and φ_3 are bounded and $\varphi_1(0,0) = \varphi_2(0,0) = 0$. Then, it follows from Lemma 3.2 that $\varphi_3(0,0) = 0$. Hence φ_1, φ_2 and φ_3 belong to the submodule $\mathcal{M}_0 = z_1 H^2 + z_2 H^2$. Then we have

$$\operatorname{ran}(T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*) \subseteq \mathcal{M}_0.$$

Further, by elementary spectral theory, we have

$$\begin{aligned}\operatorname{ran}(T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*)^{1/2} &\subseteq \overline{\operatorname{ran}}(T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*)\\ &\subseteq \overline{\mathcal{M}_0} = \mathcal{M}_0.\end{aligned}$$

Hence, it follows from the Douglas range inclusion theorem that there exists a constant $c > 0$ such that

$$0 \leq T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^* \leq cP_{\mathcal{M}_0},$$

where $P_{\mathcal{M}_0}$ denotes the orthogonal projection of H^2 onto $\mathcal{M}_0$. By Lemma 3.1, we may take $c = \|T\|$. Hence we have

$$0 \leq T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^* \leq \|T\|P_{\mathcal{M}_0} = \|T\|(T_{z_1}T_{z_1}^* + T_{z_2}T_{z_2}^* - T_{z_1z_2}T_{z_1z_2}^*)$$

by Example 2.2. In particular,

$$\begin{aligned}&(|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 - |\varphi_3(\lambda)|^2)k_\lambda(\lambda)\\ &= \langle (T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*)k_\lambda, k_\lambda\rangle\\ &\leq \langle \|T\|(T_{z_1}T_{z_1}^* + T_{z_2}T_{z_2}^* - T_{z_1z_2}T_{z_1z_2}^*)k_\lambda, k_\lambda\rangle\\ &= \|T\|(|\lambda_1|^2 + |\lambda_2|^2 - |\lambda_1\lambda_2|^2)k_\lambda(\lambda)\end{aligned}$$

for any $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{D}^2$. This concludes the proof. □

Lemma 3.4. *If* (ψ_1, ψ_2) *is an analytic self-map on* $\mathbb{D}^2$*, then* $(\psi_1, \psi_2, \psi_1\psi_2)$ *belongs to* $\mathcal{P}(\mathbb{D}^2; 2, 1)$*.*

Proof. Since $\|\psi_j\|_\infty \leq 1$ for $j = 1, 2$, we have

$$T_{\psi_1}T_{\psi_1}^* + T_{\psi_2}T_{\psi_2}^* - T_{\psi_1\psi_2}T_{\psi_1\psi_2}^* = T_{\psi_1}T_{\psi_1}^* + T_{\psi_2}(I - T_{\psi_1}T_{\psi_1}^*)T_{\psi_2}^* \geq 0.$$

Hence $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{P}(\mathbb{D}^2; 2, 1)$. □

The following theorem is a bidisk version of the Schwarz lemma.

Theorem 3.5. *If* $\psi = (\psi_1, \psi_2)$ *is an analytic self-map on* $\mathbb{D}^2$ *and* $\psi(0,0) = (0,0)$, *then*

$$0 \le |\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 \le \|T\|(|z_1|^2 + |z_2|^2 - |z_1 z_2|^2)$$

for any $z = (z_1, z_2)$ *in* $\mathbb{D}^2$, *where we set*

$$T = T_{\psi_1} T_{\psi_1}^* + T_{\psi_2} T_{\psi_2}^* - T_{\psi_1\psi_2} T_{\psi_1\psi_2}^*.$$

Proof. By Lemma 3.3 and Lemma 3.4, we have the conclusion. □

Proposition 3.6. *Let* $(\varphi_1, \varphi_2, \varphi_3)$ *be a triplet in* $\mathcal{S}(\mathbb{D}^2; 2, 1)$. *If* $\varphi_1(0,0) = \varphi_2(0,0) = 0$, *then*

$$0 \le |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_3(z)|^2 \le |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

for any $z = (z_1, z_2)$ *in* $\mathbb{D}^2$.

Proof. If $(\varphi_1, \varphi_2, \varphi_3)$ is bounded, then we have the conclusion immediately by Lemma 3.3. Suppose that $(\varphi_1, \varphi_2, \varphi_3)$ is unbounded. Setting $\psi_r(z_1, z_2) = (rz_1, rz_2)$ for $0 < r < 1$, $(\varphi_1 \circ \psi_r, \varphi_2 \circ \psi_r, \varphi_3 \circ \psi_r)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$ by Corollary 2.6 and Example 2.2. Moreover, $\varphi_1 \circ \psi_r$, $\varphi_2 \circ \psi_r$ and $\varphi_3 \circ \psi_r$ are bounded on $\mathbb{D}^2$, and $\varphi_1 \circ \psi_r(0,0) = \varphi_2 \circ \psi_r(0,0) = 0$. Hence we have

$$\begin{aligned} 0 &\le |\varphi_1(rz)|^2 + |\varphi_2(rz)|^2 - |\varphi_3(rz)|^2 \\ &= |\varphi_1 \circ \psi_r(z)|^2 + |\varphi_2 \circ \psi_r(z)|^2 - |\varphi_3 \circ \psi_r(z)|^2 \\ &\le |z_1|^2 + |z_2|^2 - |z_1 z_2|^2 \end{aligned}$$

by Lemma 3.3. Letting r tend to 1, we have the conclusion for unbounded triplets. □

Theorem 3.7. *Suppse that* $\psi = (\psi_1, \psi_2)$ *is an analytic self-map on* $\mathbb{D}^2$ *and* $(\psi_1, \psi_2, \psi_1\psi_2)$ *belongs to* $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. *If* $\psi(0,0) = (0,0)$, *then* $(\psi_1, \psi_2, \psi_1\psi_2)$ *belongs to* $\mathcal{S}(\mathbb{D}^2; 2, 1)$ *and*

$$0 \le |\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 \le |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

for any $z = (z_1, z_2)$ *in* $\mathbb{D}^2$. *Moreover, equality*

$$|\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 = |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

holds on some open set if and only if $\psi = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$ *or* $(e^{i\theta_2} z_2, e^{i\theta_1} z_1)$.

Proof. First, by Lemma 3.4, $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$. Hence, we have the inequality by Theorem 3.5. Next, we suppose that

$$|\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 = |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

on an open set V. Then, by the polarization (see p. 28 in Agler-McCarthy [1] or p. 2762 in Knese [12]), we have

$$\overline{\psi_1(\lambda)}\psi_1(z) + \overline{\psi_2(\lambda)}\psi_2(z) - \overline{\psi_1(\lambda)\psi_2(\lambda)}\psi_1(z)\psi_2(z) = \overline{\lambda_1} z_1 + \overline{\lambda_2} z_2 - \overline{\lambda_1\lambda_2} z_1 z_2$$

on $\overline{V} \times V$, and this identity can be extended to $\mathbb{D}^2 \times \mathbb{D}^2$. Then, for $j = 1, 2$, we have

$$\left|\frac{\partial \psi_1}{\partial z_j}\right|^2 + \left|\frac{\partial \psi_2}{\partial z_j}\right|^2 - \left|\frac{\partial \psi_1 \psi_2}{\partial z_j}\right|^2 = \left|\frac{\partial z_1}{\partial z_j}\right|^2 + \left|\frac{\partial z_2}{\partial z_j}\right|^2 - \left|\frac{\partial z_1 z_2}{\partial z_j}\right|^2.$$

Hence we have

$$\left|\frac{\partial \psi_1}{\partial z_j}(0,0)\right|^2 + \left|\frac{\partial \psi_2}{\partial z_j}(0,0)\right|^2 = 1. \tag{3.1}$$

Similarly, we have

$$\left|\frac{\partial^2 \psi_1}{\partial z_j^2}(0,0)\right|^2 + \left|\frac{\partial^2 \psi_2}{\partial z_j^2}(0,0)\right|^2 - 4\left|\frac{\partial \psi_1}{\partial z_j}(0,0)\frac{\partial \psi_2}{\partial z_j}(0,0)\right|^2 = 0. \tag{3.2}$$

It follows from (3.1) that

$$\|\psi_1\|^2 + \|\psi_2\|^2 \geq \left|\frac{\partial \psi_1}{\partial z_1}(0,0)\right|^2 + \left|\frac{\partial \psi_1}{\partial z_2}(0,0)\right|^2 + \left|\frac{\partial \psi_2}{\partial z_1}(0,0)\right|^2 + \left|\frac{\partial \psi_2}{\partial z_2}(0,0)\right|^2 = 2.$$

Hence, $\|\psi_1\| = 1$ and $\|\psi_2\| = 1$ and

$$\psi_i = c_{i1} z_1 + c_{i2} z_2 \quad (|c_{i1}|^2 + |c_{i2}|^2 = 1).$$

Further, by (3.2), we have

$$\frac{\partial \psi_1}{\partial z_j}(0,0)\frac{\partial \psi_2}{\partial z_j}(0,0) = 0,$$

that is, $c_{1j} c_{2j} = 0$. This concludes the proof. □

Corollary 3.8. *Let f be an analytic function on $\mathbb{D}^2$. If $\|f\|_\infty \leq 1$ and $f(0,0) = 0$, then*

$$0 \leq |f(z)|^2 \leq |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

for any $z = (z_1, z_2)$ in $\mathbb{D}^2$.

Proof. Set $\psi = (\psi_1, \psi_2) = (f, 0)$. Then ψ is an analytic self-map, $\psi(0,0) = (0,0)$ and $(\psi_1, \psi_2, \psi_1\psi_2) = (f, 0, 0)$ belongs to $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. □

Remark 3.9. Suppose that $\psi = (\psi_1, \psi_2)$ is an analytic self-map on $\mathbb{D}^2$ and $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$. Then, the proof of Theorem 1 in Jury [10] can be applied and we have that the composition operator C_ψ is contractive on H^2. As its corollary, the inequality in Theorem 3.7 is obtained.

Remark 3.10 (Kreĭn space geometry and $\mathbb{D}^2$)**.** We introduce a Kreĭn space structure into $\mathbb{C}^3$ as follows:

$$\langle z, w\rangle_{\mathcal{K}} = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3} \quad (z = (z_1, z_2, z_3), w = (w_1, w_2, w_3) \in \mathbb{C}^3).$$

Let $\mathcal{K}$ denote the Kreĭn space $(\mathbb{C}^3, \langle\cdot,\cdot\rangle_{\mathcal{K}})$, and let Φ be the map defined as follows:

$$\Phi : \mathbb{D}^2 \to \mathcal{K}, \quad (z_1, z_2) \mapsto (z_1, z_2, z_1 z_2).$$

Moreover, we set

$$\begin{aligned}\Omega &= \{(z_1, z_2) \in \mathbb{C}^2 : 0 \leq |z_1|^2 + |z_2|^2 - |z_1 z_2|^2 < 1\} \\ &= \{z \in \mathbb{C}^2 : 0 \leq \langle \Phi(z), \Phi(z) \rangle_{\mathcal{K}} < 1\}.\end{aligned}$$

Then, since

$$|z_1|^2 + |z_2|^2 - |z_1 z_2|^2 = 1 - (1 - |z_1|^2)(1 - |z_2|^2),$$

$\mathbb{D}^2$ is the bounded connected component of Ω, and $\partial\mathbb{D}^2$, the topological boundary of $\mathbb{D}^2$, is equal to the subset

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 - |z_1 z_2|^2 = 1\} = \{z \in \mathbb{C}^2 : \langle \Phi(z), \Phi(z) \rangle_{\mathcal{K}} = 1\}.$$

4. Indefinite Schwarz-Pick inequalities

Let $q_1 = q_1(z_1)$ and $q_2 = q_2(z_2)$ be inner functions of single variable. Then

$$\mathcal{M} = q_1 H^2 + q_2 H^2$$

is a submodule of H^2. This submodule was introduced by Izuchi-Nakazi-S [9], and is called a submodule of INS-type. In this section, we shall give an application of spectral theory on submodules of INS type. In the general theory of Hilbert modules in H^2, the core (defect) operator of a submodule $\mathcal{M}$ in H^2 is defined as follows:

$$\Delta_{\mathcal{M}} = P_{\mathcal{M}} - T_{z_1} P_{\mathcal{M}} T_{z_1}^* - T_{z_2} P_{\mathcal{M}} T_{z_2}^* + T_{z_1 z_2} P_{\mathcal{M}} T_{z_1 z_2}^*,$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection of H^2 onto $\mathcal{M}$. For a submodule of INS-type, it is known that

$$\Delta_{\mathcal{M}} = q_1 \otimes q_1 + q_2 \otimes q_2 - (q_1 q_2) \otimes (q_1 q_2),$$

where $\otimes$ denotes the Schatten form. Core operators were introduced and studied by Guo-Yang [8] and Yang [16] in detail, and which are devices connecting reproducing kernels and submodules. In particular, the following formula is useful:

$$k_\lambda(\Delta_{\mathcal{M}} k_\lambda) = P_{\mathcal{M}} k_\lambda. \tag{4.1}$$

Further, core operators of finite rank were discussed by Yang [17]. Let $\mathcal{M}$ be a submodule whose core operator $\Delta_{\mathcal{M}}$ is of finite rank. Then the rank of $\Delta_{\mathcal{M}}$ is odd. Moreover, if the rank of $\Delta_{\mathcal{M}}$ is $2n+1$, then the signature of $\Delta_{\mathcal{M}}$ is $(n+1, n)$. Hence $\Delta_{\mathcal{M}}$ has the following representation:

$$\Delta_{\mathcal{M}} = \sum_{j=1}^{n+1} \eta_j \otimes \eta_j - \sum_{j=n+2}^{2n+1} \eta_j \otimes \eta_j. \tag{4.2}$$

By application of those facts, Lemma 3.3 is generalized as follows.

Lemma 4.1. *Let $\mathcal{M}$ be a submodule of finite rank. We suppose that the core operator of $\mathcal{M}$ has the representation (4.2). If $(\varphi_1, \varphi_2, \varphi_3)$ is a bounded triplet in $\mathcal{P}(\mathbb{D}^2; 2, 1)$, and φ_1 and φ_2 belong to $\mathcal{M}$, then*

$$0 \leq |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_3(z)|^2 \leq \|T\| \left(\sum_{j=1}^{n+1} |\eta_j(z)|^2 - \sum_{j=n+2}^{2n+1} |\eta_j(z)|^2 \right)$$

for any z in $\mathbb{D}^2$, where we set

$$T = T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^*.$$

In particular, if $\mathcal{M} = q_1 H^2 + q_2 H^2$ for inner functions $q_1 = q_1(z_1)$ and $q_2 = q_2(z_2)$ of single variable, then

$$0 \leq |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_3(z)|^2 \leq \|T\| (|q_1(z_1)|^2 + |q_2(z_2)|^2 - |q_1(z_1) q_2(z_2)|^2)$$

for any $z = (z_1, z_2)$ in $\mathbb{D}^2$.

Proof. By the same argument as the first half of the proof of Lemma 3.3, we have

$$0 \leq T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^* \leq \|T\| P_{\mathcal{M}}.$$

Then, for any $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{D}^2$, we have

$$\begin{aligned}
&(|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 - |\varphi_3(\lambda)|^2) k_\lambda(\lambda) \\
&= \langle (T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^*) k_\lambda, k_\lambda \rangle \\
&\leq \langle \|T\| P_{\mathcal{M}} k_\lambda, k_\lambda \rangle \\
&= \|T\| \langle k_\lambda (\Delta_{\mathcal{M}} k_\lambda), k_\lambda \rangle \\
&= \|T\| \left\langle k_\lambda \left(\sum_{j=1}^{n+1} \eta_j \otimes \eta_j - \sum_{j=n+2}^{2n+1} \eta_j \otimes \eta_j \right) k_\lambda, k_\lambda \right\rangle \\
&= \|T\| \left(\sum_{j=1}^{n+1} |\eta_j(\lambda)|^2 - \sum_{j=n+2}^{2n+1} |\eta_j(\lambda)|^2 \right) k_\lambda(\lambda)
\end{aligned}$$

by (4.1). This concludes the proof. □

For $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{D}^2$, we set

$$b_{w_j}(z_j) = \frac{z_j - w_j}{1 - \overline{w_j} z_j} \quad (j = 1, 2).$$

Then, we note that

$$\begin{aligned}
&|b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1) b_{w_2}(z_2)|^2 \\
&= 1 - (1 - |b_{w_1}(z_1)|^2)(1 - |b_{w_2}(z_2)|^2) > 0.
\end{aligned}$$

Hence

$$d(z, w) = \sqrt{|b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1) b_{w_2}(z_2)|^2}$$

is defined. It should be mentioned here that d is a distance on $\mathbb{D}^2$ by Lemma 9.9 in Agler-McCarthy [1].

Theorem 4.2. *Let $\psi = (\psi_1, \psi_2)$ be an analytic self-map on $\mathbb{D}^2$. Then,*

$$0 \le d(\psi(z), \psi(w)) \le \sqrt{2}d(z, w) < \sqrt{2}$$

for any z and w in $\mathbb{D}^2$.

Proof. For $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{D}^2$, we set

$$\varphi_j(z) = b_{\psi_j(w)} \circ \psi(z) = \frac{\psi_j(z) - \psi_j(w)}{1 - \overline{\psi_j(w)}\psi_j(z)}.$$

Then, (φ_1, φ_2) is an analytic self-map on $\mathbb{D}^2$, and $(\varphi_1, \varphi_2, \varphi_1\varphi_2)$ belongs to $\mathcal{P}(\mathbb{D}^2; 2, 1)$ by Lemma 3.4. Further, since $\varphi_1(w) = \varphi_2(w) = 0$, φ_1 and φ_2 belong to the submodule $b_{w_1}(z_1)H^2 + b_{w_2}(z_2)H^2$. Hence, by Lemma 4.1, we have

$$\begin{aligned}
0 &\le |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_1(z)\varphi_2(z)|^2 \\
&\le \|T\|(|b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1)b_{w_2}(z_2)|^2) \\
&\le 2(|b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1)b_{w_2}(z_2)|^2) \\
&< 2.
\end{aligned}$$

This concludes the proof. □

Theorem 4.3. *Suppose that $\psi = (\psi_1, \psi_2)$ is an analytic self-map on $\mathbb{D}^2$ and $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. Then*

$$0 \le d(\psi(z), \psi(w)) \le d(z, w) < 1$$

for any z and w in $\mathbb{D}^2$. Moreover, equality

$$d(\psi(z), \psi(w)) = d(z, w) \quad (z, w \in V)$$

holds on some open set V if and only if ψ belongs to $\mathrm{Aut}(\mathbb{D}^2)$.

Proof. We shall use the same notations as those in the proof of Theorem 4.2. By the assumption and Corollary 2.6, $(\varphi_1, \varphi_2, \varphi_1\varphi_2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$. Hence we have $\|T\| \le 1$. Thus we have the first half. Next, suppose that

$$d(\psi(z), \psi(w)) = d(z, w) \quad (z, w \in V)$$

holds on some open set V. We fix a point w in V. Then we have

$$|\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_1(z)\varphi_2(z)|^2 = |b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1)b_{w_2}(z_2)|^2$$

for any z in V. Setting $\beta(z) = (b_{-w_1}(z_1), b_{-w_2}(z_2))$, $(\varphi_1 \circ \beta, \varphi_2 \circ \beta, (\varphi_1\varphi_2) \circ \beta)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$ by Corollary 2.6, $\varphi \circ \beta(0, 0) = (0, 0)$ by the definition of φ, and

$$|\varphi_1 \circ \beta(z)|^2 + |\varphi_2 \circ \beta(z)|^2 - |(\varphi_1\varphi_2) \circ \beta(z)|^2 = |z_1|^2 + |z_2|^2 - |z_1z_2|^2$$

for any z in $\beta^{-1}(V)$. Hence, by Theorem 3.7, we have

$$(\varphi_1 \circ \beta(z), \varphi_2 \circ \beta(z)) = (e^{i\theta_1}z_1, e^{i\theta_2}z_2) \quad \text{or} \quad (e^{i\theta_2}z_2, e^{i\theta_1}z_1).$$

This concludes the second half. □

Corollary 4.4. *Let f be an analytic function on $\mathbb{D}^2$. If $\|f\|_\infty \leq 1$, then*

$$0 \leq \left| \frac{f(z) - f(w)}{1 - \overline{f(w)} f(z)} \right| \leq d(z, w)$$

for any z and w in $\mathbb{D}^2$.

Proof. In the proof of Corollary 3.8, we showed that $(f, 0, 0)$ belongs to $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. □

Acknowledgements

The author is grateful to the referee for valuable comments and suggestions which have contributed to the preparation of the paper.

References

[1] Agler, Jim; McCarthy, John E. Pick interpolation and Hilbert function spaces. Graduate Studies in Mathematics, 44. *American Mathematical Society, Providence, RI*, 2002. xx+308 pp. ISBN: 0-8218-2898-3. MR1882259, Zbl 1010.47001, doi: 10.1090/gsm/044. 122, 125

[2] Anderson, J. Milne; Dritschel, Michael A.; Rovnyak, James. Schwarz–Pick inequalities for the Schur–Agler class on the polydisk and unit ball. *Comput. Methods Funct. Theory* **8** (2008), no. 1–2, 339–361. MR2419482, Zbl 1157.30019, arXiv:math/0702269, doi: 10.1007/BF03321692. 116

[3] Anderson, J. Milne; Rovnyak, James. On generalized Schwarz–Pick estimates. *Mathematika* **53** (2006), no. 1, 161–168. MR2304058, Zbl 1120.30001, doi: 10.1112/S0025579300000085. 116

[4] Ando, T. De Branges spaces and analytic operator functions. Hokkaido University, Sapporo, 1990. Zbl 0717.47003. 120

[5] Dineen, Seán. The Schwarz lemma. Oxford Mathematical Monographs. Oxford Science Publications. *The Clarendon Press, Oxford University Press, New York*, 1989. x+248 pp. ISBN: 0-19-853571-6. MR1033739, Zbl 0708.46046. 116

[6] Drury, Stephen W. Remarks on von Neumann's inequality. *Banach spaces, harmonic analysis, and probability theory* (Storrs, Conn., 1980/1981), 14–32, Lecture Notes in Math., 995. *Springer, Berlin*, 1983. MR0717226, Zbl 0523.47006, doi: 10.1007/BFb0061886. 118

[7] Fillmore, Peter A.; Williams, James P. On operator ranges. *Advances in Math.* **7** (1971), 254–281. MR0293441, Zbl 0224.47009, doi: 10.1016/S0001-8708(71)80006-3. 120

[8] Guo, Kunyu; Yang, Rongwei. The core function of submodules over the bidisk. *Indiana Univ. Math. J.* **53** (2004), no. 1, 205–222. MR2048190, Zbl 1062.47009, doi: 10.1512/iumj.2004.53.2327. 124

[9] Izuchi, Keiji; Nakazi, Takahiko; Seto, Michio. Backward shift invariant subspaces in the bidisc. II. *J. Operator Theory* **51** (2004), no. 2, 361–376. MR2074186, Zbl 1055.47009, doi: 10.14943/83803. 117, 124

[10] Jury, Michael T. Reproducing kernels, de Branges-Rovnyak spaces, and norms of weighted composition operators. *Proc. Amer. Math. Soc.* **135** (2007), no. 11, 3669–3675. MR2336583, Zbl 1137.47019, arXiv:0707.3426, doi: 10.1090/S0002-9939-07-08931-9. 123

[11] Jury, Michael T. Norms and spectral radii of linear fractional composition operators on the ball. *J. Funct. Anal.* **254** (2008), no. 9, 2387–2400. MR2409166, Zbl 1146.47014, doi: 10.1016/j.jfa.2008.01.016. 118, 119

[12] Knese, Greg. A Schwarz lemma on the polydisk. *Proc. Amer. Math. Soc.* **135** (2007), no. 9, 2759–2768. MR2317950, Zbl 1116.32003, arXiv:math/0512452, doi: 10.1090/S0002-9939-07-08766-7. 116, 122

[13] MacCluer, Barbara D.; Stroethoff, Karel; Zhao, Ruhan. Generalized Schwarz–Pick estimates. *Proc. Amer. Math. Soc.* **131** (2003), no. 2, 593–599. MR1933351, Zbl 1012.30015, doi: 10.1090/S0002-9939-02-06588-7. 116

[14] MacCluer, Barbara D.; Stroethoff, Karel; Zhao, Ruhan. Schwarz–Pick type estimates. *Complex Var. Theory Appl.* **48** (2003), no. 8, 711–730. MR2002120, Zbl 1031.30012, doi: 10.1080/0278107031000154867. 116

[15] Wu, Yue; Seto, Michio; Yang, Rongwei. Kreĭn space representation and Lorentz groups of analytic Hilbert modules. *Sci. China Math.* **61** (2018), no. 4, 745–768. MR3776465, Zbl 06879974, doi: 10.1007/s11425-016-9009-x. 118, 119

[16] Yang, Rongwei. The core operator and congruent submodules. *J. Funct. Anal.* **228** (2005), no. 2, 469–489. MR2175415, Zbl 1094.47007, doi: 10.1016/j.jfa.2005.06.022. 118, 124

[17] Yang, Rongwei. A note on classification of submodules in $H^2(\mathbb{D}^2)$. *Proc. Amer. Math. Soc.* **137** (2009), no. 8, 2655–2659. MR2497478, Zbl 1166.47010, doi: 10.1090/S0002-9939-09-09893-1.

118, 124

(Michio Seto) National Defense Academy, Yokosuka 239-8686, Japan
mseto@nda.ac.jp

This paper is available via http://nyjm.albany.edu/j/2020/26-5.html.