

Harmonic cocycles, von Neumann algebras, and irreducible affine isometric actions

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ABSTRACT. Let G be a compactly generated locally compact group and (π, \mathcal{H}) a unitary representation of G . The 1-cocycles with coefficients in π which are harmonic (with respect to a suitable probability measure on G) represent classes in the first reduced cohomology $\bar{H}^1(G, \pi)$. We show that harmonic 1-cocycles are characterized inside their reduced cohomology class by the fact that they span a minimal closed subspace of \mathcal{H} . In particular, the affine isometric action given by a harmonic cocycle b is irreducible (in the sense that \mathcal{H} contains no nonempty, proper closed invariant affine subspace) if and only if the linear span of $b(G)$ is dense in \mathcal{H} . Our approach exploits the natural structure of the space of harmonic 1-cocycles with coefficients in π as a Hilbert module over the von Neumann algebra $\pi(G)'$, which is the commutant of $\pi(G)$. Using operator algebras techniques, such as the von Neumann dimension, we give a necessary and sufficient condition for a factorial representation π without almost invariant vectors to admit an irreducible affine action with π as linear part.

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1. Introduction, statement of results, and background material

Let G be a locally compact group and (π, \mathcal{H}) a continuous unitary (or orthogonal) representation of G on a complex (or real) Hilbert space \mathcal{H} . Recall that a 1-cocycle with coefficients in π is a continuous map $b : G \rightarrow \mathcal{H}$ such that $b(gh) = b(g) + \pi(g)b(h)$ for all $g, h \in G$ and that a 1-cocycle is a *coboundary* if it is of the form ∂_v for some $v \in \mathcal{H}$, where $\partial_v(g) = \pi(g)v - v$ for $g \in G$. The space $Z^1(G, \pi)$ of 1-cocycles with coefficients in π is a vector space containing the space $B^1(G, \pi)$ of coboundaries as linear subspace. The 1-cohomology $H^1(G, \pi)$ is the quotient $Z^1(G, \pi)/B^1(G, \pi)$.

The space $B^1(G, \pi)$ is not necessarily closed in $Z^1(G, \pi)$, endowed with the topology of uniform convergence on compact subsets of G (see Proposition 1), and the *reduced 1-cohomology* with coefficients in π is defined as $\overline{H}^1(G, \pi) = Z^1(G, \pi)/\overline{B^1(G, \pi)}$.

Assume now that G is compactly generated, that is, $G = \cup_{n \in \mathbb{Z}} Q^n$ for a compact subset Q , which we can assume to be a neighbourhood of the identity $e \in G$ and to be symmetric ($Q^{-1} = Q$).

Harmonic 1-cocycles in $Z^1(G, \pi)$, with respect to an appropriate probability measure on G , form a set of representatives for the classes in the reduced cohomology $\overline{H}^1(G, \pi)$, as we will shortly explain. Such cocycles appear in [BeV] in the case where π is the regular representation of a discrete group G , in relation with the first ℓ^2 -Betti number of G ; they play an important role in Ozawa's recent proof of Gromov's polynomial growth theorem ([Oza]) as well as in the work [ErO] and [GoJ].

Harmonic 1-cocycles were implicitly introduced in [Gui, Theorem 2]; it was observed there that $Z^1(G, \pi)$ can be identified with a closed subspace of the Hilbert space $L^2(Q, \mathcal{H}, m_G)$, where m_G is a (left) Haar measure on G and so $\overline{H}^1(G, \pi)$ corresponds to the orthogonal complement $B^1(G, \pi)^\perp$ of $B^1(G, \pi)$ in $Z^1(G, \pi)$. Following [ErO], we prefer to embed $Z^1(G, \pi)$ in a more general Hilbert space, defined by a class of appropriate probability measures similar to those appearing there. For this, we consider the word length on G associated to Q , that is, the map $g \mapsto |g|_Q$, where

$$|g|_Q = \min\{n \in \mathbb{N} : g \in Q^n\}.$$

Definition 1. A probability measure μ on G is *cohomologically adapted* (or, more precisely, 1-cohomologically adapted) if it has the following properties:

- μ is symmetric;
- μ is absolutely continuous with respect to the Haar measure m_G ;

- μ is adapted: the support of μ is a generating set for G ;
- μ has a second moment: $\int_G |x|_Q^2 d\mu(x) < \infty$.

Observe that the class of cohomologically adapted measures is independent of the generating compact set Q , since the length functions associated to two compact generating sets are bi-Lipschitz equivalent.

We consider the Hilbert space $L^2(G, \mathcal{H}, \mu)$ of measurable square-integrable maps $F : G \rightarrow \mathcal{H}$. Then $Z^1(G, \pi)$ is a subset of $L^2(G, \mathcal{H}, \mu)$ (see Section 2). Moreover, the linear operator

$$\partial : \mathcal{H} \rightarrow Z^1(G, \pi), v \mapsto \partial_v$$

is bounded, has $B^1(G, \pi)$ as range, and it is straightforward to check that its adjoint is $-\frac{1}{2}M_\mu$, where

$$M_\mu : Z^1(G, \pi) \rightarrow \mathcal{H}, b \mapsto \int_G b(x)d\mu(x).$$

So, the orthogonal complement $B^1(G, \pi)^\perp$ of $B^1(G, \pi)$ in $Z^1(G, \pi)$ can be identified with the space of harmonic cocycles in the sense of the following definition. In particular, the reduced cohomology $\overline{H}^1(G, \pi)$ can be identified with $\text{Har}_\mu(G, \pi)$.

Definition 2. A cocycle $b \in Z^1(G, \pi)$ is μ -harmonic if $M_\mu(b) = 0$, that is, $\int_G b(x)d\mu(x) = 0$. We denote by $\text{Har}_\mu(G, \pi)$ the space of μ -harmonic cocycles in $Z^1(G, \pi)$ and by

$$P_{\text{Har}} : L^2(G, \mathcal{H}, \mu) \rightarrow \text{Har}_\mu(G, \pi)$$

the orthogonal projection on $\text{Har}_\mu(G, \pi)$.

Observe that, by the cocycle relation, $b \in Z^1(G, \pi)$ is μ -harmonic if and only if it has the mean value property

$$b(g) = \int_G b(gx)d\mu(x) \quad \text{for all } g \in G.$$

In our opinion, the Hilbert space structure of $\overline{H}^1(G, \pi)$ given by its realization as a space of harmonic cocycles, together with its module structure over the von Neumann algebra $\pi(G)'$ (see below), deserves more attention than it has received so far in the literature. Our aim in this paper is to use this structure in relation with a natural notion of irreducibility for affine isometric actions (see Definition 3).

Our first result shows that harmonic 1-cocycles b are characterized by a remarkable minimality property of the space $\overline{\text{span}(b(G))}$, the closure of the linear span of $b(G)$.

Theorem 1. *Let G be a compactly generated locally compact group. Let (π, \mathcal{H}) be an orthogonal or unitary representation of G and μ a cohomologically adapted probability measure on G . Let $b \in \text{Har}_\mu(G, \pi)$ be a μ -harmonic*

cocycle. We have

$$\overline{\text{span}(b(G))} = \bigcap_{b'} \overline{\text{span}(b'(G))},$$

where b' runs over the 1-cocycles in the class of b in $\overline{H}^1(G, \pi)$.

In particular, Theorem 1 shows that, for a μ -harmonic cocycle b , the closed linear subspace spanned by $b(G)$ only depends on the reduced cohomology class of b and not on the choice of μ .

Recall that, given a cocycle $b \in Z^1(G, \pi)$, a continuous action $\alpha_{\pi, b}$ of G on \mathcal{H} by affine isometries is defined by the formula

$$\alpha_{\pi, b}(g)v = \pi(g)v + b(g) \quad \text{for all } g \in G, v \in \mathcal{H}.$$

Conversely, let α be a continuous action of G on \mathcal{H} by affine isometries. Denote by $\pi(g)$ and $b(g)$ the linear part and the translation part of $\alpha(g)$ for $g \in G$. Then π is a unitary (or orthogonal) representation of G on \mathcal{H} , b is a 1-cocycle in $Z^1(G, \pi)$, and $\alpha = \alpha_{\pi, b}$. For all this, see Chapter 2 in [BHV].

The following notion of irreducibility of affine actions was introduced in [Ner] and further studied in [BPV].

Definition 3. An affine isometric action α of G on the complex or real Hilbert space \mathcal{H} is *irreducible* if \mathcal{H} has no nonempty, closed and proper $\alpha(G)$ -invariant affine subspace.

First examples of irreducible affine isometric actions arise as actions $\alpha_{\pi, b}$, where π is an irreducible unitary representation of G with non trivial 1-cohomology and $b \in Z^1(G, \pi)$ a cocycle which is not a coboundary. By [Sha1, Theorem 0.2], such a pair (π, b) always exists, provided G does not have Kazhdan's Property (T). A remarkable feature of irreducible affine isometric actions of a locally compact group G is that they remain irreducible under restriction to "most" lattices in G (see [Ner, 3.6], [BPV, Theorem 4.2]), whereas this is not true in general for irreducible unitary representations.

Let $b \in Z^1(G, \pi)$. Observe that $\text{span}(b(G))$ is $\alpha_{\pi, b}(G)$ -invariant. So, for $\alpha_{\pi, b}$ to be irreducible, it is necessary that $\text{span}(b(G))$ is dense in \mathcal{H} . This condition is not sufficient (see [BPV, Example 2.4]; however, see also Proposition 3 below). The following corollary of Theorem 1 relates harmonic cocycles to this question.

Corollary 1. Let $G, (\pi, \mathcal{H})$, and μ be as in Theorem 1. Let $b \in Z^1(G, \pi)$ and $P_{\text{Har}}b$ its projection on $\text{Har}_{\mu}(G, \pi)$.

- (i) If $\text{span}(P_{\text{Har}}b(G))$ is dense in \mathcal{H} , then the affine action $\alpha_{\pi, b}$ is irreducible.
- (ii) Assume that $B^1(G, \pi)$ is closed; if the affine action $\alpha_{\pi, b}$ is irreducible, then $\text{span}(P_{\text{Har}}b(G))$ is dense in \mathcal{H} .

Remark 1.

- (a) Point (ii) in Corollary 1 does not hold in general when $B^1(G, \pi)$ is not closed; indeed, let $G = \mathbb{F}_2$ denote the free group on 2 generators. On the one hand, $H^1(G, \pi) \neq 0$ for every unitary representation π of G (see [Gui, §9, Example 1]). On the other hand, there exists an irreducible unitary representation π of G with $\overline{H}^1(G, \pi) = 0$ (see [MaV, Theorem 1.1]), so that $\text{Har}_\mu(G, \pi) = 0$ for any cohomologically adapted probability measure μ on G . Now, let b be a 1-cocycle in $Z^1(G, \pi)$ which is not a coboundary. Then the affine action $\alpha_{\pi,b}$ is irreducible.
- (b) Although we will not need it, we will give an explicit formula for the projection

$$P_{\text{Har}} : Z^1(G, \pi) \rightarrow \text{Har}_\mu(G, \mu)$$

in the case where $B^1(G, \pi)$ is closed (see Proposition 4 below).

In view of Corollary 1, it is of interest to know when $B^1(G, \pi)$ is closed. Write $\mathcal{H} = \mathcal{H}^G \oplus \mathcal{H}^0$, where \mathcal{H}^G is the space of $\pi(G)$ -invariant vectors in \mathcal{H} and \mathcal{H}^0 its orthogonal complement. Let π^0 denote the restriction of π to \mathcal{H}^0 . Observe that $B^1(G, \pi^0) = B^1(G, \pi)$ and that $Z^1(G, \pi^0)$ is closed in $Z^1(G, \pi)$; so, the following result is both a (slight) strengthening and a consequence of Théorème 1 in [Gui].

Proposition 1 ([Gui]). *Let (π, \mathcal{H}) be an orthogonal or unitary representation of the σ -compact locally compact group G . Then $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$ if and only if (π^0, \mathcal{H}^0) does not weakly contain the trivial representation 1_G .*

Our approach to the proof of Theorem 1 uses the fact, observed in [BPV, §3.1] and [BeV] that $\overline{H}^1(G, \pi)$ is, in a natural way, a module over the (real or complex) von Neumann algebra $\pi(G)'$, which is the commutant of $\pi(G)$ in $\mathcal{B}(\mathcal{H})$; see Section 2. Viewing, as we do, $\overline{H}^1(G, \pi)$ as the Hilbert space $\text{Har}_\mu(G, \pi)$, one is lead to the study of $\text{Har}_\mu(G, \pi)$ as a Hilbert module over $\pi(G)'$.

For instance, if $\mathcal{M} := \pi(G)'$ is a finite von Neumann algebra (that is, if there exists a faithful finite trace on \mathcal{M}) then, we can define (as in [GHJ, Definition p.138] or [Bek, p. 327]) the *von Neumann dimension* of $\overline{H}^1(G, \pi)$ as

$$\dim_{\mathcal{M}} \overline{H}^1(G, \pi) := \dim_{\mathcal{M}} \text{Har}_\mu(G, \pi) \in [0, +\infty) \cup \{+\infty\};$$

for more details, see Section 2. It is worth mentioning that in case π is the regular representation of a discrete group G , $\dim_{\mathcal{M}} \overline{H}^1(G, \pi)$ coincides with $\beta_2^1(G)$, the first ℓ^2 -Betti number of G (see [BeV, Proposition 2]).

We now give some applications of von Neumann techniques to the problem of the existence of an irreducible affine isometric action of G with a given linear part π .

Let $b \in Z^1(G, \pi)$. It was shown in Item (A3) of Proposition 2.3 in [BPV] that $\alpha_{\pi,b}$ is irreducible if and only if $P_{\mathcal{K}}b \notin B^1(G, \pi_0)$ for every nonzero

subrepresentation (π_0, \mathcal{K}) of (π, \mathcal{H}) , where

$$P_{\mathcal{K}}b := P_{\mathcal{K}} \circ b \in Z^1(G, \pi_0)$$

is the orthogonal projection of b on \mathcal{K} . The following definition as well as the statement of Proposition 2 were suggested by the referee.

Definition 4. Let $b \in Z^1(G, \pi)$. The affine isometric action $\alpha_{\pi, b}$ is *strongly irreducible* if $P_{\mathcal{K}}b \notin \overline{B^1(G, \pi_0)}$ for every nonzero subrepresentation (π_0, \mathcal{K}) of (π, \mathcal{H}) .

Recall that a vector v in a Hilbert module over a von Neumann algebra \mathcal{M} is a *separating vector* for \mathcal{M} if $Tv = 0$ for $T \in \mathcal{M}$ implies $T = 0$.

Proposition 2. For $G, (\pi, \mathcal{H})$, and μ as in Theorem 1, and $b \in \text{Har}_{\mu}(G, \pi)$, the following properties are equivalent:

- (i) $\alpha_{\pi, b}$ is irreducible.
- (ii) $\alpha_{\pi, b}$ is strongly irreducible.
- (iii) b is a separating vector for $\mathcal{M} = \pi(G)'$.
- (iv) $\text{span}(b(G))$ is dense in \mathcal{H} .

We mention that, as shown in [BPV, Corollary 3.7], the equivalence of (i) and (iii) in Proposition 2 holds more generally for any $b \in Z^1(G, \pi)$. For an application of this equivalence in the case where G is a discrete finitely generated group and π a subrepresentation of a multiple of the regular representation of G , see [BPV, Theorem 4.25]. We extend this result to arbitrary factor representations, that is, to unitary representations (π, \mathcal{H}) such that the von Neumann subalgebra $\pi(G)''$ of $\mathcal{B}(\mathcal{H})$ generated by $\pi(G)$ is a factor (equivalently, such that $\pi(G)'$ is a factor). Concerning general facts about factors, such as their type classification, see [Dix1].

Theorem 2. Let (π, \mathcal{H}) be a factor representation of the compactly generated locally compact group G on the separable complex Hilbert space \mathcal{H} . Assume that $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$. Set $\mathcal{M} := \pi(G)'$ and let μ be a cohomologically adapted probability measure on G . Depending on the type of \mathcal{M} , there exists $b \in Z^1(G, \pi)$ such that $\alpha_{\pi, b}$ is irreducible if and only if:

- (i) The factor \mathcal{M} is of type I_{∞} or of type II_{∞} and its commutant in $\mathcal{B}(\text{Har}_{\mu}(G, \pi))$ is of infinite type (that is, of type I_{∞} or II_{∞} , respectively).
- (ii) The factor \mathcal{M} is of finite type (that is, of type I_n for $n \in \mathbb{N}$ or of type II_1) and $\dim_{\mathcal{M}} \text{Har}_{\mu}(G, \pi) \geq 1$.
- (iii) The factor \mathcal{M} is of type III and $\text{Har}_{\mu}(G, \pi) \neq \{0\}$.

Remark 2. Let (π, \mathcal{H}) be a unitary representation of G such that $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$; let

$$\pi = \int_{\Omega}^{\oplus} \pi_{\omega} d\nu(\omega)$$

be the central integral decomposition of π , so that the π_ω 's are mutually disjoint factor representations of G (see [Dix2, Theorem 8.4.2]). One checks that one has a corresponding decomposition of $\text{Har}_\mu(G, \pi)$ as a direct integral of Hilbert spaces:

$$\text{Har}_\mu(G, \pi) = \int_\Omega^\oplus \text{Har}_\mu(G, \pi_\omega) d\nu(\omega).$$

Moreover, $B^1(G, \pi_\omega)$ is closed in $Z^1(G, \pi_\omega)$ and there exists a separating vector for $\pi(G)'$ in $\text{Har}_\mu(G, \pi)$ if and only if there exists a separating vector for $\pi_\omega(G)'$ in $\text{Har}_\mu(G, \pi_\omega)$ for ν -almost every ω . So, Theorem 2 can be used to check the existence of an irreducible affine with *any* unitary representation π as linear part (provided $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$).

As an illustration of the use of Theorem 2, we will treat the example of a wreath product of the form $\Gamma = G \wr \mathbb{Z}$ and a unitary representation π of Γ which factorizes through a representation of G ; the reduced cohomology of such groups was considered in [Sha2, §5.4].

Theorem 3. *Let G be a finitely generated group, and let (π, \mathcal{H}) be a unitary representation of the wreath product $\Gamma = G \wr \mathbb{Z}$ in the separable Hilbert space \mathcal{H} . Assume that π factorizes through G and that $H^1(G, \pi) = 0$.*

- (i) *For a suitable cohomologically adapted probability measure μ on Γ , the space $\text{Har}_\mu(\Gamma, \mu)$ can be identified, as a module over $\pi(\Gamma)'$, with the Hilbert space \mathcal{H} .*
- (ii) *There exists an irreducible affine action of Γ with linear part π if and only if the representation (π, \mathcal{H}) is cyclic.*

Remark 3.

- (i) When π is a factor representation, a necessary and sufficient condition for the existence of a cyclic vector for $\pi(G)$ (equivalently, a separating vector for $\pi(G)'$) in \mathcal{H} is given in Theorem 2, with \mathcal{H} replacing $\text{Har}_\mu(G, \mu)$ there.
- (ii) By the Delorme–Guichardet theorem ([BHV, Theorem 2.12.4]), the condition $H^1(G, \pi) = 0$ is satisfied for every unitary representation π of G if (and only if) G has Kazhdan’s property (T).
- (iii) Assume that G is not virtually abelian (that is, G does not have an abelian normal subgroup of finite index). Then G has a factorial representation π for which $\pi(G)'$ is of any possible type. Indeed, G is not of type I, by Thoma’s theorem ([Tho, Satz 6]); the result follows then from Glimm’s theorem [Gli, Theorem 2]).

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2. The space of harmonic cocycles as a von Neumann algebra module

Let G be a locally compact group which is generated by a compact subset Q , which we assume to be a symmetric neighbourhood of the identity $e \in G$. Let (π, \mathcal{H}) be an orthogonal or unitary representation of G . The map

$$b \mapsto \|b\|_Q = \sup_{x \in Q} \|b(x)\|$$

is a norm which generates the topology of uniform convergence on compact subsets and for which $Z^1(G, \pi)$ is a Banach space.

Let $\mathcal{M} := \pi(G)'$ be the commutant of $\pi(G)$ in $\mathcal{B}(\mathcal{H})$, that is,

$$\mathcal{M} = \{T \in \mathcal{B}(\mathcal{H}) : T\pi(g) = \pi(g)T \text{ for all } g \in G\};$$

\mathcal{M} is a (real or complex) von Neumann algebra, that is, \mathcal{M} is a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed for the weak (or strong) operator topology.

As observed in [BPV, §3.1]), $H^1(G, \pi)$ is a module over \mathcal{M} ; indeed, if $b \in Z^1(G, \pi)$ and $T \in \pi(G)'$, then $Tb \in Z^1(G, \pi)$, where Tb is defined by

$$Tb(g) = T(b(g)) \quad \text{for all } g \in G;$$

moreover, $T\partial_v = \partial_{Tv}$ for every vector $v \in \mathcal{H}$.

Let μ be a cohomologically adapted probability measure on G (Definition 1). We consider the Hilbert space $L^2(G, \mathcal{H}, \mu)$ of measurable mappings $F : G \rightarrow \mathcal{H}$ such that

$$\|F\|_2^2 := \int_G \|F(x)\|^2 d\mu(x) < \infty.$$

Then every $b \in Z^1(G, \pi)$ belongs to $L^2(G, \mathcal{H}, \mu)$; indeed, the cocycle relation shows that

$$\|b(x)\| \leq |x|_Q \|b\|_Q \quad \text{for all } x \in G,$$

and hence

$$\|b\|_2^2 \leq \|b\|_Q^2 \int_G |x|_Q^2 d\mu(x) < \infty.$$

In fact, the norms $\|\cdot\|_2$ and $\|\cdot\|_Q$ on $Z^1(G, \pi)$ are equivalent (see [ErO, Lemma 2.1]). So, we can (and will) identify $Z^1(G, \pi)$ with a closed subspace of the Hilbert space $L^2(G, \mathcal{H}, \mu)$.

The von Neumann algebra \mathcal{M} acts on \mathcal{H} in the tautological way and on $L^2(G, \mathcal{H}, \mu)$ by

$$TF(g) = T(F(g)) \quad \text{for all } T \in \pi(G)', F \in L^2(G, \mathcal{H}, \mu), g \in G,$$

preserving $Z^1(G, \pi)$ and $B^1(G, \pi)$. Since the operator

$$M_\mu : Z^1(G, \mu) \rightarrow \mathcal{H}$$

is equivariant for these actions, the space $\text{Har}_\mu(G, \pi) = \ker M_\mu$ as well as its orthogonal complement $\overline{B^1(G, \pi)}$ are modules over \mathcal{M} .

The image of \mathcal{M} in $\mathcal{B}(L^2(G, \mathcal{H}, \mu)) \cong \mathcal{B}(L^2(G, \mu)) \otimes \mathcal{B}(\mathcal{H})$ is

$$\widetilde{\mathcal{M}} = I \otimes \pi(G)',$$

which is a von Neumann algebra isomorphic to \mathcal{M} . The orthogonal projection $P_{\text{Har}} : L^2(G, \mathcal{H}, \mu) \rightarrow \text{Har}_\mu(G, \pi)$ belongs to the commutant

$$\widetilde{\mathcal{M}}' = \mathcal{B}(L^2(G, \mu)) \otimes \pi(G)''$$

of \mathcal{M} in $\mathcal{B}(L^2(G, \mathcal{H}, \mu))$, where $\pi(G)''$ is the von Neumann algebra generated by $\pi(G)$ in $\mathcal{B}(\mathcal{H})$. The commutant of \mathcal{M} in $\text{Har}_\mu(G, \pi)$ is then the reduced von Neumann algebra (see Chap.1, §2, Proposition 1 in [Dix1])

$$P_{\text{Har}} \widetilde{\mathcal{M}}' P_{\text{Har}} = P_{\text{Har}} (\mathcal{B}(L^2(G, \mu)) \otimes \pi(G)'') P_{\text{Har}}.$$

Assume now that \mathcal{M} is a finite von Neumann algebra, with faithful normalized trace τ . Let $L^2(\mathcal{M})$ be the Hilbert space obtained from τ by the GNS construction. We identify \mathcal{M} with the subalgebra of $\mathcal{B}(L^2(\mathcal{M}))$ of operators given by left multiplication with elements from \mathcal{M} . The commutant of \mathcal{M} in $\mathcal{B}(L^2(\mathcal{M}))$ is $\mathcal{M}' = J\mathcal{M}J$, where $J : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ is the conjugate linear isometry which extends the map $\mathcal{M} \rightarrow \mathcal{M}, x \mapsto x^*$. The trace on \mathcal{M}' , again denoted by τ , is defined by $\tau(JxJ) = \tau(x)$ for $x \in \mathcal{M}$.

As every \mathcal{M} -module, $L^2(G, \mathcal{H}, \mu)$ can be identified with an \mathcal{M} -submodule of $L^2(\mathcal{M}) \otimes \ell^2$, with \mathcal{M} acting on $L^2(\mathcal{M}) \otimes \ell^2$ by $T \mapsto T \otimes I$. The orthogonal projection $Q : L^2(\mathcal{M}) \otimes \ell^2 \rightarrow L^2(G, \mathcal{H}, \mu)$ belongs to the commutant of \mathcal{M} in $\mathcal{B}(L^2(\mathcal{M}) \otimes \ell^2)$, which is $\mathcal{M}' \otimes \mathcal{B}(\ell^2)$. The projection $P = P_{\text{Har}} \circ Q$ belongs therefore to the commutant of \mathcal{M} in $\mathcal{B}(L^2(\mathcal{M}) \otimes \ell^2)$.

Let $\{e_n\}_n$ be a basis of ℓ^2 and let $(P_{ij})_{i,j}$ be the matrix of P with respect to the decomposition $L^2(\mathcal{M}) \otimes \ell^2 = \oplus_i (L^2(\mathcal{M}) \otimes e_i)$. Then every P_{ij} belongs to \mathcal{M}' and the von Neumann dimension of the \mathcal{M} -module $\text{Har}_\mu(G, \pi)$ is

$$\dim_{\mathcal{M}} \mathcal{H} = \sum_i \tau(P_{ii}).$$

3. Proofs of the main results

3.1. Proof of Theorem 1. Let $b_0 \in \text{Har}_\mu(G, \pi)$. Let $b_1 \in \overline{B^1(G, \pi)}$ and set $b := b_0 + b_1$. We claim that $b_0(G)$ is contained in the closure of $\text{span}(b(G))$.

Indeed, let \mathcal{K} denote the closure of $\text{span}(b(G))$ and $P_{\mathcal{K}} : \mathcal{H} \rightarrow \mathcal{K}$ the corresponding orthogonal projection. Since \mathcal{K} is $\pi(G)$ -invariant, $P_{\mathcal{K}}$ belongs to the commutant $\pi(G)'$ of $\pi(G)$. Therefore (see Section 2), $\mathcal{P}_{\mathcal{K}} b_0$ is contained in $\text{Har}_\mu(G, \pi)$ and $\mathcal{P}_{\mathcal{K}} b_1$ is contained in $\overline{B^1(G, \pi)}$. On the other hand, since b take its values in \mathcal{K} , we have that

$$P_{\mathcal{K}} b = b = b_0 + b_1.$$

It follows that $\mathcal{P}_{\mathcal{K}} b_0 = b_0$ and $\mathcal{P}_{\mathcal{K}} b_1 = b_1$. Therefore,

$$b_0(G) \subset \mathcal{K} = \overline{\text{span}(b(G))},$$

as claimed. □

3.2. A characterization of irreducible affine isometric actions. We will repeatedly use one of the several characterizations of irreducible affine actions from Proposition 2.1 in [BPV]; for the convenience of the reader, we give a direct and short argument.

Proposition 3 ([BPV]). *For $b \in Z^1(G, \pi)$, the following properties are equivalent:*

- (i) *The action $\alpha = \alpha_{\pi, b}$ is irreducible.*
- (ii) *the linear span of $(b + \partial_v)(G)$ is dense in \mathcal{H} for every $v \in \mathcal{H}$.*

Proof. Observe that

$$\alpha_{\pi, b + \partial_v}(g) = t_{-v} \circ \alpha_{\pi, b}(g) \circ t_v \quad \text{for all } g \in G, v \in \mathcal{H},$$

where t_v is the translation by v . So, $\alpha_{\pi, b}$ is irreducible if and only if $\alpha_{\pi, b + \partial_v}$ is irreducible. This shows that (i) implies (ii).

To show the converse implication, let F be a non empty closed $\alpha_{\pi, b}(G)$ -invariant affine subspace of \mathcal{H} . Then $F = v + \mathcal{K}$ for a vector $v \in \mathcal{H}$ and a closed linear subspace \mathcal{K} of \mathcal{H} . Set $b_0 := b + \partial_v$. Then

$$v + b_0(g) = \alpha_{\pi, b}(g)v \in F \quad \text{for all } g \in G,$$

and $b_0(G)$ is hence contained in \mathcal{K} . Therefore, $\mathcal{K} = \mathcal{H}$, since $\text{span}(b_0(G))$ is dense in \mathcal{H} . \square

3.3. Proof of Corollary 1. Let $b \in Z^1(G, \pi)$ and set

$$b_0 := P_{\text{Har}} b \in \text{Har}_{\mu}(G, \pi).$$

(i) Assume that $\text{span}(b_0(G))$ is dense in \mathcal{H} . By Theorem 1, the linear span of $(b + \partial_v)(G)$ is dense for every $v \in \mathcal{H}$, and Proposition 3 shows that $\alpha_{\pi, b}$ is irreducible.

(ii) Assume that $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$ and that $\alpha_{\pi, b}$ is irreducible. Write $b = b_0 + \partial_{v_0}$ for $b_0 = P_{\text{Har}} b$ and $v_0 \in \mathcal{H}$. Then $\alpha_{\pi, b_0} = \alpha_{\pi, b - \partial_{v_0}}$ is also irreducible, by Proposition 3; hence, $\text{span}(b_0(G))$ is dense. \square

3.4. Proof of Proposition 2. Let $b \in \text{Har}_{\mu}(G, \pi)$ and set $\mathcal{M} = \pi(G)'$. We establish the following sequence of implications:

$$(i) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (i).$$

Assume that $\alpha_{\pi, b}$ is irreducible; then $\text{span}(b(G))$ is dense in \mathcal{H} , by Proposition 3. This shows the implication (i) \Rightarrow (iv).

Assume that $\text{span}(b(G))$ is dense in \mathcal{H} and let (π_0, \mathcal{K}) be a nonzero subrepresentation of (π, \mathcal{H}) . On the one hand, the orthogonal projection $P_{\mathcal{K}}$ on \mathcal{K} belongs to \mathcal{M} and hence $P_{\mathcal{K}} b = P_{\mathcal{K}} \circ b$ belongs to $\text{Har}_{\mu}(G, \pi)$. On the other hand, we have $P_{\mathcal{K}} b \neq 0$, since $\text{span}(b(G))$ is dense in \mathcal{H} . Hence, $P_{\mathcal{K}} b \notin \overline{B^1(G, \pi)}$. This shows the implication (iv) \Rightarrow (ii).

The fact that (ii) \Rightarrow (i) being obvious, assume that $\alpha_{\pi, b}$ is irreducible. Let $T \in \mathcal{M}$ be such that $Tb = 0$, that is, $T(b(g)) = 0$ for all $g \in G$. Then $T = 0$, since $\text{span}(b(G))$ is dense in \mathcal{H} , by Theorem 1. So, b is a separating vector for \mathcal{M} . This shows the implication (i) \Rightarrow (iii)

Assume that b is a separating vector for \mathcal{M} . The orthogonal projection P on $b(G)^\perp$ belongs to \mathcal{M} (see the proof of Theorem 1); since $Pb = 0$, we have $P = 0$. Hence, $\text{span}(b(G))$ is dense in \mathcal{H} . So, $\alpha_{\pi,b}$ is irreducible by Corollary 1(i). This shows the implication (iii) \Rightarrow (i). \square

3.5. Proof of Theorem 2. Let (π, \mathcal{H}) be a unitary representation of G ; we assume that $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$. Let μ be a cohomologically adapted probability measure on G .

In view of Proposition 2, we have to investigate under which conditions $\mathcal{M} = \pi(G)'$ has a separating vector in $\text{Har}_\mu(G, \pi)$. We may assume that $\text{Har}_\mu(G, \pi) \neq \{0\}$.

Observe that a vector in $\text{Har}_\mu(G, \pi)$ is separating for \mathcal{M} if and only if it is cyclic for the commutant \mathcal{N} of \mathcal{M} in $\mathcal{B}(\text{Har}_\mu(G, \pi))$. Three cases can occur.

Case 1. \mathcal{N} is an infinite factor. Then \mathcal{M} always has a separating vector (see Corollaire 11 in Chap. III, §8 of [Dix1]).

Case 2. \mathcal{N} is a finite factor and \mathcal{M} is an infinite factor. Then \mathcal{N} has a cyclic vector in $\text{Har}_\mu(G, \pi)$ if and only if $\dim_{\mathcal{N}} \text{Har}_\mu(G, \pi) \leq 1$ (see [Bek, Corollary 1]). For this to happen a necessary condition is that \mathcal{M} is a finite factor. So, \mathcal{M} has no separating vector.

Case 3. \mathcal{N} and \mathcal{M} are finite factors. In this case, we have (see [GHJ, Prop. 3.2.5])

$$\dim_{\mathcal{M}} \text{Har}_\mu(G, \pi) \dim_{\mathcal{N}} \text{Har}_\mu(G, \pi) = 1;$$

hence, \mathcal{M} has a separating vector in $\text{Har}_\mu(G, \pi)$ if and only if

$$\dim_{\mathcal{M}} \text{Har}_\mu(G, \pi) \geq 1.$$

Claims (i), (ii), and (iii) follow from this discussion. \square

3.6. Proof of Theorem 3. We first consider the general case of the wreath product $\Gamma = G \wr H$ of two finitely generated groups G and H . Recall that $\Gamma = G \ltimes H^{(G)}$, for $H^{(G)} = \bigoplus_{g \in G} H$ and G acts on $H^{(G)}$ by shifting the copies of H . We view H as a subgroup of Γ , by identifying it with the copy of H inside $H^{(G)}$ indexed by e .

Let S_1 and S_2 finite symmetric generating sets for G and H , respectively. Then $S_1 \cup S_2$ is a finite symmetric generating set for Γ . Let μ_1 and μ_2 be cohomologically adapted probability measures on G and H respectively. Then $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ is a cohomologically adapted probability measure on Γ .

Let (π, \mathcal{H}) be a unitary representation of G , viewed as representation of Γ . We have an orthogonal $\pi(\Gamma)$ -invariant decompositions

$$\ell^2(\Gamma, \mathcal{H}, \mu) = \ell^2(G, \mathcal{H}, \mu_1) \oplus \ell^2(H, \mathcal{H}, \mu_2).$$

This decomposition gives rise to a decomposition

$$\text{Har}_\mu(\Gamma, \pi) = \text{Har}_{\mu_1}(G, \pi) \oplus \text{Har}_{\mu_2}(H, \pi).$$

Indeed, it is clear that $\text{Har}_\mu(\Gamma, \pi)$ is contained in $\text{Har}_{\mu_1}(G, \pi) \oplus \text{Har}_{\mu_2}(H, \pi)$; so, we only have to show the converse inclusion.

Let $b_1 \in \text{Har}_{\mu_1}(G, \pi)$. Define $b : \Gamma \rightarrow \mathcal{H}$ by $b(g, n) = b_1(g)$ for $(g, n) \in \Gamma$. Then b extends b_1 and one checks that $b \in Z^1(\Gamma, \pi)$ and that $b \in \text{Har}_\mu(\Gamma, \pi)$.

Let $b_2 \in \text{Har}_{\mu_2}(H, \pi)$. Define $b : \Gamma \rightarrow \mathcal{H}$ by

$$b(g, \oplus_{x \in G} h_x) = \oplus_{x \in G} \pi(x^{-1})(b_2(h_{g^{-1}x})) \quad \text{for } (g, \oplus_{x \in G} h_x) \in \Gamma.$$

Then b extends b_2 and one checks that $b \in Z^1(\Gamma, \pi)$ and that $b \in \text{Har}_\mu(\Gamma, \pi)$.

Since π is trivial on H , the space $Z^1(H, \pi)$ coincides with $\text{Hom}(H, \mathcal{H})$, the set of homomorphisms $H \rightarrow \mathcal{H}$. Observe that every $b \in \text{Hom}(H, \mathcal{H})$ is automatically μ_2 -harmonic, since

$$\sum_{h \in H} b(h)\mu_2(h) = \sum_{h \in H} b(h^{-1})\mu_2(h) = - \sum_{h \in H} b(h)\mu_2(h).$$

Hence, $\text{Har}_{\mu_2}(H, \pi) = \text{Hom}(H, \mathcal{H})$ (alternatively, this follows from the fact that $B^1(H, \pi) = B^1(H, 1_H)$ is trivial); therefore, we have

$$\text{Har}_\mu(\Gamma, \pi) = \text{Har}_{\mu_1}(G, \pi) \oplus \text{Hom}(H, \mathcal{H}).$$

We specialize by taking $H = \mathbb{Z}$; then $\text{Hom}(H, \mathcal{H})$ can be identified with \mathcal{H} and we have

$$\text{Har}_\mu(\Gamma, \pi) = \text{Har}_{\mu_1}(G, \pi) \oplus \mathcal{H};$$

moreover, the action of the von Neumann algebra

$$\pi(\Gamma)' = \pi(G)'$$

on $\text{Har}_\mu(G, \mu)$ corresponds to the direct sum of the actions of $\pi(G)'$ on $\text{Har}_{\mu_1}(G, \mu_1)$ and on \mathcal{H} .

In particular, when the 1-cohomology $H^1(G, \pi)$ is trivial, we have

$$\text{Har}_\mu(\Gamma, \pi) = \mathcal{H},$$

so that Claim (i) is proved. Claim (ii) follows from Proposition 2. \square

4. An explicit formula for the projection on harmonic cocycles

We give an explicit formula for the orthogonal projection P_{Har} in terms of an averaging (or Markov) operator associated to μ , in the case where $B^1(G, \pi)$ is closed.

Consider the operator $\pi^0(\mu) \in \mathcal{B}(\mathcal{H}^0)$ defined by

$$\pi^0(\mu)v = \int_G \pi(x)v d\mu(x) \quad \text{for all } v \in \mathcal{H}^0.$$

The operator $\pi^0(\mu) - I : \mathcal{H}^0 \rightarrow \mathcal{H}^0$ is invertible if and only if π^0 does not weakly contain the trivial representation 1_G (see Proposition G.4.2 in [BHV]); in view of Proposition 1, this is the case if and only if $B^1(G, \pi)$ is closed.

Proposition 4. *Assume that $B^1(G, \pi)$ is closed. For $b \in Z^1(G, \pi)$, we have $P_{\text{Har}}b = b - \partial_v$, where*

$$v = (\pi^0(\mu) - I)^{-1}(M_\mu(b)).$$

Proof. Indeed, observe first that $M_\mu(b) \in \mathcal{H}^0$; indeed, for every $w \in \mathcal{H}^G$, we have

$$\begin{aligned} \langle M_\mu(b), w \rangle &= \int_G \langle b(x), w \rangle d\mu(x) = \int_G \langle b(x), \pi(x)w \rangle d\mu(x) \\ &= \int_G \langle \pi(x^{-1})b(x), w \rangle d\mu(x) = - \int_G \langle b(x^{-1}), w \rangle d\mu(x) \\ &= - \int_G \langle b(x), w \rangle d\mu(x) = -\langle M_\mu(b), w \rangle. \end{aligned}$$

Moreover, for $v = (\pi^0(\mu) - I)^{-1}(M_\mu(b))$, we have

$$M_\mu(\partial_v) = \int_G (\pi(x)v - v) d\mu(x) = (\pi^0(\mu) - I)v = M_\mu(b). \quad \square$$

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