

Constructing Poincaré series for number theoretic applications

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ABSTRACT. We give a general method for constructing Poincaré series on higher rank groups satisfying automorphic differential equations, by winding up solutions to differential equations of the form $(\Delta - \lambda)^\nu u = \theta$ on the underlying Riemannian symmetric space G/K , where Δ is the Laplacian, λ is a complex parameter, ν is an integral power, and θ a compactly supported distribution. To obtain formulas that are as explicit as possible we restrict ourselves to the case in which G is a *complex* semi-simple Lie group, and we consider two simple choices for θ , namely $\theta = \delta$, the Dirac delta distribution at the basepoint, and $\theta = S_b$, the distribution that integrates along a shell of radius b around the basepoint. We develop a global zonal spherical Sobolev theory, which enables us to use the harmonic analysis of spherical functions to obtain integral representations for the solutions. In the case $\theta = \delta$, we obtain an explicit expression for the solution, allowing relatively easy estimation of its behavior in the eigenvalue parameter λ , necessary for applications involving the associated Poincaré series. The behavior of the solution corresponding to $\theta = S_b$ is considerably subtler, even in the simplest possible higher rank cases; nevertheless, global automorphic Sobolev theory ensures the existence and uniqueness of an automorphic spectral expansion for the associated Poincaré series in a global automorphic Sobolev space, which is sufficient for the applications we have in mind.

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1. Introduction

1.1. Context and motivation: applications in number theory. The subconvexity results of Diaconu and Goldfeld [11, 12] and Diaconu and Garrett [8, 9] and the Diaconu–Garrett–Goldfeld prescription for spectral identities involving second moments of L-functions [10], rely critically on a Poincaré series, whose data, in contrast to classical Poincaré series, is neither smooth nor compactly supported. The data was chosen to imitate Good’s kernel in [22], but in hindsight, can be understood as the solution to a differential equation, $(\Delta - \lambda)u = \theta_H$, on the free space G/K , where λ is a complex parameter and θ_H the distribution that integrates a function along a subgroup H . The Poincaré series is then itself a solution to the corresponding automorphic differential equation and therefore has a heuristically immediate spectral expansion in terms of cusp forms, Eisenstein series, and residues of Eisenstein series. This provides motivation for constructing higher rank Poincaré series from solutions to differential equations of the form $(\Delta - \lambda)^\nu u = \theta$, where Δ is the Laplacian on a symmetric space, θ a distribution, λ a complex parameter, and ν a positive integral power.

A second motivation lies in constructing eigenfunctions for pseudo-Laplacians. Colin de Verdière’s proof of the meromorphic continuation of Eisenstein series used the fact that a function is an eigenfunction for the (self-adjoint) Friedrichs extension of a certain restriction Δ_a of the Laplacian on $SL_2(\mathbb{Z}) \backslash \mathfrak{H}$ if and only if it is a solution to the differential equation $(\Delta - \lambda)u = T_a$, where T_a is the distribution that evaluates the constant term at height a [5, 19]. While it would be desirable to construct a self-adjoint Friedrichs extension for a suitable restriction of the Laplacian such that eigenfunctions for this pseudo-Laplacian would be solutions to $(\Delta - \lambda)u = \delta$, where δ is Dirac delta at a base point, the details of the Friedrichs construction make this impossible, as can be shown with global automorphic Sobolev theory [20]. Replacing δ with S_b , the distribution that integrates along a shell of radius b , avoids this technicality.

Classical Poincaré series producing Kloosterman sums were generalized by Bump, Friedberg, and Goldfeld for $GL_n(\mathbb{R})$ and by Stevens for $GL_n(\mathbb{A})$ [3, 21, 29]. Other higher rank Poincaré series include those constructed by Miatello and Wallach, the singular Poincaré series constructed by Oda and Tsuzuki, and Thillainatesan’s Poincaré series producing multiple Dirichlet series of cusp forms on $GL_n(\mathbb{R})$ [26, 27, 28, 30].

1.2. Overview of main results. Motivated by the applications discussed above, we aim to obtain explicit formulas for solutions to differential equations of the form $(\Delta - \lambda)^\nu u = \theta$, where Δ is the Laplacian on a Riemannian symmetric space G/K , λ is a complex parameter, ν is an integral power, and

θ a compactly supported distribution, and to derive Poincaré series representations for solutions to corresponding automorphic differential equations by averaging over an arithmetic subgroup Γ .

In this paper, to obtain formulas that are as explicit as possible, we restrict ourselves to the case in which G is a *complex* semi-simple Lie group. We consider two simple choices for θ , namely $\theta = \delta$, the Dirac delta distribution at the basepoint, and $\theta = S_b$, the distribution that integrates along a shell of radius b around the basepoint. Global zonal spherical Sobolev theory ensures that the harmonic analysis of spherical functions produces solutions.

With $\theta = \delta$, the Dirac delta distribution at the basepoint in G/K , we obtain an explicit formula for the fundamental solution u_z for $(\Delta - \lambda_z)^\nu$, where $\lambda_z = z(z - 1), z \in \mathbb{C}$. For a derivation of the fundamental solution in the case $G = SL_2(\mathbb{C})$, *assuming* a suitable global zonal spherical Sobolev theory, see [16, 17]. Our results for the general case are sketched in [18]. The following theorem appears in Section 3 as Theorem 3.1; please see its context for the technical notation.

Theorem. *For an integer $\nu > \dim(G/K)/2 = n/2 + d$, where d is the number of positive roots, counted without multiplicities, and $n = \dim(\mathfrak{a})$ is the rank, u_z can be expressed in terms of a K -Bessel function:*

$$u_z(a) = \frac{2(-1)^\nu}{\pi^+(\rho)\Gamma(\nu)} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\log a)} \cdot \left(\frac{|\log a|}{2z}\right)^{\nu-d-\frac{n}{2}} \cdot K_{\nu-d-\frac{n}{2}}(z|\log a|).$$

In the odd rank case, with $\nu = m + d + (n + 1)/2$, where m is any nonnegative integer, $u_z(a)$ is given by

$$\frac{(-1)^{m+d+\frac{n+1}{2}} \pi^{\frac{n+1}{2}}}{(m+d+\frac{n-1}{2})! \pi^+(\rho)} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\alpha(\log a))} \cdot \frac{e^{-z|\log a|}}{z} \cdot P(|\log a|, z^{-1})$$

where P is a degree m polynomial in $|\log a|$ and a degree $2m$ polynomial in z^{-1} . In particular, choosing ν minimally, i.e., $\nu = d + \frac{n+1}{2}$,

$$u_z(a) = \frac{(-1)^{d+\frac{n+1}{2}} \pi^{(n+1)/2}}{\pi^+(\rho) \Gamma(d + \frac{n+1}{2})} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\alpha(\log a))} \cdot \frac{e^{-z|\log a|}}{z}.$$

When G is of even rank, and ν is minimal, i.e., $\nu = d + \frac{n}{2} + 1$,

$$u_z(a) = \frac{(-1)^{d+\frac{n}{2}+1} \pi^{\frac{n}{2}}}{\pi^+(\rho)\Gamma(d + \frac{n}{2} + 1)} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\alpha(\log a))} \cdot \frac{|\log a|}{z} \cdot K_1(z|\log a|).$$

Wallach derives a similar, though less explicit, formula in Section 4 of [32], and the formula can also be obtained by multiplying the Euclidean fundamental solution by $J^{-1/2} = \prod \frac{\alpha}{\sinh(\alpha)}$, using Hall and Mitchell’s “intertwining” formula relating $\Delta = \Delta_{G/K}$ and $\Delta_{\mathfrak{p}}$ [23].

Note that the formula is particularly simple when G is of odd rank and ν is the minimal power needed to ensure continuity. This allows relatively

easy estimation of its behavior in the eigenvalue parameter, proving L^2 -convergence and continuity of the associated Poincaré series $Pé_{u_z}$ and making it possible to determine the vertical growth of the Poincaré series in the eigenvalue parameter, as is needed for applications. The Poincaré series $Pé_{u_z}$ is used to obtain an explicit formula relating the number of lattice points in an expanding region in G/K to the automorphic spectrum [7]. Further, the two-variable Poincaré series $Pé_{u_z}(y^{-1}x)$ produces identities involving moments of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ Rankin–Selberg L-functions [6].

The second example, motivated by application to eigenvalues of pseudo-Laplacians, is the solution corresponding to $\theta = S_b$, the distribution that integrates along a shell of radius b around the basepoint in G/K . The explicit formula for this solution is given in Theorem 3.2, which we state here and prove in Section 3. Please refer to the context of the theorem in Section 3 for the notation.

Theorem. *For $\nu > (n + 2d + 1)/4$, the solution to $(\Delta - \lambda_z)^\nu v_z = S_b$ is*

$$v_z(a) = \frac{(-1)^\nu \pi^{\frac{n}{2}}}{2^{\nu - \frac{n}{2} - 1} \Gamma(\nu) \prod \sinh(\alpha(\log a))} \cdot \int_{|H|=b} \left(\frac{|\log a - H|}{z} \right)^{\nu - \frac{n}{2}} K_{\nu - \frac{n}{2}}(z|\log a - H|) \prod_{\alpha \in \Sigma^+} \sinh(\alpha(H)) dH.$$

In particular, when $n = \dim \mathfrak{a}^$ is odd,*

$$v_z(a) = \frac{(-1)^\nu \pi^{\frac{n+1}{2}} \Gamma(\nu - \frac{n-1}{2})}{\Gamma(\nu) \prod \sinh(\alpha(\log a))} \cdot \int_{|H|=b} \frac{P_{\nu - \frac{n+1}{2}}(z|\log a - H|) e^{-z|\log a - H|}}{z^{2\nu - n}} \prod_{\alpha \in \Sigma^+} \sinh(\alpha(H)) dH$$

where $P_\ell(x)$ is a degree ℓ polynomial with coefficients given by

$$a_k = \frac{(2\ell - k)!}{2^{2\ell - k} \ell! (\ell - k)! k!}.$$

The behavior of the free space solution v_z along the walls of the Weyl chambers is very subtle, even in the simplest possible higher rank cases, namely G complex of odd rank, making it difficult to verify the hypotheses ensuring that the associated Poincaré series converges. Nevertheless, a distributional Poincaré series may be constructed via an *averaging map*, and global automorphic Sobolev theory ensures the existence and uniqueness of an automorphic spectral expansion for the Poincaré series, in terms of cusp forms, Eisenstein series, and residues of Eisenstein series, in a global automorphic Sobolev space. Moreover, by construction, the automorphic spectral expansion of the associated Poincaré series is *immediate*, given the analytic framework of global automorphic Sobolev spaces. This appears

as Theorem 4.1, in Section 4; please see the context for explanation of the technical notation.

Theorem. *If the solution v_z is of sufficient rapid decay, the Poincaré series $\text{Pé}_{v_z}(g) = \sum_{\gamma \in \Gamma} v_z(\gamma \cdot g)$ converges absolutely and uniformly on compact sets, to a continuous function of moderate growth, square-integrable modulo Γ . Moreover, it has an automorphic spectral expansion, converging uniformly pointwise:*

$$\text{Pé}_z = \int_{\Xi}^{\oplus} \frac{\frac{\pi^+(\rho)}{\pi^+(-i\xi)} \left(\int_{|H|=b} e^{-i\langle \xi, H \rangle} \prod_{\alpha \in \Sigma^+} \sinh(\alpha H) dH \right) \overline{\Phi}_\xi(x_0) \cdot \Phi_\xi}{(-1)^\nu (|\xi|^2 + z^2)^\nu}$$

where $\{\Phi_\xi\}$ denotes a suitable spectral family of spherical automorphic forms (cusp forms, Eisenstein series, and residues of Eisenstein series) and

$$\lambda_\xi = -(|\xi|^2 + |\rho|^2)$$

is the Casimir eigenvalue of Φ_ξ .

If desired, uniform pointwise convergence (or any degree of C^k -convergence) of the spectral expansion can be obtained by choosing the parameter ν sufficiently large. However, for constructing eigenfunctions for pseudo-Laplacians, weaker than C^0 -convergence is desired, since eigenfunctions for the pseudo-Laplacian lie in a global automorphic Sobolev space potentially much larger than C^0 . In the case of $\theta = S_b$, the desired convergence is guaranteed for $\nu = 1$.

The difficulty, in all but the simplest possible higher rank case, namely G complex of odd rank and $\theta = \delta$, of ascertaining whether the free space solution to $(\Delta - \lambda)^\nu u = \theta$ is of sufficiently rapid decay along the walls of the Weyl chambers, where $\prod \sinh(\alpha(\log a))$ blows up, is reason to question whether an explicit “geometric” Poincaré series representation is actually needed in a given application or whether the automorphic spectral expansion suffices. Global automorphic Sobolev theory provides a robust framework for discussing the convergence of automorphic spectral expansions without reference to explicit geometric Poincaré series representations.

1.3. Outline of paper. In Section 2, we develop the necessary analytic framework for solving the free space differential equations using the harmonic analysis of spherical functions: global zonal spherical Sobolev theory. To our knowledge, this is the first construction of Sobolev spaces of bi- K -invariant distributions; an introduction to positively indexed Sobolev spaces of bi- K -invariant functions can be found in [2]. Our discussion closely parallels the global automorphic Sobolev theory developed in [7]; once a suitable foundation has been laid, many of the results follow readily using the same arguments, mutatis mutandis. In Section 3, we use the harmonic analysis of spherical functions to derive integral representations of the solutions to the differential equation $(\Delta - \lambda)^\nu u = \theta$, in the two cases $\theta = \delta$ and $\theta = S_b$, discussed above, and, in the $\theta = \delta$ case obtain an explicit formula for the

solution. In Section 4, we construct the associated Poincaré series and describe their automorphic spectral expansions. In Appendix A, we give a new, direct proof of the harmonicity of the π^+ function in the formulas for spherical functions on a complex semi-simple Lie group; this fact is needed for evaluating the integral representing the solution corresponding to the $\delta = \theta$ case. Finally, Appendix B carries out an explicit computation that is referenced in the derivations of the formulas for the free space solutions, evaluating an integral over \mathbb{R}^n in terms of K-Bessel functions.

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2. Spherical transforms, global zonal spherical Sobolev spaces, and differential equations on G/K

Global automorphic and global zonal spherical Sobolev spaces provide a robust framework for decisively treating many analytic issues that arise in constructing and manipulating the Poincaré series discussed in this paper. In this section, we discuss global zonal spherical Sobolev theory and its application to solving differential equations on G/K . Due to the many parallels with the theory of global automorphic Sobolev spaces, which is carefully discussed in Section 2 of [7], we abbreviate the discussion here and frequently refer the reader to proofs of corresponding results in that paper.

2.1. Spherical transform and inversion. Let G be a complex semi-simple Lie group with finite center and K a maximal compact subgroup. Let $G = NAK$, $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ be corresponding Iwasawa decompositions. Let Σ denote the set of roots of \mathfrak{g} with respect to \mathfrak{a} , let Σ^+ denote the subset of positive roots (for the ordering corresponding to \mathfrak{n}), and let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$, m_α denoting the multiplicity of α . Let $\mathfrak{a}_\mathbb{C}^*$ denote the set of complex-valued linear functions on \mathfrak{a} . Let $X = K \backslash G/K$ and $\Xi = \mathfrak{a}^*/W \approx \mathfrak{a}_+$. The spherical transform of Harish-Chandra and Berezin integrates a bi- K -invariant against a zonal spherical function:

$$\mathcal{F}f(\xi) = \int_G f(g) \overline{\varphi}_{\rho+i\xi}(g) dg.$$

Zonal spherical functions $\varphi_{\rho+i\xi}$ are eigenfunctions for Casimir (restricted to bi- K -invariant functions) with eigenvalue $\lambda_\xi = -(|\xi|^2 + |\rho|^2)$. The inverse transform is

$$\mathcal{F}^{-1}f = \int_\Xi f(\xi) \varphi_{\rho+i\xi} |\mathbf{c}(\xi)|^{-2} d\xi$$

where $\mathbf{c}(\xi)$ is the Harish-Chandra \mathbf{c} -function and $d\xi$ is the usual Lebesgue measure on $\mathfrak{a}^* \approx \mathbb{R}^n$. For brevity, denote $L^2(\Xi, |\mathbf{c}(\xi)|^{-2})$ by $L^2(\Xi)$. The Plancherel theorem asserts that the spectral transform and its inverse are isometries between $L^2(X)$ and $L^2(\Xi)$.

2.2. Characterizations of Sobolev spaces. We define positive index zonal spherical Sobolev spaces as left K -invariant subspaces of completions of $C_c^\infty(G/K)$ with respect to a topology induced by seminorms associated to derivatives from the universal enveloping algebra, as follows. Let $\mathcal{U}\mathfrak{g}^{\leq \ell}$ be the finite dimensional subspace of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ consisting of elements of degree less than or equal to ℓ . Each $\alpha \in \mathcal{U}\mathfrak{g}$ gives a seminorm $\nu_\alpha(f) = \|\alpha f\|_{L^2(G/K)}^2$ on $C_c^\infty(G/K)$.

Definition 2.1. Consider the space of smooth functions that are bounded with respect to these seminorms:

$$\{f \in C^\infty(G/K) : \nu_\alpha f < \infty \text{ for all } \alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}\}.$$

Let $H^\ell(G/K)$ be the completion of this space with respect to the topology induced by the family $\{\nu_\alpha : \alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}\}$. The *global zonal spherical Sobolev space* $H^\ell(X) = H^\ell(G/K)^K$ is the subspace of left- K -invariant functions in $H^\ell(G/K)$.

Proposition 2.1. *The space of test functions $C_c^\infty(X)$ is dense in $H^\ell(X)$.*

Proof. We approximate a smooth function $f \in H^\ell(X)$ by pointwise products with smooth cut-off functions, whose construction (given by [15], Lemma 6.1.7) is as follows. Let $\sigma(g)$ be the geodesic distance between the cosets $1 \cdot K$ and $g \cdot K$ in G/K . For $R > 0$, let B_R denote the ball

$$B_R = \{g \in G : \sigma(g) < R\}.$$

Let η be a nonnegative smooth bi- K -invariant function, supported in $B_{1/4}$, such that $\eta(g) = \eta(g^{-1})$, for all $g \in G$. Let $\text{char}_{R+1/2}$ denote the characteristic function of $B_{R+1/2}$, and let $\eta_R = \eta * \text{char}_{R+1/2} * \eta$. As shown in [15], η_R is smooth, bi- K -invariant, takes values between zero and one, is identically one on B_R and identically zero outside B_{R+1} , and, for any $\gamma \in \mathcal{U}\mathfrak{g}$, there is a constant C_γ such that

$$\sup_{g \in G} |(\gamma \eta_R)(g)| \leq C_\gamma.$$

We will show that the pointwise products $\eta_R \cdot f$ approach f in the ℓ^{th} Sobolev topology, i.e., for any $\gamma \in \mathcal{U}\mathfrak{g}^{\leq \ell}$, $\nu_\gamma(\eta_R \cdot f - f) \rightarrow 0$ as $R \rightarrow \infty$. By definition,

$$\nu_\gamma(\eta_R \cdot f - f) = \|\gamma(\eta_R \cdot f - f)\|_{L^2(G/K)}^2.$$

Leibnitz' rule implies that $\gamma(\eta_R \cdot f - f)$ is a finite linear combination of terms of the form $\alpha(\eta_R - 1) \cdot \beta f$ where $\alpha, \beta \in \mathcal{U}\mathfrak{g}^{\leq \ell}$. When $\text{deg}(\alpha) = 0$,

$$\|\alpha(\eta_R - 1) \cdot \beta f\|_{L^2(G/K)}^2 \ll \|(\eta_R - 1) \cdot \beta f\|_{L^2(G/K)}^2 \leq \int_{\sigma(g) \geq R} |(\beta f)(g)|^2 dg.$$

Otherwise, $\alpha(\eta_R - 1) = \alpha\eta_R$, and

$$\begin{aligned} \|\alpha(\eta_R - 1) \cdot \beta f\|_{L^2(G/K)}^2 &= \|\alpha\eta_R \cdot \beta f\|_{L^2(G/K)}^2 \\ &\ll \sup_{g \in G} |\alpha \eta_R(g)|^2 \cdot \int_{\sigma(g) \geq R} |(\beta f)(g)|^2 dg \\ &\ll \int_{\sigma(g) \geq R} |(\beta f)(g)|^2 dg. \end{aligned}$$

Let B be any bounded set containing all of the (finitely many) β that appear as a result of applying Leibniz' rule. Then

$$\nu_\gamma(\eta_R \cdot f - f) \ll \sup_{\beta \in B} \int_{\sigma(g) \geq R} |(\beta f)(g)|^2 dg.$$

Since B is bounded and $f \in H^\ell(X)$, the right hand side approaches zero as $R \rightarrow \infty$. □

Proposition 2.2. *Let Ω be the Casimir operator in the center of $\mathcal{U}\mathfrak{g}$. The norm $\|\cdot\|_{2\ell}$ on $C_c^\infty(G/K)^K$ given by*

$$\|f\|_{2\ell}^2 = \|f\|^2 + \|(1 - \Omega) f\|^2 + \|(1 - \Omega)^2 f\|^2 + \dots + \|(1 - \Omega)^\ell f\|^2$$

where $\|\cdot\|$ is the usual norm on $L^2(G/K)$, induces a topology on $C_c^\infty(G/K)^K$ that is equivalent to the topology induced by the family $\{\nu_\alpha : \alpha \in \mathcal{U}\mathfrak{g}^{\leq 2\ell}\}$ of seminorms and with respect to which $H^{2\ell}(X)$ is a Hilbert space.

Proof. Let $\{X_i\}$ be a basis for \mathfrak{g} subordinate to the Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$. Then $\Omega = \sum_i X_i X_i^*$, where $\{X_i^*\}$ denotes the dual basis, with respect to the Killing form. Let $\Omega_{\mathfrak{p}}$ and $\Omega_{\mathfrak{k}}$ denote the subsums corresponding to \mathfrak{p} and \mathfrak{k} respectively. Then $\Omega_{\mathfrak{p}}$ is a nonpositive operator, while $\Omega_{\mathfrak{k}}$ is nonnegative.

Lemma 2.1. *For any nonnegative integer r , let Σ_r denote the finite set of possible K -types of γf , for $\gamma \in \mathcal{U}\mathfrak{g}^{\leq r}$ and $f \in C_c^\infty(G/K)^K$, and let C_r be a constant greater than all of the finitely many eigenvalues λ_σ for $\Omega_{\mathfrak{k}}$ on the K -types $\sigma \in \Sigma_r$. For any $\varphi \in C_c^\infty(G/K)$ of K -type $\sigma \in \Sigma_m$ and $\beta = x_1 \dots x_n$ a monomial in $\mathcal{U}\mathfrak{g}$ with $x_i \in \mathfrak{p}$,*

$$\langle \beta \varphi, \beta \varphi \rangle \leq \langle (-\Omega + C_{m+n-1})^n \varphi, \varphi \rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $L^2(G/K)$.

Proof. We proceed by induction on $n = \deg \beta$. For $n = 1$, $\beta = x \in \mathfrak{p}$. Let $\{X_i\}$ be a self-dual basis for \mathfrak{p} such that $X_1 = x$. Then,

$$\begin{aligned} \langle x\varphi, x\varphi \rangle &\leq \sum_i \langle X_i \varphi, X_i \varphi \rangle = - \sum_i \langle X_i^2 \varphi, \varphi \rangle = \langle -\Omega_{\mathfrak{p}} \varphi, \varphi \rangle \\ &= \langle (-\Omega + \Omega_{\mathfrak{k}}) \varphi, \varphi \rangle \leq \langle (-\Omega + C_m) \varphi, \varphi \rangle = \langle (-\Omega + C_{m+n-1}) \varphi, \varphi \rangle. \end{aligned}$$

For $n > 1$, write $\beta = x\gamma$, where $x = x_1$ and $\gamma = x_2 \dots x_n$. Then the K -type of $\gamma\varphi$ lies in Σ_{m+n-1} , and by the above argument,

$$\langle x\gamma\varphi, x\gamma\varphi \rangle \leq \langle (-\Omega + C_{m+n-1})\gamma\varphi, \gamma\varphi \rangle.$$

Let $C_c^\infty(G/K)_{\Sigma_r}$ be the subspace of $C_c^\infty(G/K)$ consisting of functions of K -type in Σ_r and $L^2(G/K)_{\Sigma_r}$ be the corresponding subspace of $L^2(G/K)$. For the moment, let $\Sigma = \Sigma_{m+n-1}$ and $C = C_{m+n-1}$. Then, by construction, $-\Omega_{\mathfrak{k}} + C$ is positive on $C_c^\infty(G/K)_\Sigma$, and thus

$$-\Omega + C = -\Omega_{\mathfrak{p}} - \Omega_{\mathfrak{k}} + C$$

is a positive densely defined symmetric operator on $L^2(G/K)_\Sigma$. Thus, by Friedrichs [13, 14], there is an everywhere defined inverse R , which is a positive symmetric *bounded* operator on $L^2(G/K)_\Sigma$, and which, by the spectral theory for bounded symmetric operators, has a positive symmetric square root \sqrt{R} in the closure of the polynomial algebra $\mathbb{C}[R]$ in the Banach space of bounded operators on $L^2(G/K)_\Sigma$. Thus $-\Omega + C$ has a symmetric positive square root, namely $(\sqrt{R})^{-1}$, defined on $C_c^\infty(G/K)_\Sigma$, commuting with all elements of $\mathcal{U}\mathfrak{g}$, and

$$\langle (-\Omega + C)\gamma\varphi, \gamma\varphi \rangle = \langle \gamma\sqrt{-\Omega + C}\varphi, \gamma\sqrt{-\Omega + C}\varphi \rangle.$$

Now the K -type of $\sqrt{-\Omega + C}\varphi$, being the same as that of φ , lies in Σ_m , so by inductive hypothesis,

$$\begin{aligned} &\langle \gamma\sqrt{-\Omega + C}\varphi, \gamma\sqrt{-\Omega + C}\varphi \rangle \\ &\leq \langle (-\Omega + C_{m+n-2})^{n-1}\sqrt{-\Omega + C}\varphi, \sqrt{-\Omega + C}\varphi \rangle \\ &= \langle (-\Omega + C_{m+n-2})^{n-1}(-\Omega + C_{m+n-1})\varphi, \varphi \rangle \\ &\leq \langle (-\Omega + C_{m+n-1})^n\varphi, \varphi \rangle \end{aligned}$$

and this completes the proof of Lemma 2.1. □

Let $\alpha \in \mathcal{U}\mathfrak{g}^{\leq 2\ell}$. By the Poincaré–Birkhoff–Witt theorem we may assume α is a monomial of the form $\alpha = x_1 \dots x_n y_1 \dots y_m$ where $x_i \in \mathfrak{p}$ and $y_i \in \mathfrak{k}$. Then, for any $f \in C_c^\infty(G/K)^K$,

$$\nu_\alpha f = \langle \alpha f, \alpha f \rangle_{L^2(G/K)} = \langle x_1 \dots x_n f, x_1 \dots x_n f \rangle_{L^2(G/K)} (x_i \in \mathfrak{p}).$$

By the lemma, there is a constant C , depending on the degree of α , such that $\nu_\alpha(f) \ll \langle (-\Omega + C)^{\deg \alpha} f, f \rangle$ for all $f \in C_c^\infty(G/K)^K$. In fact, for bi- K -invariant functions, $(-\Omega + C)^{\deg \alpha} f = (-\Omega_{\mathfrak{p}} + C)^{\deg \alpha} f$. Since $\Omega_{\mathfrak{p}}$ is positive semi-definite, multiplying by a positive constant does not change the topology. Thus, we may take $C = 1$. That is, the subfamily $\{\nu_\alpha : \alpha = (1 - \Omega)^k, k \leq \ell\}$ of seminorms on $C_c^\infty(G/K)^K$ dominates the family $\{\nu_\alpha : \alpha \in \mathcal{U}\mathfrak{g}^{\leq 2\ell}\}$ and thus induces an equivalent topology. This completes the proof of Proposition 2.2. □

It will be necessary to have another description of Sobolev spaces. Let

$$W^{2,\ell}(G/K) = \{f \in L^2(G/K) : \alpha f \in L^2(G/K) \text{ for all } \alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}\}$$

where the action of $\mathcal{U}\mathfrak{g}$ on $L^2(G/K)$ is by distributional differentiation. Give $W^{2,\ell}(G/K)$ the topology induced by the seminorms $\nu_\alpha f = \|\alpha f\|_{L^2(G/K)}^2$, $\alpha \in \mathcal{U}\mathfrak{g}^{\leq \ell}$. Let $W^{2,\ell}(X)$ be the subspace of left K -invariants. These spaces are equal to the corresponding Sobolev spaces: $W^{2,\ell}(G/K) = H^\ell(G/K)$ and $W^{2,\ell}(X) = H^\ell(X)$. The proof of this is very similar to the proof of Proposition 2.3 in [7]. By Proposition 2.2, $H^{2\ell}(X) = W^{2,2\ell}(X)$ is a Hilbert space with norm

$$\|f\|_{2\ell}^2 = \|f\|^2 + \|(1 - \Omega) f\|^2 + \dots + \|(1 - \Omega)^\ell f\|^2$$

where $\|\cdot\|$ is the usual norm on $L^2(G/K)$, and $(1 - \Omega)^k f$ is a distributional derivative.

2.3. Spherical transforms and differentiation on Sobolev spaces.

Let $\ell \geq 0$. By an argument very similar to the proof of Proposition 2.4 in [7], the Laplacian extends to a continuous linear map $H^{2\ell+2}(X) \rightarrow H^{2\ell}(X)$; the spherical transform extends to a map on $H^{2\ell}(X)$; and

$$\mathcal{F}((1 - \Delta)f) = (1 - \lambda_\xi) \cdot \mathcal{F}f \quad \text{for all } f \in H^{2\ell+2}(X).$$

Let μ be the multiplication map $\mu(v)(\xi) = (1 - \lambda_\xi) \cdot v(\xi) = (1 + |\rho|^2 + |\xi|^2) \cdot v(\xi)$ where ρ is the half sum of positive roots. For $\ell \in \mathbb{Z}$, the weighted L^2 -spaces $V^{2\ell} = \{v \text{ measurable} : \mu^\ell(v) \in L^2(\Xi)\}$ with norms

$$\|v\|_{V^{2\ell}}^2 = \|\mu^\ell(v)\|_{L^2(\Xi)}^2 = \int_\Xi (1 + |\rho|^2 + |\xi|^2)^{2\ell} |v(\xi)|^2 |\mathbf{c}(\xi)|^{-2} d\xi$$

are Hilbert spaces with $V^{2\ell+2} \subset V^{2\ell}$ for all ℓ . In fact, these are dense inclusions, since truncations are dense in all $V^{2\ell}$ -spaces. The multiplication map μ is a Hilbert space isomorphism $\mu : V^{2\ell+2} \rightarrow V^{2\ell}$, since for $v \in V^{2\ell+2}$,

$$\|\mu(v)\|_{V^{2\ell}} = \|\mu^{\ell+1}(v)\|_{L^2(\Xi)} = \|v\|_{V^{2\ell+2}}.$$

The negatively indexed spaces are the Hilbert space duals of their positively indexed counterparts, by integration. The adjoints to inclusion maps are genuine inclusions, since $V^{2\ell+2} \hookrightarrow V^{2\ell}$ is dense for all $\ell \geq 0$, and, under the identification $(V^{2\ell})^* = V^{-2\ell}$ the adjoint map $\mu^* : (V^{2\ell})^* \rightarrow (V^{2\ell+2})^*$ is the multiplication map $\mu : V^{-2\ell} \rightarrow V^{-2\ell-2}$. For $\ell \geq 0$, the spherical transform is an *isometric isomorphism* $H^{2\ell}(X) \rightarrow V^{2\ell}$; see the proof of Proposition 2.5 in [7]. This Hilbert space isomorphism $\mathcal{F} : H^{2\ell} \rightarrow V^{2\ell}$ gives a *spectral* characterization of the $2\ell^{\text{th}}$ Sobolev space, namely the preimage of $V^{2\ell}$ under \mathcal{F} :

$$H^{2\ell}(X) = \{f \in L^2(X) : (1 - \lambda_\xi)^\ell \cdot \mathcal{F}f(\xi) \in L^2(\Xi)\}.$$

2.4. Negatively indexed Sobolev spaces and distributions. Negatively indexed Sobolev spaces allow the use of spectral theory for solving differential equations involving certain *distributions*.

Definition 2.2. For $\ell > 0$, the Sobolev space $H^{-\ell}(X)$ is the Hilbert space dual of $H^\ell(X)$.

Since the space of test functions is a dense subspace of $H^\ell(X)$ with $\ell > 0$, dualizing gives an inclusion of $H^{-\ell}(X)$ into the space of distributions. The adjoints of the dense inclusions $H^\ell \hookrightarrow H^{\ell-1}$ are inclusions $H^{-\ell+1}(X) \hookrightarrow H^{-\ell}(X)$, and the self-duality of $H^0(X) = L^2(X)$ implies that $H^\ell(X) \hookrightarrow H^{\ell-1}$ for all $\ell \in \mathbb{Z}$. The spectral transform extends to an isometric isomorphism on negatively indexed Sobolev spaces $\mathcal{F} : H^{-2\ell} \rightarrow V^{-2\ell}$, and for any $u \in H^{2\ell}(X)$, $\ell \in \mathbb{Z}$, $\mathcal{F}((1 - \Delta)u) = (1 - \lambda_\xi) \cdot \mathcal{F}u$. Please see the proof of Proposition 2.6 in [7].

Recall that, for a smooth manifold M , the positively indexed *local* Sobolev spaces $H_{\text{loc}}^\ell(M)$ consist of functions f on M such that for all points $x \in M$, all open neighborhoods U of x small enough that there is a diffeomorphism $\Phi : U \rightarrow \mathbb{R}^n$ with $\Omega = \Phi(U)$ having compact closure, and all test functions φ with support in U , the function $(f \cdot \varphi) \circ \Phi^{-1} : \Omega \rightarrow \mathbb{C}$ is in the Euclidean Sobolev space $H^\ell(\Omega)$. The Sobolev embedding theorem for local Sobolev spaces states that $H_{\text{loc}}^{\ell+k}(M) \subset C^k(M)$ for $\ell > \dim(M)/2$. A global version of Sobolev embedding also holds; since the proof is similar to that of Proposition 2.7 in [7], we state the theorem without proof here.

Proposition 2.3 (Global Sobolev embedding). *For $\ell > \dim(G/K)/2$,*

$$H^{\ell+k}(X) \subset H^{\ell+k}(G/K) \subset C^k(G/K).$$

This embedding of global Sobolev spaces into C^k -spaces is used to prove that the integral defining spectral inversion for test functions can be extended to sufficiently highly indexed Sobolev spaces, i.e., the abstract isometric isomorphism $\mathcal{F}^{-1} \circ \mathcal{F} : H^\ell(X) \rightarrow H^\ell(X)$ is given by an integral that is convergent uniformly pointwise, when $\ell > \dim(G/K)/2$. This result will be needed later, but its proof is parallel to the proof of Proposition 2.8 in [7], so we state the result here without proof.

Proposition 2.4. *For $f \in H^s(X)$, $s > k + \dim(G/K)/2$,*

$$f = \int_{\Xi} \mathcal{F}f(\xi) \varphi_{\rho+i\xi} |\mathbf{c}(\xi)|^{-2} d\xi \quad \text{in } H^s(X) \text{ and } C^k(X).$$

The embedding of global Sobolev spaces into C^k -spaces also implies that *compactly supported* distributions lie in global Sobolev spaces. Specifically, a compactly supported distribution of order k lies in $H^{-s}(X)$ for all $s > k + \dim(G/K)/2$. The proof of this is similar to that of Proposition 2.9 in [7]. Thus the spectral transform of a compactly supported distribution is defined (by isometric isomorphism, as discussed above) and, in particular, is obtained by evaluating the distribution at the elementary spherical function, as stated in the following proposition, whose proof is similar to the proof of Proposition 2.10 in [7].

Proposition 2.5. *For a compactly supported distribution u of order k , $\mathcal{F}u = u(\overline{\varphi}_{\rho+i\xi})$ in V^{-s} where $s > k + \dim(G/K)/2$.*

Remark 2.1. In particular, since the Dirac delta distribution at the base point $x_o = 1 \cdot K$ in G/K is a compactly supported distribution of order zero, it lies in $H^{-\ell}(X)$ for all $\ell > \dim(G/K)/2$, and its spherical transform is $\mathcal{F}\delta = \varphi_{\rho+i\xi}(1) = 1$.

3. Free space solutions

3.1. Fundamental solutions. Let G be a complex semi-simple Lie group with finite center and K a maximal compact subgroup. Let $G = NAK$, $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ be corresponding Iwasawa decompositions. Let Σ denote the set of roots of \mathfrak{g} with respect to \mathfrak{a} , let Σ^+ denote the subset of positive roots (for the ordering corresponding to \mathfrak{n}), and let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$, m_α denoting the multiplicity of α . Since G is complex, $m_\alpha = 2$, for all $\alpha \in \Sigma^+$, so $\rho = \sum_{\alpha \in \Sigma^+} \alpha$. Let $\mathfrak{a}_\mathbb{C}^*$ denote the set of complex-valued linear functions on \mathfrak{a} . Consider the differential equation on the symmetric space $X = G/K$:

$$(\Delta - \lambda_z)^\nu u_z = \delta_{1 \cdot K}$$

where the Laplacian Δ is the image of the Casimir operator for \mathfrak{g} , λ_z is $z^2 - |\rho|^2$ for a complex parameter z , ν is an integer, and $\delta_{1 \cdot K}$ is Dirac delta at the basepoint $x_o = 1 \cdot K \in G/K$. Since $\delta_{1 \cdot K}$ is also *left-K*-invariant, we construct a left- K -invariant solution on G/K using the harmonic analysis of spherical functions.

Proposition 3.1. *For integral $\nu > \dim(G/K)/2$, u_z is a continuous left- K -invariant function on G/K with the following spectral expansion:*

$$u_z(g) = \int_{\Xi} \frac{(-1)^\nu}{(|\xi|^2 + z^2)^\nu} \varphi_{\rho+i\xi}(g) |\mathbf{c}(\xi)|^{-2} d\xi.$$

Proof. As mentioned above in Remark 2.1, $\delta_{1 \cdot K}$ lies in the global zonal spherical Sobolev spaces $H^{-\ell}(X)$ for all $\ell > \dim(G/K)/2$. Thus there is a solution $u_z \in H^{-\ell+2\nu}(X)$. The solution u_z is unique in Sobolev spaces, since any u'_z satisfying the differential equation must necessarily satisfy $\mathcal{F}(u'_z) = \mathcal{F}(\delta_{1 \cdot K})/(\lambda_\xi - \lambda_z)^\nu = (-1)^\nu/(|\xi|^2 + z^2)^\nu$. For $\nu > \dim(G/K)/2$, the solution is continuous by Proposition 2.3, and by Propositions 2.4 and 2.5, has the stated spectral expansion. □

Remark 3.1. As the proof shows, the condition on ν is necessary only if uniform pointwise convergence of the spectral expansion is desired. In general, there is a solution, unique in global zonal spherical Sobolev spaces, whose spectral expansion, given above, converges in the corresponding Sobolev topologies.

For a *complex* semi-simple Lie group, the zonal spherical functions are *elementary*. The spherical function associated with the principal series I_χ with $\chi = e^{\rho+i\lambda}$, $\lambda \in \mathfrak{a}_\mathbb{C}^*$ is

$$\varphi_{\rho+i\lambda} = \frac{\pi^+(\rho)}{\pi^+(i\lambda)} \frac{\sum \operatorname{sgn}(w) e^{i w \lambda}}{\sum \operatorname{sgn}(w) e^{w \rho}}$$

where the sums are taken over the elements w of the Weyl group, and the function π^+ is the product $\pi^+(\mu) = \prod_{\alpha>0} \langle \alpha, \mu \rangle$ over positive roots, *without* multiplicities. The ratio of $\pi^+(\rho)$ to $\pi^+(i\lambda)$ is the \mathbf{c} -function, $\mathbf{c}(\lambda)$. The denominator can be rewritten

$$\sum_{w \in W} \text{sgn}(w) e^{w\rho} = \prod_{\alpha \in \Sigma^+} 2 \sinh(\alpha).$$

Proposition 3.2. *The fundamental solution u_z has the following integral representation:*

$$u_z = \frac{(-1)^\nu (-i)^d}{\pi^+(\rho) \prod 2 \sinh \alpha} \cdot \int_{\mathfrak{a}^*} \frac{\pi^+(\lambda) e^{i\lambda}}{(|\lambda|^2 + z^2)^\nu} d\lambda.$$

Proof. In the case of complex semi-simple Lie groups, the inverse spherical transform has an elementary form. Since the function π^+ is a homogeneous polynomial of degree d , equal to the number of positive roots, counted without multiplicity, and is W -equivariant by the sign character, $\mathcal{F}^{-1}f$ is

$$\int_{\mathfrak{a}^*/W} f(\lambda) \varphi_{\rho+i\lambda} |\mathbf{c}(\lambda)|^{-2} d\lambda = \frac{(-i)^d}{\pi^+(\rho) \prod 2 \sinh \alpha} \cdot \int_{\mathfrak{a}^*} f(\lambda) \pi^+(\lambda) e^{i\lambda} d\lambda.$$

By Proposition 3.1, u_z has the stated integral representation. □

Proposition 3.3. *The integral in Proposition 3.2 can be expressed in terms of a K -Bessel function:*

$$\begin{aligned} & \int_{\mathfrak{a}^*} \frac{\pi^+(\lambda) e^{i\langle \lambda, \log a \rangle}}{(|\lambda|^2 + z^2)^\nu} d\lambda \\ &= \frac{\pi^{n/2} i^d \pi^+(\log a)}{2^{\nu-(1+d+n/2)} \Gamma(\nu)} \cdot \left(\frac{|\log a|}{z} \right)^{\nu-d-n/2} K_{\nu-d-n/2}(|\log a|z) \end{aligned}$$

where $n = \dim \mathfrak{a}$, d is the number of positive roots, counted without multiplicity, and $\nu > n/2 + d$.

Proof. Let $I(\log a)$ denote the integral to be evaluated. Using the Γ -function and changing variables $\lambda \rightarrow \frac{\lambda}{\sqrt{t}}$,

$$\begin{aligned} I(\log a) &= \frac{1}{\Gamma(\nu)} \cdot \int_0^\infty \int_{\mathfrak{a}^*} t^\nu e^{-t(|\lambda|^2+z^2)} \pi^+(\lambda) e^{i\lambda} d\lambda \frac{dt}{t} \\ &= \frac{1}{\Gamma(\nu)} \cdot \int_0^\infty t^{\nu-(d+n)/2} e^{-tz^2} \int_{\mathfrak{a}^*} e^{-|\lambda|^2} \pi^+(\lambda) e^{-i\langle \lambda, -\log a/\sqrt{t} \rangle} d\lambda \frac{dt}{t}. \end{aligned}$$

The polynomial π^+ is in fact *harmonic*. See, for example, Lemma 2 in [31] or, for a more direct proof, Theorem A.1, below. Thus the integral over \mathfrak{a}^* is the Fourier transform of the product of a Gaussian and a harmonic polynomial, and by Hecke’s identity,

$$\int_{\mathfrak{a}^*} e^{-|\lambda|^2} \pi^+(\lambda) e^{-i\langle \lambda, -\log a/\sqrt{t} \rangle} d\lambda = i^d t^{-d/2} \pi^+(\log a) e^{-|\log a|^2/t}.$$

Returning to the main integral,

$$I(\log a) = \frac{i^d \pi^+(\log a)}{\Gamma(\nu)} \cdot \int_0^\infty t^{\nu-d} e^{-tz^2} (t^{-n/2} e^{-|\log a|^2/t}) \frac{dt}{t}.$$

Replacing the Gaussian by its Fourier transform and using the Γ -function identity again,

$$I(\log a) = i^d \pi^+(\log a) \cdot \frac{\Gamma(\nu - d)}{\Gamma(\nu)} \cdot \int_{\mathfrak{a}^*} \frac{e^{i\langle \lambda, \log a \rangle}}{(|\lambda|^2 + z^2)^{\nu-d}} d\lambda.$$

This integral can be written as a K-Bessel function (see Section B) yielding the desired conclusion. \square

The explicit formula for u_z follows immediately. Choosing ν to be the minimal integer required for C^0 -convergence yields a particularly simple expression, as described in the following theorem.

Theorem 3.1. *For an integer $\nu > \dim(G/K)/2 = n/2 + d$, where d is the number of positive roots, counted without multiplicities, and $n = \dim(\mathfrak{a})$ is the rank, u_z can be expressed in terms of a K-Bessel function:*

$$u_z(a) = \frac{2(-1)^\nu}{\pi^+(\rho)\Gamma(\nu)} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\log a)} \cdot \left(\frac{|\log a|}{2z} \right)^{\nu-d-\frac{n}{2}} \cdot K_{\nu-d-\frac{n}{2}}(z |\log a|).$$

In the odd rank case, with $\nu = m + d + (n + 1)/2$, where m is any nonnegative integer, $u_z(a)$ is given by

$$\frac{(-1)^{m+d+\frac{n+1}{2}} \pi^{\frac{n+1}{2}}}{(m + d + \frac{n-1}{2})! \pi^+(\rho)} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\alpha(\log a))} \cdot \frac{e^{-z|\log a|}}{z} \cdot P(|\log a|, z^{-1})$$

where P is a degree m polynomial in $|\log a|$ and a degree $2m$ polynomial in z^{-1} . In particular, choosing ν minimally, i.e., $\nu = d + \frac{n+1}{2}$,

$$u_z(a) = \frac{(-1)^{d+\frac{n+1}{2}} \pi^{(n+1)/2}}{\pi^+(\rho) \Gamma(d + \frac{n+1}{2})} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\alpha(\log a))} \cdot \frac{e^{-z|\log a|}}{z}.$$

When G is of even rank, and ν is minimal, i.e., $\nu = d + \frac{n}{2} + 1$,

$$u_z(a) = \frac{(-1)^{d+\frac{n}{2}+1} \pi^{\frac{n}{2}}}{\pi^+(\rho)\Gamma(d + \frac{n}{2} + 1)} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\alpha(\log a))} \cdot \frac{|\log a|}{z} \cdot K_1(z |\log a|).$$

Remark 3.2. For fixed α , large $|z|$, and $\mu = 4\alpha^2$ (see [1], 9.7.2),

$$K_\alpha(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{\mu - 1}{8z} + \frac{(\mu - 1)(\mu - 9)}{2!(8z)^2} + \frac{(\mu - 1)(\mu - 9)(\mu - 25)}{3!(8z)^3} + \dots \right)$$

for $|\arg z| < \frac{3\pi}{2}$. Thus, for ν minimal, in the even rank case the fundamental solution has the following asymptotic:

$$u_z(a) \approx \frac{(-1)^{d+(n/2)+1}\pi^{(n+1)/2}}{\sqrt{2}\pi^+(\rho)\Gamma(d+(n/2)+1)} \cdot \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\alpha(\log a))} \cdot \sqrt{\frac{|\log a|}{z}} \cdot \frac{e^{-z|\log a|}}{z}.$$

Remark 3.3. Recall from Proposition 3.1 that zonal spherical Sobolev theory ensures the continuity of u_z for ν chosen as in the theorem. For $G = SL_2(\mathbb{C})$, the continuity is visible, since fundamental solution is, up to a constant,

$$u_z(a_r) = \frac{r e^{-(2z-1)r}}{(2z-1) \sinh r} \quad \text{where} \quad a_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}.$$

3.1.1. Using Hall and Mitchell’s Intertwining Formula. The symmetric space fundamental solution can also be obtained by multiplying the Euclidean fundamental solution by $\prod \frac{\alpha}{\sinh(\alpha)}$, using Hall and Mitchell’s “intertwining” formula relating $\Delta = \Delta_{G/K}$ and $\Delta_{\mathfrak{p}}$ [23] as follows. (See also Helgason’s discussion of the wave equation on G/K in [25].)

Again, G is a complex semi-simple Lie group with maximal compact K . We identify G/K with \mathfrak{p} via the exponential mapping. Then

$$\Delta f = J^{-1/2}(\Delta_{\mathfrak{p}} - \|\rho\|^2)(J^{1/2} f)$$

where $\Delta = \Delta_{G/K}$ is the (non-Euclidean) Laplacian on G/K ,

$$J^{-1/2} = \prod \frac{\alpha}{\sinh \alpha},$$

where the product ranges over positive roots, f is a bi- K -invariant function on G , $\Delta_{\mathfrak{p}}$ is the (Euclidean) Laplacian on \mathfrak{p} . Thus,

$$(\Delta - \lambda_z)^\nu f = J^{-1/2}(\Delta_{\mathfrak{p}} - z^2)^\nu J^{1/2} f.$$

Let w_z be a solution of the Euclidean differential equation

$$(\Delta_{\mathfrak{p}} - z^2)^\nu w_z = \varphi.$$

Then the function $u_z = J^{-1/2}w_z$ is a solution to the corresponding differential equation on G/K : $(\Delta - \lambda_z)^\nu u_z = J^{-1/2}\varphi$, since

$$\begin{aligned} (\Delta - \lambda_z)^\nu (J^{-1/2}w_z) &= J^{-1/2}(\Delta_{\mathfrak{p}} - z^2)^\nu J^{1/2}(J^{-1/2}w_z) \\ &= J^{-1/2}(\Delta_{\mathfrak{p}} - z^2)^\nu w_z = J^{-1/2}\varphi. \end{aligned}$$

If $J^{-1/2} \equiv 1$ on the support of φ , as in the case at hand, $\varphi = \delta$, the function $u_z = J^{-1/2}w_z$ is the solution of $(\Delta - \lambda_z)^\nu u_z = \delta$. Thus, to obtain a formula for the fundamental solution for $(\Delta - \lambda_z)^\nu$ on G/K , one can simply multiply the Euclidean fundamental solution for $(\Delta_{\mathfrak{p}} - z^2)^\nu$ by $J^{-1/2}$. This does in fact yield the formula given in Theorem 3.1.

3.2. Integrating along shells. Now we replace the Dirac delta distribution with S_b , the distribution that integrates a function along a shell of radius b around the basepoint, by which we mean

$$K \cdot \{a = \exp(H) : H \in \mathfrak{a}_+ \text{ with } |H| = b\} \cdot K/K.$$

Note that, for $SL_2(\mathbb{C})$, this is a spherical shell of radius b , centered at the basepoint $1 \cdot K$, in hyperbolic 3-space. Arguing as in the previous case (see the proof of Proposition 3.1), since S_b is a compactly supported distribution, the differential equation $(\Delta - \lambda_z)^\nu v_z = S_b$ has a unique solution in global zonal spherical Sobolev spaces. The spherical inversion formula of Harish-Chandra and Berezin gives an integral representation for v_z , in terms of the spherical transform of S_b . The integral representation is convergent (uniformly pointwise) for ν sufficiently large, by the global Sobolev embedding theorem. Since the distribution S_b lies in $H^s(X)$ for all $s < -1/2$, choosing $\nu > (\dim(G/K) + 1)/4$ suffices to ensure uniform pointwise convergence. If desired, convergence in the C^k -topology can be obtained by choosing $\nu > (\dim(G/K) + 1)/4 + k/2$.

On the other hand, for some applications, a *weaker* convergence is desired: e.g., for applications involving pseudo-Laplacians, what is needed H^1 -convergence, since eigenfunctions for the Friedrichs extension of (a restriction of) the Laplacian must lie in the domain of the Friedrichs extension, which, by construction, lies in $H^1(X)$. In this case H^1 -convergence is guaranteed for $\nu = 1$, regardless of the dimension of G/K .

Remark 3.4. We might hope to obtain an explicit formula for the solution by simply multiplying the corresponding Euclidean solution by $J^{-1/2}$, as in the case of the fundamental solution. However, since $J^{-1/2}$ is *not* identically one on the shell of radius b , this does not succeed. (See Section 3.1.1.)

Theorem 3.2. *For $\nu > (n + 2d + 1)/4$, the solution to $(\Delta - \lambda_z)^\nu v_z = S_b$ is*

$$v_z(a) = \frac{(-1)^\nu \pi^{\frac{n}{2}}}{2^{\nu - \frac{n}{2} - 1} \Gamma(\nu) \prod \sinh(\alpha(\log a))} \cdot \int_{|H|=b} \left(\frac{|\log a - H|}{z} \right)^{\nu - \frac{n}{2}} K_{\nu - \frac{n}{2}}(z|\log a - H|) \prod_{\alpha \in \Sigma^+} \sinh(\alpha(H)) dH.$$

In particular, when $n = \dim \mathfrak{a}^$ is odd,*

$$v_z(a) = \frac{(-1)^\nu \pi^{\frac{n+1}{2}} \Gamma(\nu - \frac{n-1}{2})}{\Gamma(\nu) \prod \sinh(\alpha(\log a))} \cdot \int_{|H|=b} \frac{P_{\nu - \frac{n+1}{2}}(z|\log a - H|) e^{-z|\log a - H|}}{z^{2\nu - n}} \prod_{\alpha \in \Sigma^+} \sinh(\alpha(H)) dH$$

where $P_\ell(x)$ is a degree ℓ polynomial with coefficients given by

$$a_k = \frac{(2\ell - k)!}{2^{2\ell - k} \ell! (\ell - k)! k!}.$$

Proof. By Proposition 2.4, the solution v_z has the following integral representation,

$$v_z(a) = \int_{\mathfrak{a}^*/W} \frac{(-1)^\nu \mathcal{F}(S_b)(\xi)}{(|\xi|^2 + z^2)^\nu} \cdot \varphi_{\rho+i\xi}(a) |\mathbf{c}(\xi)|^{-2} d\xi.$$

As in the derivation of the fundamental solution, we use the W -equivariance of π^+ by the sign character and the degree d homogeneity of π^+ to rewrite this as

$$v_z(a) = \frac{(-1)^\nu (-i)^d}{\pi^+(\rho) \prod 2 \sinh(\alpha(\log a))} \int_{\mathfrak{a}^*} \frac{\mathcal{F}S_b(\xi)}{(|\xi|^2 + z^2)^\nu} e^{i\langle \xi, \log a \rangle} \pi^+(\xi) d\xi.$$

The spherical transform is

$$\mathcal{F}(S_b)(\xi) = S_b(\bar{\varphi}_{\rho+i\xi}) = \int_{b\text{-shell}} \bar{\varphi}_{\xi+i\rho}(g) dg.$$

Writing $g \in G$ as $g = k a k' = k \exp(H) k'$, we reduce to an integral over a Euclidean sphere in \mathfrak{a} ,

$$\mathcal{F}(S_b)(\xi) = \int_{|H|=b} \frac{\pi^+(\rho)}{\pi^+(-i\xi)} \frac{\sum \operatorname{sgn} w e^{-iw\xi(H)}}{\sum \operatorname{sgn} w e^{w\rho(H)}} \prod_{\alpha \in \Sigma^+} \sinh^2(\alpha(H)) dH.$$

Using the fact that $\sum_{w \in W} \operatorname{sgn} w e^{w\rho(H)} = \prod_{\alpha \in \Sigma^+} 2 \sinh(\alpha(H))$ and Weyl group invariance,

$$\mathcal{F}(S_b)(\xi) = i^d \frac{\pi^+(\rho)}{\pi^+(\xi)} \int_{|H|=b} e^{-i\langle \xi, H \rangle} \prod_{\alpha \in \Sigma^+} 2 \sinh(\alpha(H)) dH.$$

Thus

$$\begin{aligned} v_z(a) &= \frac{(-1)^\nu}{\prod 2 \sinh(\alpha(\log a))} \int_{\mathfrak{a}^*} \int_{|H|=b} \frac{e^{i\langle \xi, \log a - H \rangle}}{(|\xi|^2 + z^2)^\nu} \prod_{\alpha \in \Sigma^+} 2 \sinh(\alpha(H)) dH d\xi \\ &= \frac{(-1)^\nu}{\prod \sinh(\alpha(\log a))} \int_{|H|=b} \left(\int_{\mathfrak{a}^*} \frac{e^{i\langle \xi, \log a - H \rangle}}{(|\xi|^2 + z^2)^\nu} d\xi \right) \prod_{\alpha \in \Sigma^+} \sinh(\alpha(H)) dH. \end{aligned}$$

The inner integral can be interpreted as an integral over \mathbb{R}^n , where $n = \dim \mathfrak{a}^*$, and can be expressed as a K-Bessel function to obtain the desired results. (See Section B.) \square

Remark 3.5. For $G = SL_2(\mathbb{C})$, with $\nu = 1$, ensuring H^1 -convergence, the solution is

$$v_z(a_r) = \frac{-\sinh(b)}{z \sinh(r)} \cdot \begin{cases} e^{-2bz} \sinh(2rz) & \text{if } r < b \\ \sinh(2bz) e^{-2rz} & \text{if } r > b, \end{cases} \quad \text{where } a_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}$$

and, with $\nu = 2$, ensuring uniform pointwise convergence, the solution is

$$v_z(a_r) = \frac{2 \sinh(b)}{z^3 \sinh(r)} \cdot \begin{cases} e^{-2bz} ((1 + 2bz) \cosh(2rz) - 2rz \sinh(2rz)) & \text{if } r < b \\ ((1 + 2rz) \cosh(2bz) - 2bz \sinh(2bz)) e^{-2rz} & \text{if } r > b. \end{cases}$$

Remark 3.6. In principle, one can also obtain a solution by convolution with the fundamental solution, u_z , discussed above. For $x = k'_x \cdot a_{r_x} \cdot k_x$ in G and $g = k'_g \cdot a_b \cdot k_g$ on the b -shell in G/K ,

$$u_z(g \cdot x^{-1}) = u_z(k'_g a_b k_g k_x^{-1} a_{r_x} (k'_x)^{-1}) = u_z(a_b k_g k_x^{-1} a_{r_x})$$

and thus

$$v_z(x) = (x \cdot S_b)(u_z) = \int_{b\text{-shell}} u_z(g \cdot x^{-1}) dg = \int_K u_z(a_b k k_x^{-1} a_{r_x}) dk$$

where dk is dg , restricted to K .

4. Poincaré series and automorphic spectral expansions

Let Γ be a discrete subgroup of G . The averaging map

$$\alpha = \alpha_\Gamma : C_c^0(G/K) \longrightarrow C_c^0(\Gamma \backslash G)^K \text{ given by } f \mapsto \sum_{\gamma \in \Gamma} \gamma \cdot f$$

is a continuous surjection, as is its extension $\alpha : \mathcal{E}'(G/K) \rightarrow \mathcal{E}'(\Gamma \backslash G)^K$, to the space of compactly supported distributions on G/K . We call $\text{Pé}_f = \alpha(f)$ the Poincaré series associated to f .

Though the automorphic spectrum consists of disparate pieces (cusp forms, Eisenstein series, residues of Eisenstein series) it will be useful to have a uniform notation. We posit a parameter space Ξ with spectral (Plancherel) measure $d\xi$ and let $\{\Phi_\xi\}_{\xi \in \Xi}$ denote the elements of the spectrum.

The Poincaré series Pé_{u_z} associated to the fundamental solution u_z discussed above is used to obtain an explicit formula relating the number of lattice points in an expanding region in G/K to the automorphic spectrum [7]. Further, the two-variable Poincaré series $\text{Pé}_{u_z}(y^{-1}x)$ produces identities involving moments of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ Rankin–Selberg L-functions [6]. The arguments given in [7] generalize as follows.

For a given compactly supported distribution θ on G/K , let $\theta^{\text{afc}} = \alpha(\theta)$, and consider the automorphic differential equation $(\Delta - \lambda)^\nu u^{\text{afc}} = \theta^{\text{afc}}$. Since θ^{afc} is compactly supported modulo Γ , it lies in a global automorphic Sobolev space. Thus there is a solution u^{afc} , unique in global automorphic Sobolev spaces, with an automorphic spectral expansion whose coefficients are obtained by $\langle \theta, \Phi_\xi \rangle$, $\xi \in \Xi$. The spectral expansion is convergent (uniformly pointwise) for sufficiently large ν . If the corresponding free-space solution u is of sufficiently rapid decay, then, by arguments involving gauges on groups, the Poincaré series Pé_u converges to a continuous function that is square integral modulo Γ . Thus it lies in a global automorphic Sobolev space, and by uniqueness, it must be pointwise equal to u^{afc} .

We now consider the Poincaré series associated to the solution to

$$(\Delta - \lambda_z)^\nu v_z = S_b, \quad \nu > (\dim(G/K) + 1)/4.$$

Theorem 4.1. *If the solution v_z is of sufficient rapid decay, the Poincaré series $\text{Pé}_{v_z}(g) = \sum_{\gamma \in \Gamma} v_z(\gamma \cdot g)$ converges absolutely and uniformly on compact sets, to a continuous function of moderate growth, square-integrable modulo Γ . Moreover, it has an automorphic spectral expansion, converging uniformly pointwise:*

$$\text{Pé}_z = \int_{\Xi}^{\oplus} \frac{\pi^+(\rho)}{\pi^+(-i\xi)} \left(\int_{|H|=b} e^{-i\langle \xi, H \rangle} \prod_{\alpha \in \Sigma^+} \sinh(\alpha H) dH \right) \frac{\overline{\Phi}_{\xi}(x_0) \cdot \Phi_{\xi}}{(-1)^{\nu} (|\xi|^2 + z^2)^{\nu}}$$

where $\{\Phi_{\xi}\}$ denotes a suitable spectral family of spherical automorphic forms (cusp forms, Eisenstein series, and residues of Eisenstein series) and

$$\lambda_{\xi} = -(|\xi|^2 + |\rho|^2)$$

is the Casimir eigenvalue of Φ_{ξ} .

Proof. Since v_z is of sufficient rapid decay the Poincaré series converges absolutely and uniformly on compact sets to a function that is of moderate growth and square integrable modulo Γ , by Proposition 3.1.1 in [7].

The automorphic spectral expansion of Pé_z can be written as a Hilbert direct integral $\int_{\Xi}^{\oplus} \frac{S_b^{\text{afc}} \overline{\Phi}_{\xi} \cdot \Phi_{\xi}}{(\lambda_{\xi} - \lambda_z)^{\nu}}$. To determine the coefficients $S_b^{\text{afc}} \overline{\Phi}_{\xi}$, we consider the effect of S_b^{afc} on an automorphic spherical eigenfunction f for Casimir. The averaging map α_K given by $\alpha_K(f) = \int_K f(kg) dk$ maps f to a constant multiple of the zonal spherical function φ_f° with the same eigenvalue as f . Since $\alpha_K(f)(1) = f(x_0)$ and φ_f° is normalized so that $\varphi_f^{\circ}(1) = 1$, $\alpha_K(f) = f(x_0) \cdot \varphi_f^{\circ}$.

$$\begin{aligned} S_b^{\text{afc}} f &= \int_{|H|=b} \int_K f(k \exp H) dk \prod_{\alpha \in \Sigma^+} 4 \sinh^2(\alpha H) dH \\ &= (x_0) \cdot \int_{|H|=b} \varphi_f^{\circ}(\exp(H)) \prod_{\alpha \in \Sigma^+} 4 \sinh^2(\alpha H) dH. \end{aligned}$$

Thus $S_b^{\text{afc}} \overline{\Phi}_{\xi}$ is

$$\begin{aligned} S_b^{\text{afc}} \overline{\Phi}_{\xi} &= \overline{\Phi}_{\xi}(x_0) \cdot \int_{|H|=b} \frac{\pi^+(\rho)}{\pi^+(-i\xi)} \frac{\sum \text{sgn} w e^{-iw\xi(H)}}{\sum \text{sgn} w e^{w\rho(H)}} \prod_{\alpha \in \Sigma^+} 4 \sinh^2(\alpha(H)) dH \\ &= \overline{\Phi}_{\xi}(x_0) \cdot \frac{\pi^+(\rho)}{\pi^+(-i\xi)} \cdot \int_{|H|=b} e^{-i\langle \xi, H \rangle} \prod_{\alpha \in \Sigma^+} 2 \sinh(\alpha(H)) dH \end{aligned}$$

and the spectral expansion of Pé_z is as stated. Global automorphic Sobolev theory ensures convergence. □

Remark 4.1. In the case $G = SL_2(\mathbb{C})$, $\Gamma = SL_2(\mathbb{Z}[i])$, with $\nu = 2$, it is clear from Remark 3.5 that v_z is of sufficient rapid decay for $\text{Re}(z) \gg 1$.

Thus its Poincaré series $\text{Pé}_z(g)$ is:

$$\frac{2 \sinh(b)}{z^3} \left(\sum_{\sigma(\gamma g) < b} \frac{((1 + 2bz) \cosh(2\sigma(\gamma g)z) - 2\sigma(\gamma g)z \sinh(2rz)) e^{-2bz}}{\sinh(\sigma(\gamma g))} + \sum_{\sigma(\gamma g) > b} \frac{((1 + 2\sigma(\gamma g)z) \cosh(2bz) - 2bz \sinh(2bz)) e^{-2\sigma(\gamma g)z}}{\sinh(\sigma(\gamma g))} \right)$$

where $\sigma(g)$ is the geodesic distance from gK to $x_0 = 1 \cdot K$. The Poincaré series has spectral expansion

$$\begin{aligned} \text{Pé}_z = & \sum_{f \in GL_2 \text{ cfm}} \frac{\sin(2bt_f) \cdot \bar{f}(x_0) \cdot f}{2t_f \sinh(b)(t_f^2 + z^2)^2} \\ & + \frac{\bar{\Phi}_0(x_0) \cdot \Phi_0}{(z^2 - \frac{1}{4})^2} + \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\sin(2bt) \cdot E_{\frac{1}{2}-it}(x_0) \cdot E_{\frac{1}{2}+it}}{2t \sinh(b)(t^2 + z^2)^2} dt \end{aligned}$$

where the sum ranges over an orthonormal basis of cusp forms, Φ_0 denotes the constant automorphic form with L^2 -norm one, and

$$-(t_f^2 + \frac{1}{4}) \quad \text{and} \quad -(t^2 + \frac{1}{4})$$

are the Casimir eigenvalues of f and $E_{\frac{1}{2}+it}$, respectively.

Remark 4.2. Regardless of the convergence of the Poincaré series, the solution v_z^{afc} to the automorphic differential equation exists, is unique in global automorphic Sobolev spaces, and has the given spectral expansion, converging in a global automorphic Sobolev space. If desired, uniform pointwise convergence of the spectral expansion can be obtained by choosing ν sufficiently large, as mentioned above. The difficulty, even in the simplest possible higher rank cases, namely G complex of odd rank, of ascertaining whether v_z is of sufficiently rapid decay along the walls of the Weyl chambers, where $\prod \sinh(\alpha(\log a))$ blows up, is reason to question whether the explicit “geometric” Poincaré series representation of v^{afc} is actually needed in a given application or whether the automorphic spectral expansion suffices.

Appendix A. The harmonicity of π^+

Let G be complex semi-simple Lie group. We will give a direct proof that the function $\pi^+ : \mathfrak{a}^* \rightarrow \mathbb{R}$ given by $\pi^+(\mu) = \prod_{\alpha > 0} \langle \alpha, \mu \rangle$ where the product is taken over all positive roots, counted without multiplicity, is harmonic with respect to the Laplacian naturally associated to the pairing on \mathfrak{a}^* . (See also [31], Lemma 2, where this result is obtained as a simple corollary of the less trivial fact that π^+ divides any polynomial that is W -equivariant by the sign character.) It is this property that enables us to use Hecke’s identity in the computations above. We will use the following lemma.

Lemma A.1. *Let I be the set of all non-orthogonal pairs of distinct positive roots, as functions on \mathfrak{a} . Then π^+ is harmonic if $\sum_{(\beta,\gamma) \in I} \frac{\langle \beta, \gamma \rangle}{\beta\gamma} = 0$.*

Proof. Considering \mathfrak{a}^* as a Euclidean space, its Lie algebra can be identified with itself. For any basis $\{x_i\}$ of \mathfrak{a}^* , the Casimir operator (Laplacian) is $\Delta = \sum_i x_i x_i^*$. For any α, β in \mathfrak{a}^* and any $\lambda \in \mathfrak{a}$

$$\begin{aligned} \Delta \langle \alpha, \lambda \rangle \langle \beta, \lambda \rangle &= \sum_i x_i (\langle \alpha, x_i^* \rangle \langle \beta, \lambda \rangle + \langle \alpha, \lambda \rangle \langle \beta, x_i^* \rangle) \\ &= \sum_i (\langle \alpha, x_i^* \rangle \langle \beta, x_i \rangle + \langle \alpha, x_i \rangle \langle \beta, x_i^* \rangle) = 2\langle \alpha, \beta \rangle. \end{aligned}$$

Thus

$$\begin{aligned} \Delta \pi^+ &= \sum_i x_i x_i^* \pi^+ = \sum_i x_i \sum_{\beta > 0} \alpha(x_i^*) \cdot \frac{\pi^+}{\beta} \\ &= \sum_i \sum_{\beta > 0} \beta(x_i^*) \cdot \left(\sum_{\gamma \neq \beta} \gamma(x_i) \cdot \frac{\pi^+}{\beta\gamma} \right) = \left(\sum_{\beta \neq \gamma} \frac{\langle \beta, \gamma \rangle}{\beta\gamma} \right) \cdot \pi^+ \end{aligned}$$

and π^+ is harmonic if the sum in the statement of the lemma is zero. □

Remark A.1. When the Lie algebra \mathfrak{g} is not simple, but merely semi-simple, i.e., $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, any pair β, γ of roots with $\beta \in \mathfrak{g}_1$ and $\gamma \in \mathfrak{g}_2$ will have $\langle \beta, \gamma \rangle = 0$, so it suffices to consider \mathfrak{g} simple.

Proposition A.1. *For $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_2$, or \mathfrak{g}_2 , the following sum over all pairs (β, γ) of distinct positive roots is zero: $\sum_{\beta \neq \gamma} \frac{\langle \beta, \gamma \rangle}{\beta\gamma} = 0$.*

Proof. The positive roots in \mathfrak{sl}_3 are α, β , and $(\alpha + \beta)$ with $\langle \alpha, \alpha \rangle = 2, \langle \beta, \beta \rangle = 2, \langle \alpha, \beta \rangle = -1$. In other words, the two simple roots have the same length and have an angle of $2\pi/3$ between them. The pairs of distinct positive roots are $(\alpha, \beta), (\alpha, \alpha + \beta)$ and $(\beta, \alpha + \beta)$, so the sum to compute is

$$\frac{\langle \alpha, \beta \rangle}{\alpha\beta} + \frac{\langle \alpha, \alpha + \beta \rangle}{\alpha(\alpha + \beta)} + \frac{\langle \beta, \alpha + \beta \rangle}{\beta(\alpha + \beta)}.$$

Clearing denominators and evaluating the parings,

$$\langle \alpha, \beta \rangle \cdot (\alpha + \beta) + \langle \alpha, \alpha + \beta \rangle \cdot \beta + \langle \beta, \alpha + \beta \rangle \cdot \alpha = -(\alpha + \beta) + \beta + \alpha = 0.$$

For \mathfrak{sp}_2 , the simple roots have lengths 1 and $\sqrt{2}$ and have an angle of $3\pi/4$ between them: $\langle \alpha, \alpha \rangle = 1, \langle \beta, \beta \rangle = 2, \langle \alpha, \beta \rangle = -1$. The other positive roots are $(\alpha + \beta)$ and $(2\alpha + \beta)$. The non-orthogonal pairs of distinct positive roots are $(\alpha, \beta), (\alpha, 2\alpha + \beta), (\beta, \alpha + \beta)$, and $(\alpha + \beta, 2\alpha + \beta)$. So the sum we must compute is

$$\frac{\langle \alpha, \beta \rangle}{\alpha\beta} + \frac{\langle \alpha, 2\alpha + \beta \rangle}{\alpha(2\alpha + \beta)} + \frac{\langle \beta, \alpha + \beta \rangle}{\beta(\alpha + \beta)} + \frac{\langle \alpha + \beta, 2\alpha + \beta \rangle}{(\alpha + \beta)(2\alpha + \beta)}.$$

Again, clearing denominators,

$$\begin{aligned} & \langle \alpha, \beta \rangle \cdot (\alpha + \beta)(2\alpha + \beta) + \langle \alpha, 2\alpha + \beta \rangle \cdot \beta(\alpha + \beta) \\ & \quad + \langle \beta, \alpha + \beta \rangle \cdot \alpha(2\alpha + \beta) + \langle \alpha + \beta, 2\alpha + \beta \rangle \cdot \alpha\beta \end{aligned}$$

and evaluating the pairings,

$$\begin{aligned} & -(\alpha + \beta)(2\alpha + \beta) + \beta(\alpha + \beta) + \alpha(2\alpha + \beta) + \alpha\beta \\ & = -(2\alpha^2 + 3\alpha\beta + \beta^2) + \alpha\beta + \beta^2 + 2\alpha^2 + \alpha\beta + \alpha\beta = 0. \end{aligned}$$

Finally we consider the exceptional Lie algebra \mathfrak{g}_2 . The simple roots have lengths 1 and $\sqrt{3}$ and have an angle of $5\pi/6$ between them: $\langle \alpha, \alpha \rangle = 1$, $\langle \beta, \beta \rangle = 3$, $\langle \alpha, \beta \rangle = -3/2$. The other positive roots are $(\alpha + \beta)$, $(2\alpha + \beta)$, $(3\alpha + \beta)$, and $(3\alpha + 2\beta)$. Notice that the roots α and $\alpha + \beta$ have the same length and have an angle of $3\pi/2$ between them. So together with their sum $2\alpha + \beta$, they form a copy of the \mathfrak{sl}_3 root system. The three terms corresponding to the three pairs of roots among these roots will cancel, as in the \mathfrak{sl}_3 case. Similarly, the roots $(3\alpha + \beta)$ and β have the same length and have an angle of $3\pi/2$ between them, so, together with their sum, $(3\alpha + 2\beta)$ they form a copy of the \mathfrak{sl}_3 root system, and the three terms in the sum corresponding to the three pairs among these roots will also cancel. The remaining six pairs of distinct, non-orthogonal positive roots are $(\alpha, 3\alpha + \beta)$, (α, β) , $(3\alpha + \beta, 2\alpha + \beta)$, $(2\alpha + \beta, 3\alpha + 2\beta)$, $(3\alpha + 2\beta, \alpha + \beta)$, and $(\alpha + \beta, \beta)$. We shall see that the six terms corresponding to these pairs cancel as a group. After clearing denominators, the relevant sum is

$$\begin{aligned} & \langle \alpha, \beta \rangle \cdot (\alpha + \beta)(2\alpha + \beta)(3\alpha + \beta)(3\alpha + 2\beta) \\ & \quad + \langle \alpha, 3\alpha + \beta \rangle \cdot \beta(\alpha + \beta)(2\alpha + \beta)(3\alpha + 2\beta) \\ & \quad + \langle 3\alpha + \beta, 2\alpha + \beta \rangle \cdot \alpha\beta(\alpha + \beta)(3\alpha + 2\beta) \\ & \quad + \langle 2\alpha + \beta, 3\alpha + 2\beta \rangle \cdot \alpha\beta(\alpha + \beta)(3\alpha + \beta) \\ & \quad + \langle 3\alpha + 2\beta, \alpha + \beta \rangle \cdot \alpha\beta(2\alpha + \beta)(3\alpha + \beta) \\ & \quad + \langle \alpha + \beta, \beta \rangle \cdot \alpha(2\alpha + \beta)(3\alpha + \beta)(3\alpha + 2\beta). \end{aligned}$$

Evaluating the pairings and factoring out $(3/2)$, this is

$$\begin{aligned} & -(\alpha + \beta)(2\alpha + \beta)(3\alpha + \beta)(3\alpha + 2\beta) + \beta(\alpha + \beta)(2\alpha + \beta)(3\alpha + 2\beta) \\ & \quad + \alpha\beta(\alpha + \beta)(3\alpha + 2\beta) + \alpha\beta(\alpha + \beta)(3\alpha + \beta) \\ & \quad + \alpha\beta(2\alpha + \beta)(3\alpha + \beta) + \alpha(2\alpha + \beta)(3\alpha + \beta)(3\alpha + 2\beta). \end{aligned}$$

Expanding,

$$\begin{aligned} & -18\alpha^4 - 45\alpha^3\beta - 40\alpha^2\beta^2 - 15\alpha\beta^3 - 2\beta^4 \\ & \quad + 6\alpha^3\beta + 13\alpha^2\beta^2 + 9\alpha\beta^3 + 2\beta^4 \\ & \quad + 3\alpha^3\beta + 5\alpha^2\beta^2 + 2\alpha\beta^3 \\ & \quad + 3\alpha^3\beta + 4\alpha^2\beta^2 + \alpha\beta^3 \\ & \quad + 6\alpha^3\beta + 5\alpha^2\beta^2 + \alpha\beta^3 \\ & \quad + 18\alpha^4 + 27\alpha^3\beta + 13\alpha^2\beta^2 + 2\alpha\beta^3. \end{aligned}$$

This sum is zero. □

Proposition A.2. *For any complex simple Lie algebra \mathfrak{g} , the following sum over all pairs (β, γ) of distinct positive roots is zero: $\sum_{\beta \neq \gamma} \frac{\langle \beta, \gamma \rangle}{\beta \gamma} = 0$.*

Proof. Let I be the indexing set $\{(\beta, \gamma)\}$ of pairs of distinct, non-orthogonal positive roots. For each $(\beta, \gamma) \in I$, let $\mathcal{R}_{\beta, \gamma}$ be the two-dimensional root system generated by β and γ . For such a root system \mathcal{R} , let $I_{\mathcal{R}}$ be the set of pairs of distinct, non-orthogonal positive roots, where positivity is inherited from the ambient \mathfrak{g} . The collection J of all such $I_{\mathcal{R}}$ is a cover of I . We refine J to a subcover J' of disjoint sets, in the following way.

For any pair $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$ of sets in J with nonempty intersection, there is a two-dimensional root system \mathcal{R}'' such that $I_{\mathcal{R}''}$ contains $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$. Indeed, letting (β, γ) and (β', γ') be pairs in I generating \mathcal{R} and \mathcal{R}' respectively, the nonempty intersection of $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$ implies that there is a pair (β'', γ'') lying in both $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$. Since \mathcal{R} and \mathcal{R}' are two-dimensional and β'' and γ'' are linearly independent, all six roots lie in a plane. Since all six roots lie in the root system for \mathfrak{g} , they generate a two-dimensional root system \mathcal{R}'' containing \mathcal{R} and \mathcal{R}' , and $I_{\mathcal{R}''} \supset I_{\mathcal{R}}, I_{\mathcal{R}'}$. Thus we refine J to a subcover J' : if $I_{\mathcal{R}}$ in J intersects any $I_{\mathcal{R}'}$ in J , replace $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$ with the set $I_{\mathcal{R}''}$ described above. The sets $I_{\mathcal{R}}$ in J' are mutually disjoint, and, for any $(\beta, \gamma) \in I$, there is a root system \mathcal{R} such that $(\beta, \gamma) \in I_{\mathcal{R}} \in J'$, thus

$$\sum_{(\beta, \gamma) \in I} \frac{\langle \beta, \gamma \rangle}{\beta \gamma} = \sum_{I_{\mathcal{R}} \in J'} \sum_{(\beta, \gamma) \in I_{\mathcal{R}}} \frac{\langle \beta, \gamma \rangle}{\beta \gamma}.$$

By the classification of complex simple Lie algebras of rank two, \mathcal{R} is isomorphic to the root system of \mathfrak{sl}_3 , \mathfrak{sp}_2 , or \mathfrak{g}_2 . Thus, by Proposition A.1, the inner sum over $I_{\mathcal{R}}$ is zero, proving that the whole sum is zero.

Note that the refinement is necessary, as there are copies of \mathfrak{sl}_3 inside \mathfrak{g}_2 . Note also that the only time the root system of \mathfrak{g}_2 appears is in the case of \mathfrak{g}_2 itself, since, by the classification, \mathfrak{g}_2 is the only root system containing roots that have an angle of $\pi/6$ or $5\pi/6$ between them. □

Remark A.2. See [24], Lemma 2, for a proof of Proposition A.2 when G is not necessarily complex.

Theorem A.1. *For a complex semi-simple Lie group G , the function $\pi^+ : \mathfrak{a}^* \rightarrow \mathbb{R}$ given by $\pi^+(\mu) = \prod_{\alpha > 0} \langle \alpha, \mu \rangle$ where the product is taken over all positive roots, counted without multiplicity, is harmonic with respect to the Laplacian naturally associated to the pairing on \mathfrak{a}^* .*

Proof. This follows immediately from Lemma A.1, Remark A.1, and Proposition A.2. □

Appendix B. Evaluating the integral

Proposition B.1. For $\nu > n/2$, $\text{Re}(z) > 0$, $x \in \mathbb{R}^n$,

$$\mathcal{I}_z(x) = \int_{\mathbb{R}^n} \frac{e^{i\langle \xi, x \rangle}}{(|\xi|^2 + z^2)^\nu} d\xi = \frac{\pi^{n/2}}{2^{\nu-n/2-1}\Gamma(\nu)} \left(\frac{|x|}{z}\right)^{\nu-n/2} K_{\nu-n/2}(|x|z).$$

In particular, when n is odd and $\nu = \frac{n+1}{2}$,

$$\mathcal{I}_z(x) = \frac{\pi^{(n+1)/2}}{(\frac{n-1}{2})!} \frac{e^{-|x|z}}{z} (\nu = \frac{n+1}{2} \in \mathbb{Z})$$

and when n is even and $\nu = \frac{n}{2} + 1$,

$$\mathcal{I}_z(x) = \frac{\pi^{n/2}}{(\frac{n}{2})!} \frac{|x|K_1(|x|z)}{z} (\nu = \frac{n}{2} + 1 \in \mathbb{Z}).$$

Proof. Since the integral is rotation-invariant, we may assume $\langle \xi, x \rangle = |x| \xi_1$, where $\xi = (\xi_1, \dots, \xi_n)$. Then, using the Gamma function, we may rewrite the integral as

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{e^{i\langle \xi, x \rangle}}{(|\xi|^2 + z^2)^\nu} d\xi &= \int_{\mathbb{R}^n} \frac{e^{i|x|\xi_1}}{(|\xi|^2 + z^2)^\nu} d\xi \\ &= \frac{\pi^{(n-1)/2}\Gamma(\nu - \frac{n-1}{2})}{\Gamma(\nu)} \int_{\mathbb{R}} \frac{e^{i|x|\xi_1}}{(\xi_1^2 + z^2)^{\nu-(n-1)/2}} d\xi_1. \end{aligned}$$

This integral can be expressed as a modified Bessel function:

$$\int_{\mathbb{R}} \frac{e^{iAt}}{(t^2 + z^2)^s} dt = \frac{\sqrt{2\pi}}{2^{s-1}\Gamma(s)} \left(\frac{A}{z}\right)^{s-1/2} K_{s-1/2}(Az)$$

for $\text{Re}(s) > \frac{1}{2}$, $\text{Re}(z) > 0$, $A > 0$. In particular, when $s = \ell + \frac{1}{2}$ is a half-integer:

$$\int_{\mathbb{R}} \frac{e^{iAt}}{(t^2 + z^2)^{\ell+1/2}} dt = \frac{2 \cdot \ell!}{(2\ell)!} \left(\frac{A}{2z}\right)^\ell K_\ell(Az)$$

for $\ell \in \mathbb{N}$, $\text{Re}(z) > 0$, $A > 0$, and, when $s = \ell + 1$ is an integer:

$$\int_{\mathbb{R}} \frac{e^{iAt}}{(t^2 + z^2)^{\ell+1}} dt = \frac{\pi e^{-Az}}{z^{2\ell+1}} P_\ell(Az)$$

for $\ell \in \mathbb{Z}_{\geq 0}$, $\text{Re}(z) > 0$, $A > 0$, where $P_\ell(x)$ is a degree ℓ polynomial with coefficients given by

$$a_k = \frac{(2\ell - k)!}{2^{2\ell-k} \ell! (\ell - k)! k!}.$$

Specializing to $\nu = \frac{n+1}{2}$ and $\nu = \frac{n}{2} + 1$ yields the desired conclusions. □

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