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A note on tetrablock contractions

Haripada Sau

Abstract. A commuting triple of operators (A, B, P) on a Hilbert space $\mathcal{H}$ is called a tetrablock contraction if the closure of the set

$$E = \left\{(a_{11}, a_{22}, \det A) : A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ with } \|A\| < 1\right\}$$

is a spectral set. In this paper, we construct a functional model and produce a set of complete unitary invariants for a pure tetrablock contraction. In this construction, the fundamental operators, which are the unique solutions of the operator equations

$$A - B^*P = D_P X_1 D_P \quad \text{and} \quad B - A^*P = D_P X_2 D_P,$$

where $X_1, X_2 \in \mathcal{B}(\mathcal{D}_P)$ play a pivotal role. As a result of the functional model, we show that every pure tetrablock isometry (A, B, P) on an abstract Hilbert space $\mathcal{H}$ is unitarily equivalent to the tetrablock contraction $(M_{G_1^*+G_2 z}, M_{G_2^*+G_1 z}, M_z)$ on $H^2_{\mathcal{D}_{P^*}}(\mathbb{D})$, where G_1 and G_2 are the fundamental operators of (A^*, B^*, P^*). We prove a Beurling–Lax–Halmos type theorem for a triple of operators $(M_{F_1^*+F_2 z}, M_{F_2^*+F_1 z}, M_z)$, where $\mathcal{E}$ is a Hilbert space and $F_1, F_2 \in \mathcal{B}(\mathcal{E})$. We also deal with a natural example of tetrablock contraction on a functions space to find out its fundamental operators.

Contents

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1. Introduction

The set *tetrablock* is defined as

$$E = \left\{ (a_{11}, a_{22}, \det A) : A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ with } \|A\| < 1 \right\}.$$

This domain was studied in [1] and [2] for its geometric properties. Let $A(E)$ be the algebra of functions holomorphic in E and continuous in $\bar{E}$. The distinguished boundary of E (denoted by $b(E)$), i.e., the Shilov boundary with respect to $A(E)$, is found in [1] and [2] to be the set

$$bE = \left\{ (a_{11}, a_{22}, \det A) : A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ whenever } A \text{ is unitary} \right\}.$$

The operator theory on tetrablock was first developed in [7].

Definition 1.1. A triple (A, B, P) of commuting bounded operators on a Hilbert space $\mathcal{H}$ is called a *tetrablock contraction* if $\overline{E}$ is a spectral set for (A, B, P), i.e., the Taylor joint spectrum of (A, B, P) is contained in $\overline{E}$ and

$$||f(A, B, P)|| \leq ||f||_{\infty, \overline{E}} = \sup\{|f(x_1, x_2, x_3)| : (x_1, x_2, x_3) \in \overline{E}\}$$

for any polynomial f in three variables.

It turns out that in case the set is polynomially convex as in the case of tetrablock, the condition that the Taylor joint spectrum lies inside the set, is redundant, see Lemma 3.3 in [7]. There are analogues of unitaries and isometries.

A *tetrablock unitary* is a commuting pair of normal operators (A, B, P) such that its Taylor joint spectrum is contained in bE.

A *tetrablock isometry* is the restriction of a tetrablock unitary to a joint invariant subspace. See [7], for several characterizations of a tetrablock unitary and a tetrablock isometry.

Consider a tetrablock contraction (A, B, P). Then it is easy to see that P is a contraction.

Fundamental equations for a tetrablock contraction are introduced in [7]. And these are

$$A - B^*P = D_P F_1 D_P, \quad \text{and} \quad B - A^*P = D_P F_2 D_P \tag{1.1}$$

where $D_P = (I - P^*P)^{\frac{1}{2}}$ is the defect operator of the contraction P and F_1, F_2 are bounded operators on $\mathcal{D}_P$, where $\mathcal{D}_P = \overline{\text{Ran}} D_P$. Theorem 3.5 in [7] says that the two fundamental equations can be solved and the solutions F_1 and F_2 are unique. The unique solutions F_1 and F_2 of (1.1) are called the *fundamental operators* of the tetrablock contraction (A, B, P). Moreover, $w(F_1)$ and $w(F_2)$ are not greater than 1, where $w(X)$, for a bounded operator X on a complex Hilbert space $\mathcal{H}$, denotes the numerical radius of X, i.e.,

$$w(X) = \{|\langle Xh, h\rangle| : \text{where } h \in \mathcal{H} \text{ with } \|h\| = 1\}.$$

The adjoint triple (A^*, B^*, P^*) is also a tetrablock contraction as can be seen from the definition. By what we stated above, there are unique $G_1, G_2 \in \mathcal{B}(\mathcal{D}_{P^*})$ such that

$$A^* - BP^* = D_{P^*}G_1D_{P^*} \quad \text{and} \quad B^* - AP^* = D_{P^*}G_2D_{P^*}. \tag{1.2}$$

Moreover, $w(G_1)$ and $w(G_2)$ are not greater than 1.

In [7] (Theorem 6.1), it was shown that the tetrablock is a complete spectral set under the conditions that F_1 and F_2 satisfy

$$[X_1, X_2] = 0 \quad \text{and} \quad [X_1, X_1^*] = [X_2, X_2^*] \tag{1.3}$$

in place of X_1 and X_2 respectively. Where $[X_1, X_2]$, for two bounded operators X_1 and X_2, denotes the commutator of X_1 and X_2, i.e., the operator $X_1X_2 - X_2X_1$. In Section 2, we show that if the contraction P has dense range, then commutativity of the fundamental operators F_1 and F_2 is enough to have a dilation of the tetrablock contraction (A, B, P). In fact, under the same hypothesis we show that G_1 and G_2 also satisfy (1.3), in place of X_1 and X_2 respectively. This is the content of Theorem 2.6.

For a Hilbert space $\mathcal{E}$, $H^2_{\mathcal{E}}(\mathbb{D})$ stands for the Hilbert space of $\mathcal{E}$-valued analytic functions on $\mathbb{D}$ with square summable Taylor series co-efficients about the point zero. When $\mathcal{E} = \mathbb{C}$, we write $H^2_{\mathcal{E}}(\mathbb{D})$ as $H^2(\mathbb{D})$. The space $H^2_{\mathcal{E}}(\mathbb{D})$ is unitarily equivalent to the space $H^2(\mathbb{D}) \otimes \mathcal{E}$ via the map $z^n\xi \to z^n \otimes \xi$, for all $n \geq 0$ and $\xi \in \mathcal{E}$. We shall identify these unitarily equivalent spaces and use them, without mention, interchangeably as per notational convenience

In [6], Beurling characterized invariant subspaces for the 'multiplication by z' operator on the Hardy space $H^2(\mathbb{D})$. In [11], Lax extended Beurling's result to the finite-dimensional vector space valued Hardy spaces. Then Halmos extended Lax's result to infinite-dimensional vector spaces in [10]. The extended result is the following.

Theorem 1.2 (Beurling–Lax–Halmos)**.** *Let $0 \neq \mathcal{M}$ be a closed subspace of $H^2_{\mathcal{E}}(\mathbb{D})$. Then $\mathcal{M}$ is invariant under M_z if and only if there exist a Hilbert space $\mathcal{E}_*$ and an inner function $(\mathcal{E}_*, \mathcal{E}, \Theta)$ such that $\mathcal{M} = \Theta H^2_{\mathcal{E}_*}(\mathbb{D})$.*

In Section 3, we prove a Beurling–Lax–Halmos type theorem for a triple of operators, which is the first main result of this paper. More explicitly, given a Hilbert space $\mathcal{E}$ and two bounded operators $F_1, F_2 \in \mathcal{B}(\mathcal{E})$, we shall see that a nonzero closed subspace $\mathcal{M}$ of $H^2_{\mathcal{E}}(\mathbb{D})$ is invariant under $(M_{F_1^*+F_2z}, M_{F_2^*+F_1z}, M_z)$ if and only if

$$(F_1^* + F_2z)\Theta(z) = \Theta(z)(G_1 + G_2^*z),$$
$$(F_2^* + F_1z)\Theta(z) = \Theta(z)(G_2 + G_1^*z),$$

for all $z \in \mathbb{D}$ for some unique $G_1, G_2 \in \mathcal{B}(\mathcal{E}_*)$, where $(\mathcal{E}_*, \mathcal{E}, \Theta)$ is the Beurling–Lax–Halmos representation of $\mathcal{M}$. Along the way we shall see that if F_1 and F_2 are such that $(M_{F_1^*+F_2z}, M_{F_2^*+F_1z}, M_z)$ on $H^2(\mathcal{E})$ is a tetrablock isometry, then $(M_{G_1+G_2^*z}, M_{G_2+G_1^*z}, M_z)$ is also a tetrablock isometry on $H^2(\mathcal{E}_*)$. This is the content of Theorem 3.1.

A contraction P on a Hilbert space $\mathcal{H}$ is called *pure* if $P^{*n} \to 0$ strongly, i.e., $\|P^{*n}h\|^2 \to 0$, for all $h \in \mathcal{H}$. A contraction P is called *completely-nonunitary* (c.n.u.) if it has no reducing sub-spaces on which its restriction is unitary. A tetrablock contraction (A, B, P) is called a *pure tetrablock contraction* if the contraction P is pure.

Sz.-Nagy and Foias developed the model theory for a contraction [13]. There have been numerous developments in model theory of commuting tuples associated with domains in $\mathbb{C}^n(n \geq 1)$ [4, 3, 8, 9, 12]. Section 4 gives a functional model of pure tetrablock contractions, the second main result of this paper. In this model theory, the fundamental operators play a pivotal role. We shall see that if (A, B, P) is a pure tetrablock contraction on a Hilbert space $\mathcal{H}$, then the operators A, B and P are unitarily equivalent to $P_{\mathcal{H}_P}(I \otimes G_1^* + M_z \otimes G_2)|_{\mathcal{H}_P}, P_{\mathcal{H}_P}(I \otimes G_2^* + M_z \otimes G_1)|_{\mathcal{H}_P}$ and $P_{\mathcal{H}_P}(M_z \otimes I_{\mathcal{D}_{P^*}})|_{\mathcal{H}_P}$ respectively, where G_1 and G_2 are fundamental operators of (A^*, B^*, P^*) and $\mathcal{H}_P$ is the model space of a pure contraction P, as in [13]. This is the content of Theorem 4.2. As a corollary to this theorem, we shall see that every pure tetrablock isometry (A, B, P) on an abstract Hilbert space $\mathcal{H}$ is unitarily equivalent to the tetrablock contraction $(M_{G_1^*+G_2z}, M_{G_2^*+G_1z}, M_z)$ on $H^2_{\mathcal{D}_{P^*}}(\mathbb{D})$, where G_1 and G_2 are the fundamental operators of (A^*, B^*, P^*).

Two equations associated with a contraction P and its defect operators that have been known from the time of Sz.-Nagy and that will come handy are

$$PD_P = D_{P^*}P \tag{1.4}$$

and its corresponding adjoint relation

$$D_P P^* = P^* D_{P^*}. \tag{1.5}$$

Proof of (1.4) and (1.5) can be found in [13, ch. 1, sec. 3].

For a contraction P, the *characteristic function* Θ_P is defined by

$$\Theta_P(z) = [-P + zD_{P^*}(I_{\mathcal{H}} - zP^*)^{-1}D_P]|_{\mathcal{D}_P} \quad \text{for all } z \in \mathbb{D}. \tag{1.6}$$

By virtue of (1.4), it follows that, for each $z \in \mathbb{D}$, the operator $\Theta_P(z)$ is an operator from $\mathcal{D}_P$ into $\mathcal{D}_{P^*}$.

In [13], Sz.-Nagy and Foias found a set of unitary invariant for c.n.u. contractions. The set consists of only one member, the characteristic function of the contraction. There are many beautiful results in this direction, see [8, 9, 12] and the references therein. In Section 5, we produce a set of unitary invariants for a pure tetrablock contraction (A, B, P). In this case the set of unitary invariants consists of three members, the characteristic function of P and the two fundamental operators of (A^*, B^*, P^*). This (Theorem 5.4) is the third major result of this paper. The result states that for two pure tetrablock contractions (A, B, P) and (A', B', P') to be unitary equivalent, it is necessary and sufficient that the characteristic functions of P and P' coincide and the fundamental operators (G_1, G_2) and (G'_1, G'_2) of (A, B, P) and (A', B', P') respectively, are unitary equivalent by the same unitary that is involved in the coincidence of the characteristic functions of P and P'.

It is very hard to compute the fundamental operators of a tetrablock contraction, in general. We now know how important the role of the fundamental operators is in the model theory of pure tetrablock contractions. So it is important to have a concrete example of fundamental operators and grasp the above model theory by dealing with them. That is what Section 6 does. In other words, we find the fundamental operators (G_1, G_2) of the adjoint of a pure tetrablock isometry (A, B, P) and the unitary operator which unitarizes (A, B, P) to the pure tetrablock isometry $(M_{G_1^*+G_2z}, M_{G_2^*+G_1z}, M_z)$ on $H^2_{\mathcal{D}_{P^*}}(\mathbb{D})$.

2. Relations between fundamental operators

In this section we prove some important relations between fundamental operators of a tetrablock contraction. Before going to state and prove the main theorem of this section, we shall recall two results, which were proved originally in [7].

Lemma 2.1. *Let (A, B, P) be a tetrablock contraction with commuting fundamental operators F_1 and F_2. Then*

$$A^*A - B^*B = D_P(F_1^*F_1 - F_2^*F_2)D_P.$$

Lemma 2.2. *The fundamental operators F_1 and F_2 of a tetrablock contraction (A, B, P) are the unique bounded linear operators on $\mathcal{D}_P$ that satisfy the pair of operator equations*

$$D_PA = X_1D_P + X_2^*D_PP \quad \text{and} \quad D_PB = X_2D_P + X_1^*D_PP.$$

Now we state and prove three relations between the fundamental operators of a tetrablock contraction, which will be used later in this paper.

Lemma 2.3. *Let (A, B, P) be a tetrablock contraction on a Hilbert space $\mathcal{H}$. Let F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then*

$$D_PF_1 = (AD_P - D_{P^*}G_2P)|_{\mathcal{D}_P} \quad \text{and} \quad D_PF_2 = (BD_P - D_{P^*}G_1P)|_{\mathcal{D}_{\mathcal{P}}}.$$

Proof. We shall prove only one of the above, proof of the other is similar. For $h \in \mathcal{H}$, we have

$$\begin{aligned}(AD_P - D_{P^*}G_2P)D_Ph &= A(I - P^*P)h - (D_{P^*}G_2D_{P^*})Ph \\ &= Ah - AP^*Ph - (B^* - AP^*)Ph \\ &= Ah - AP^*Ph - B^*Ph + AP^*Ph \\ &= (A - B^*P)h = (D_PF_1)D_Ph.\end{aligned}$$

□

Lemma 2.4. *Let (A,B,P) be a tetrablock contraction on a Hilbert space $\mathcal{H}$. Let F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then*

$$PF_i = G_i^* P|_{\mathcal{D}_P} \quad \text{for } i{=}1,2.$$

Proof. We shall prove only for $i = 1$, the proof for $i = 2$ is similar. Note that the operators on both sides are from $\mathcal{D}_P$ to $\mathcal{D}_{P^*}$. Let $h, h' \in \mathcal{H}$ be any two elements. Then

$$\begin{aligned}
&\langle (PF_1 - G_1^*P)D_P h, D_{P^*}h'\rangle \\
&= \langle D_{P^*}PF_1D_Ph, h'\rangle - \langle D_{P^*}G_1^*PD_Ph, h'\rangle \\
&= \langle P(D_PF_1D_P)h, h'\rangle - \langle (D_{P^*}G_1^*D_{P^*})Ph, h'\rangle \\
&= \langle P(A - B^*P)h, h'\rangle - \langle (A - PB^*)Ph, h'\rangle \\
&= \langle (PA - PB^*P - AP + PB^*P)h, h'\rangle = 0.
\end{aligned}$$

□

Lemma 2.5. *Let (A, B, P) be a tetrablock contraction on a Hilbert space $\mathcal{H}$. Let F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then*

$$\begin{aligned}
(F_1^*D_PD_{P^*} - F_2P^*)|_{\mathcal{D}_{P^*}} &= D_PD_{P^*}G_1 - P^*G_2^*, \\
(F_2^*D_PD_{P^*} - F_1P^*)|_{\mathcal{D}_{P^*}} &= D_PD_{P^*}G_2 - P^*G_1^*.
\end{aligned}$$

Proof. For $h \in \mathcal{H}$, we have

$$\begin{aligned}
&(F_1^*D_PD_{P^*} - F_2P^*)D_{P^*}h \\
&= F_1^*D_P(I - PP^*)h - F_2P^*D_{P^*}h \\
&= F_1^*D_Ph - F_1^*D_PPP^*h - F_2D_PP^*h \\
&= F_1^*D_Ph - (F_1^*D_PP + F_2D_P)P^*h \\
&= F_1^*D_Ph - D_PBP^*h \quad \text{[by Lemma 2.2]} \\
&= (AD_P - D_{P^*}G_2P)^*h - D_PBP^*h \quad \text{[by Lemma 2.3]} \\
&= D_PA^*h - P^*G_2^*D_{P^*}h - D_PBP^*h \\
&= D_P(A^* - BP^*)h - P^*G_2^*D_{P^*}h \\
&= D_PD_{P^*}G_1D_{P^*}h - P^*G_2^*D_{P^*}h \\
&= (D_PD_{P^*}G_1 - P^*G_2^*)D_{P^*}h.
\end{aligned}$$

Proof of the other relation is similar and hence is skipped. □

Now we prove the main result of this section.

Theorem 2.6. *Let F_1 and F_2 be fundamental operators of a tetrablock contraction (A, B, P) on a Hilbert space $\mathcal{H}$. And let G_1 and G_2 be fundamental operators of the tetrablock contraction (A^*, B^*, P^*). If $[F_1, F_2] = 0$ and P has dense range, then:*

(i) $[F_1, F_1^*] = [F_2, F_2^*]$.
(ii) $[G_1, G_2] = 0$.

(iii) $[G_1, G_1^*] = [G_2, G_2^*]$.

Proof. (i) From Lemma 2.2 we have $D_P A = F_1 D_P + F_2^* D_P P$. This gives after multiplying by F_2 from the left in both sides,

$$\begin{aligned}
&F_2 D_P A = F_2 F_1 D_P + F_2 F_2^* D_P P \\
&\Rightarrow D_P F_2 D_P A = D_P F_2 F_1 D_P + D_P F_2 F_2^* D_P P \\
&\Rightarrow (B - A^* P) A = D_P F_2 F_1 D_P + D_P F_2 F_2^* D_P P \\
&\Rightarrow BA - A^* AP = D_P F_2 F_1 D_P + D_P F_2 F_2^* D_P P.
\end{aligned}$$

Similarly, multiplying by F_1 from the left in both sides of

$$D_P B = F_2 D_P + F_1^* D_P P$$

and proceeding as above we get

$$AB - B^* BP = D_P F_1 F_2 D_P + D_P F_1 F_1^* D_P P.$$

Subtracting these two equations we get

$$(A^* A - B^* B) P = D_P [F_1, F_2] D_P + D_P (F_1 F_1^* - F_2 F_2^*) D_P P.$$

Eliminating A and B by Lemma 2.1, we have

$$\begin{aligned}
&D_P (F_1^* F_1 - F_2^* F_2) D_P P = D_P [F_1, F_2] D_P + D_P (F_1 F_1^* - F_2 F_2^*) D_P P \\
&\Rightarrow D_P ([F_1, F_1^*] - [F_2, F_2^*]) D_P P = 0 \text{ [since } [F_1, F_2] = 0.] \\
&\Rightarrow D_P ([F_1, F_1^*] - [F_2, F_2^*]) D_P = 0 \text{ [since } \operatorname{Ran} P \text{ is dense in } \mathcal{H}.] \\
&\Rightarrow [F_1, F_1^*] = [F_2, F_2^*].
\end{aligned}$$

(ii) From Lemma 2.4, we have that $PF_i = G_i^* P|_{\mathcal{D}_P}$ for $i = 1$ and 2. So we have

$$\begin{aligned}
&PF_1 F_2 D_P = G_1^* P F_2 D_P \\
&\Rightarrow PF_2 F_1 D_P = G_1^* P F_2 D_P \text{ [since } F_1 \text{ and } F_2 \text{ commute]} \\
&\Rightarrow G_2^* G_1^* P D_P = G_1^* G_2^* P D_P \text{ [applying Lemma 2.4]} \\
&\Rightarrow [G_1^*, G_2^*] D_{P^*} P = 0 \Rightarrow [G_1, G_2] = 0 \text{ [since } \operatorname{Ran} P \text{ is dense in } \mathcal{H}].
\end{aligned}$$

(iii) From Lemma 2.3, we have $D_P F_1 = (A D_P - D_{P^*} G_2 P)|_{\mathcal{D}_P}$, which gives after multiplying $F_2 D_P$ from right in both sides

$$\begin{aligned}
&D_P F_1 F_2 D_P = A D_P F_2 D_P - D_{P^*} G_2 P F_2 D_P \\
&\Rightarrow D_P F_1 F_2 D_P = A(B - A^* P) - D_{P^*} G_2 G_2^* P D_P \text{ [applying Lemma 2.4]} \\
&\Rightarrow D_P F_1 F_2 D_P = AB - AA^* P - D_{P^*} G_2 G_2^* P D_P.
\end{aligned}$$

Similarly, multiplying by $F_1 D_P$ from the right on both sides of

$$D_P F_2 = (B D_P - D_{P^*} G_1 P)|_{\mathcal{D}_{\mathcal{P}}},$$

we get

$$D_P F_2 F_1 D_P = BA - BB^* P - D_{P^*} G_1 G_1^* P D_P.$$

Subtracting these two equations we get

$$D_P[F_1, F_2]D_P = D_{P^*}(G_1G_1^* - G_2G_2^*)D_{P^*}P - (AA^* - BB^*)P.$$

Now applying Lemma 2.1 for the tetrablock contraction (A^*, B^*, P^*) and re-arranging terms, we get

$$\begin{aligned}&D_P[F_1, F_2]D_P = D_{P^*}([G_1, G_1^*] - [G_2, G_2^*])D_{P^*}P\\ &\Rightarrow D_{P^*}([G_1, G_1^*] - [G_2, G_2^*])D_{P^*}P = 0 \text{ [since } [F_1, F_2] = 0.]\\ &\Rightarrow [G_1, G_1^*] = [G_2, G_2^*] \text{ [since } \operatorname{Ran} P \text{ is dense in } \mathcal{H}].\end{aligned}$$

□

We would like to mention a corollary to Theorem 2.6 which gives a sufficient condition of when commutativity of the fundamental operators of (A, B, P) is necessary and sufficient for the commutativity of the fundamental operators of (A^*, B^*, P^*).

Corollary 2.7. *Let (A, B, P) be a tetrablock contraction on a Hilbert space $\mathcal{H}$ such that P is invertible. Let F_1, F_2, G_1 and G_2 be as in Theorem 2.6. Then $[F_1, F_2] = 0$ if and only if $[G_1, G_2] = 0$.*

Proof. Suppose that $[F_1, F_2] = 0$. Since P has dense range, by part (ii) of Theorem 2.6, we get $[G_1, G_2] = 0$. Conversely, let $[G_1, G_2] = 0$. Since P is invertible, P^* has dense range too. So applying Theorem 2.6 for the tetrablock contraction (A^*, B^*, P^*), we get $[F_1, F_2] = 0$. □

We conclude this section with another relation between the fundamental operators which will be used in the next section.

Lemma 2.8. *Let F_1 and F_2 be fundamental operators of a tetrablock contraction (A, B, P) and G_1 and G_2 be fundamental operators of the tetrablock contraction (A^*, B^*, P^*). Then*

$$(F_1^* + F_2z)\Theta_{P^*}(z) = \Theta_{P^*}(z)(G_1 + G_2^*z), \tag{2.1}$$

$$(F_2^* + F_1z)\Theta_{P^*}(z) = \Theta_{P^*}(z)(G_2 + G_1^*z), \tag{2.2}$$

for all $z \in \mathbb{D}$.

Proof. We prove Equation (2.1) only. The proof of Equation (2.2) is similar. By definition of Θ_{P^*} we have

$$(F_1^* + F_2z)\Theta_{P^*}(z) = (F_1^* + F_2z)\left(-P^* + \sum_{n=0}^{\infty} z^{n+1}D_PP^nD_{P^*}\right),$$

which after a re-arrangement of terms gives

$$-F_1^*P^* + z(-F_2P^* + F_1^*D_PD_{P^*}) + \sum_{n=2}^{\infty} z^n(F_1^*D_PP + F_2D_P)P^{n-2}D_{P^*},$$

which by Lemma 2.2, 2.4 and 2.5 is equal to

$$-P^*G_1 + z(D_PD_{P^*}G_1 - P^*G_2^*) + \sum_{n=2}^{\infty} z^nD_PBP^{n-2}D_{P^*}.$$

On the other hand

$$\Theta_{P^*}(z)(G_1 + G_2^* z) = \left(-P^* + \sum_{n=0}^{\infty} z^{n+1} D_P P^n D_{P^*}\right)(G_1 + G_2^* z),$$

which after a re-arrangement of terms gives

$$-P^* G_1 + z(D_P D_{P^*} G_1 - P^* G_2^*) + \sum_{n=2}^{\infty} z^n D_P P^{n-2}(P D_{P^*} G_1 + D_{P^*} G_2^*),$$

which by Lemma 2.2 is equal to

$$\begin{aligned} & - P^* G_1 + z(D_P D_{P^*} G_1 - P^* G_2^*) + \sum_{n=2}^{\infty} z^n D_P P^{n-2} B D_{P^*} \\ & = -P^* G_1 + z(D_P D_{P^*} G_1 - P^* G_2^*) + \sum_{n=2}^{\infty} z^n D_P B P^{n-2} D_{P^*}. \end{aligned}$$

Hence $(F_1^* + F_2 z)\Theta_{P^*}(z) = \Theta_{P^*}(z)(G_1 + G_2^* z)$ for all $z \in \mathbb{D}$. □

3. Beurling–Lax–Halmos representation for a triple of operators

In this section we prove a Beurling–Lax–Halmos type theorem for the triple of operators $(M_{F_1^*+F_2 z}, M_{F_2^*+F_1 z}, M_z)$ on $H^2_{\mathcal{E}}(\mathbb{D})$, where $\mathcal{E}$ is a Hilbert space and $F_1, F_2 \in \mathcal{B}(\mathcal{E})$. The triple $(M_{F_1^*+F_2 z}, M_{F_2^*+F_1 z}, M_z)$ is not commuting triple in general, but we shall show that when they commute an interesting thing happens.

Theorem 3.1. *Let $F_1, F_2 \in \mathcal{B}(\mathcal{E})$ be two operators. Then a nonzero closed subspace $\mathcal{M}$ of $H^2_{\mathcal{E}}(\mathbb{D})$ is $(M_{F_1^*+F_2 z}, M_{F_2^*+F_1 z}, M_z)$-invariant if and only if*

$$\begin{aligned} (F_1^* + F_2 z)\Theta(z) &= \Theta(z)(G_1 + G_2^* z), \\ (F_2^* + F_1 z)\Theta(z) &= \Theta(z)(G_2 + G_1^* z), \end{aligned}$$

for all $z \in \mathbb{D}$, for some unique $G_1, G_2 \in \mathcal{B}(\mathcal{E}_*)$, where $(\mathcal{E}_*, \mathcal{E}, \Theta)$ is the Beurling–Lax–Halmos representation of $\mathcal{M}$.

Moreover, if the triple $(M_{F_1^*+F_2 z}, M_{F_2^*+F_1 z}, M_z)$ on $H^2_{\mathcal{E}}(\mathbb{D})$ is a tetrablock isometry, then the triple $(M_{G_1+G_2^* z}, M_{G_2+G_1^* z}, M_z)$ is also a tetrablock isometry on $H^2(\mathcal{E}_*)$.

Proof. So let $\{0\} \neq \mathcal{M} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ be a $(M_{F_1^*+F_2 z}, M_{F_2^*+F_1 z}, M_z)$-invariant subspace. Let $\mathcal{M} = M_\Theta H^2_{\mathcal{E}_*}(\mathbb{D})$ be the Beurling–Lax–Halmos representation of $\mathcal{M}$, where $(\mathcal{E}_*, \mathcal{E}, \Theta)$ is an inner analytic function and $\mathcal{E}_*$ is an auxiliary Hilbert space. Since $\mathcal{M}$ is $M_{F_1^*+F_2 z}$ and $M_{F_2^*+F_1 z}$ invariant also, we have

$$\begin{aligned} M_{F_1^*+F_2 z} M_\Theta H^2_{\mathcal{E}_*}(\mathbb{D}) &\subseteq M_\Theta H^2_{\mathcal{E}_*}(\mathbb{D}), \\ M_{F_2^*+F_1 z} M_\Theta H^2_{\mathcal{E}_*}(\mathbb{D}) &\subseteq M_\Theta H^2_{\mathcal{E}_*}(\mathbb{D}). \end{aligned}$$

Now let us define two operators X and Y on $H^2(\mathcal{E}_*)$ by the following way:

$$M_{F_1^*+F_2z}M_\Theta = M_\Theta X,$$
$$M_{F_2^*+F_1z}M_\Theta = M_\Theta Y.$$

That X and Y are well defined and unique, follows from the fact that Θ is an inner analytic function, hence M_Θ is an isometry, (see [13, ch. V, prop. 2.2].)

$$\begin{aligned} M_{F_1^*+F_2z}M_\Theta = M_\Theta X \quad &\Rightarrow \quad M_\Theta^* M_{F_1^*+F_2z}^* M_\Theta = X^* \text{ [as } M_\Theta \text{ is an isometry]} \\ &\Rightarrow \quad M_z^* M_\Theta^* M_{F_1^*+F_2z}^* M_\Theta = M_z^* X^* \\ &\Rightarrow \quad M_\Theta^* M_{F_1^*+F_2z}^* M_\Theta M_z^* = M_z^* X^* \\ &\Rightarrow \quad X^* M_z^* = M_z^* X^*. \end{aligned}$$

Hence X commutes with M_z. Similarly one can prove that Y commutes with M_z. So $X = M_\Phi$ and $Y = M_\Psi$, for some $\Phi, \Psi \in H^\infty(\mathcal{B}(\mathcal{E}_*))$. Therefore we have

$$M_{F_1^*+F_2z}M_\Theta = M_\Theta M_\Phi, \tag{3.1}$$
$$M_{F_2^*+F_1z}M_\Theta = M_\Theta M_\Psi. \tag{3.2}$$

Multiplying M_Θ^* from left of (3.1) and (3.2) and using the fact that M_Θ is an isometry, we get

$$M_\Theta^* M_{F_1^*+F_2z}M_\Theta = M_\Phi, \tag{3.3}$$
$$M_\Theta^* M_{F_2^*+F_1z}M_\Theta = M_\Psi. \tag{3.4}$$

Multiplying M_z^* from left of (3.3) we get, $M_\Theta^* M_{F_2^*+F_1z}^* M_\Theta = M_z^* M_\Phi$, here we have used the fact that M_Θ and M_z commute. Hence

$$M_\Psi = M_\Theta^* M_\Theta M_\Psi = M_\Theta^* M_{F_2^*+F_1z} M_\Theta = M_\Phi^* M_z.$$

Similarly dealing with Equation (3.4), we get $M_\Phi = M_\Psi^* M_z$. Considering the power series expression of Φ and Ψ and using that $M_\Phi = M_\Psi^* M_z$ and $M_\Psi = M_\Phi^* M_z$, we get Φ and Ψ to be of the form $\Phi(z) = G_1 + G_2^* z$ and $\Psi(z) = G_2 + G_1^* z$ for some $G_1, G_2 \in \mathcal{B}(\mathcal{E}_*)$. Uniqueness of G_1 and G_2 follows from the fact that X and Y are unique. The converse part is trivial. Hence the proof of the first part of the theorem.

Moreover, suppose that $(M_{F_1^*+F_2z}, M_{F_2^*+F_1z}, M_z)$ is a tetrablock isometry. To show that $(M_{G_1+G_2^*z}, M_{G_2+G_1^*z}, M_z)$ is also a tetrablock isometry we first show that they commute with each other. Commutativity of $M_{G_1+G_2^*z}$ and

$M_{G_2+G_1^*z}$ with M_z is clear. Now

$$\begin{aligned}
&M_{G_1+G_2^*z}M_{G_2+G_1^*z}\\
&= M_\Theta^* M_{F_1^*+F_2z}M_\Theta M_\Theta^* M_{F_2^*+F_1z}M_\Theta \text{ [using Equations (3.3) and (3.4)]}\\
&= M_\Theta^* M_{F_1^*+F_2z}M_{F_2^*+F_1z}M_\Theta \text{ [by Equation (3.2)]}\\
&= M_\Theta^* M_{F_2^*+F_1z}M_{F_1^*+F_2z}M_\Theta \text{ [since } M_{F_1^*+F_2z} \text{ and } M_{F_2^*+F_1z} \text{ commute]}\\
&= M_\Theta^* M_{F_2^*+F_1z}M_\Theta M_\Theta^* M_{F_1^*+F_2z}M_\Theta \text{ [by Equation (3.1)]}\\
&= M_{G_2+G_1^*z}M_{G_1+G_2^*z}.
\end{aligned}$$

Since $(M_{F_1^*+F_2z}, M_{F_2^*+F_1z}, M_z)$ is a tetrablock isometry, we have by part (3) of Theorem 5.7 in [7] that $||M_{F_2^*+F_1z}|| \leq 1$, . From the operator equation

$$M_{G_2+G_1^*z} = M_\Theta^* M_{F_2^*+F_1z}M_\Theta$$

we get that $||M_{G_2+G_1^*z}|| \leq 1$. From the proof of the first part, we have that $M_\Phi = M_\Psi^* M_z$ Hence $(M_{G_1+G_2^*z}, M_{G_2+G_1^*z}, M_z)$ is a tetrablock isometry invoking part (3) of Theorem 5.7 in [7]. □

Now we use Lemma 2.8 to prove the following result which is a consequence of Theorem 3.1.

Corollary 3.2. *Let F_1, F_2 and G_1, G_2 be fundamental operators of (A, B, P) and (A^*, B^*, P^*) respectively. Then the triple $(M_{G_1+G_2^*z}, M_{G_2+G_1^*z}, M_z)$ is a tetrablock isometry whenever $(M_{F_1^*+F_2z}, M_{F_2^*+F_1z}, M_z)$ is a tetrablock isometry, provided P^* is pure, i.e., $P^n \to 0$ strongly as $n \to \infty$.*

Proof. Note that while proving the last part of Theorem 3.1, we used the fact that the multiplier M_Θ is an isometry. Since P^* is pure, by virtu of Proposition 3.5 of chapter VI in [13], we note that the multiplier $M_{\Theta_{P^*}}$ is an isometry. From Lemma 2.8, we have

$$\begin{aligned}
(F_1^* + F_2z)\Theta_{P^*}(z) &= \Theta_{P^*}(z)(G_1 + G_2^*z),\\
(F_2^* + F_1z)\Theta_{P^*}(z) &= \Theta_{P^*}(z)(G_2 + G_1^*z),
\end{aligned}$$

for all $z \in \mathbb{D}$. Invoking the last part of Theorem 3.1, we get the result as stated. □

4. Functional model

In this section we find a functional model of pure tetrablock contractions. We first need to recall the functional model of pure contractions from [13].

The characteristic function as in (1.6) induces a multiplication operator M_{Θ_P} from $H^2(\mathbb{D}) \otimes \mathcal{D}_P$ into $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$, defined by

$$M_{\Theta_P} f(z) = \Theta_P(z)f(z), \text{ for all } f \in H^2(\mathbb{D}) \otimes \mathcal{D}_P \text{ and } z \in \mathbb{D}.$$

Note that $M_{\Theta_P}(M_z \otimes I_{\mathcal{D}_P}) = (M_z \otimes I_{\mathcal{D}_{P^*}})M_{\Theta_P}$. Let us define

$$\mathcal{H}_P = (H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}) \ominus M_{\Theta_P}(H^2(\mathbb{D}) \otimes \mathcal{D}_P).$$

In [13], Sz.-Nagy and Foias showed that every pure contraction P defined on an abstract Hilbert space $\mathcal{H}$ is unitarily equivalent to the operator $P_{\mathcal{H}_P}(M_z \otimes I_{\mathcal{D}_{P^*}})_{\mathcal{H}_P}$, where the Hilbert space $\mathcal{H}_P$ is as defined above and $P_{\mathcal{H}_P}$ is the projection of $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ onto $\mathcal{H}_P$. Now we mention an interesting and well-known result, a proof of which can be found in [8, Lemma 3.6]. There it is proved for a commuting contractive d-tuple, for d ≥ 1. We shall write the proof here for the sake of completeness. Define $W : \mathcal{H} \to H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ by

$$W(h) = \sum_{n=0}^{\infty} z^n \otimes D_{P^*}P^{*n}h, \text{ for all } h \in \mathcal{H}.$$

It is easy to check that W is an isometry when P is pure and its adjoint is given by

$$W^*(z^n \otimes \xi) = P^n D_{P^*}\xi, \text{ for all } \xi \in \mathcal{D}_{P^*} \text{ and } n \geq 0.$$

Lemma 4.1. *For every contraction P, the identity*

$$WW^* + M_{\Theta_P}M_{\Theta_P}^* = I_{H^2(\mathbb{D})\otimes\mathcal{D}_{P^*}} \tag{4.1}$$

holds.

Proof. As observed by Arveson in the proof of Theorem 1.2 in [5], the operator W^* satisfies the identity

$$W^*(k_z \otimes \xi) = (I - \bar{z}P)^{-1}D_{P^*}\xi \text{ for } z \in \mathbb{D} \text{ and } \xi \in \mathcal{D}_{P^*},$$

where $k_z(w) := (1 - \langle w, z\rangle)^{-1}$ for all $w \in \mathbb{D}$. Therefore we have

$$\begin{aligned}
&\langle (WW^* + M_{\Theta_P}M_{\Theta_P}^*)(k_z \otimes \xi), (k_w \otimes \eta)\rangle \\
&= \langle W^*(k_z \otimes \xi), W^*(k_w \otimes \eta)\rangle + \langle M_{\Theta_P}^*(k_z \otimes \xi), M_{\Theta_P}^*(k_w \otimes \eta)\rangle \\
&= \langle (I - \bar{z}P)^{-1}D_{P^*}\xi, (I - \bar{w}P)^{-1}D_{P^*}\eta\rangle + \langle k_z \otimes \Theta_P(z)^*\xi, k_w \otimes \Theta_P(w)^*\eta\rangle \\
&= \langle D_{P^*}(I - wP^*)^{-1}(I - \bar{z}P)^{-1}D_{P^*}\xi, \eta\rangle + \langle k_z, k_w\rangle\langle \Theta_P(w)\Theta_P(z)^*\xi, \eta\rangle \\
&= \langle k_z \otimes \xi, k_w \otimes \eta\rangle \text{ for all } z, w \in \mathbb{D} \text{ and } \xi, \eta \in \mathcal{D}_{P^*}.
\end{aligned}$$

Here, the last equality follows from the following well-known identity

$$I - \Theta_P(w)\Theta_P(z)^* = (1 - w\bar{z})D_{P^*}(I - wP^*)^{-1}(I - \bar{z}P)^{-1}D_{P^*}.$$

Now using the fact that $\{k_z : z \in \mathbb{D}\}$ forms a total set of $H^2(\mathbb{D})$, the assertion follows. □

The following theorem is the main result of this section.

Theorem 4.2. *Let (A, B, P) be a pure tetrablock contraction on a Hilbert space $\mathcal{H}$. Then the operators A, B and P are unitarily equivalent to*

$$P_{\mathcal{H}_P}(I \otimes G_1^* + M_z \otimes G_2)|_{\mathcal{H}_P},$$
$$P_{\mathcal{H}_P}(I \otimes G_2^* + M_z \otimes G_1)|_{\mathcal{H}_P},$$
$$P_{\mathcal{H}_P}(M_z \otimes I_{\mathcal{D}_{P^*}})|_{\mathcal{H}_P},$$

respectively, where G_1, G_2 are the fundamental operators of (A^*, B^*, P^*).

Proof. Since W is an isometry, WW^* is the projection onto $\operatorname{Ran} W$ and since P is pure, M_{Θ_P} is also an isometry. So by Lemma 4.1, we have

$$W(\mathcal{H}_P) = (H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}) \ominus M_{\Theta_P}(H^2(\mathbb{D}) \otimes \mathcal{D}_P).$$

For every $\xi \in \mathcal{D}_{P^*}$ and $n \geq 0$, we have

$$\begin{aligned} W^*(I \otimes G_1^* + M_z \otimes G_2)(z^n \otimes \xi) &= W^*(z^n \otimes G_1^*\xi) + W^*(z^{n+1} \otimes G_2\xi) \\ &= P^n D_{P^*} G_1^* \xi + P^{n+1} D_{P^*} G_2 \xi \\ &= P^n (D_{P^*} G_1^* + P D_{P^*} G_2)\xi \\ &= P^n A D_{P^*} \xi \ \ \text{[by Lemma 2.2]} \\ &= A P^n D_{P^*} \xi = A W^*(z^n \otimes \xi). \end{aligned}$$

Therefore we have $W^*(I \otimes G_1^* + M_z \otimes G_2) = AW^*$ on the set

$$\{z^n \otimes \xi : \text{ where } n \geq 0 \text{ and } \xi \in \mathcal{D}_{P^*}\},$$

which spans $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$ and hence we have $W^*(I \otimes G_1^* + M_z \otimes G_2) = AW^*$, which implies $W^*(I \otimes G_1^* + M_z \otimes G_2)W = A$. Therefore A is unitarily equivalent to $P_{\mathcal{H}_P}(I \otimes G_1^* + M_z \otimes G_2)|_{\mathcal{H}_P}$. Also we have for every $\xi \in \mathcal{D}_{P^*}$ and $n \geq 0$,

$$\begin{aligned} W^*(I \otimes G_2^* + M_z \otimes G_1)(z^n \otimes \xi) &= W^*(z^n \otimes G_2^*\xi) + W^*(z^{n+1} \otimes G_1\xi) \\ &= P^n D_{P^*} G_2^* \xi + P^{n+1} D_{P^*} G_1 \xi \\ &= P^n (D_{P^*} G_2^* + P D_{P^*} G_1)\xi \\ &= P^n B D_{P^*} \xi \ \ \text{[by Lemma 2.2]} \\ &= B P^n D_{P^*} \xi = B W^*(z^n \otimes \xi). \end{aligned}$$

Hence by the same argument as above, we have

$$W^*(I \otimes G_2^* + M_z \otimes G_1) = BW^*.$$

Therefore B is unitarily equivalent to $P_{\mathcal{H}_P}(I \otimes G_2^* + M_z \otimes G_1)|_{\mathcal{H}_P}$. And finally,

$$W^*(M_z \otimes I)(z^n \otimes \xi) = W^*(z^{n+1} \otimes \xi) = P^{n+1} D_{P^*} \xi = PW^*(z^n \otimes \xi).$$

Therefore P is unitarily equivalent to $P_{\mathcal{H}_P}(M_z \otimes I_{\mathcal{D}_{P^*}})|_{\mathcal{H}_P}$. Note that the unitary operator which unitarizes A, B and P to their model operators is $W : \mathcal{H} \to \mathcal{H}_P$. □

We end this section with an important result which gives a functional model for a special class of tetrablock contractions, viz., pure tetrablock isometries. This is a consequence of Theorem 4.2. This is important because this gives a relation between the fundamental operators G_1 and G_2 of adjoint of a pure tetrablock isometry.

Corollary 4.3. *Let (A, B, P) be a pure tetrablock isometry. Then (A, B, P) is unitarily equivalent to $(M_{G_1^*+G_2 z}, M_{G_2^*+G_1 z}, M_z)$, where G_1 and G_2 are* *the fundamental operators of* (A^*, B^*, P^*). *Moreover,* G_1 *and* G_2 *satisfy Equation* (1.3).

Proof. Note that for an isometry P, the defect space $\mathcal{D}_P$ is zero, hence the charateristic function Θ_P is also zero. So for an isometry P, the space $\mathcal{H}_P$ becomes $H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}$. So by Theorem 4.2, we have the result. From the commutativity of the triple $(M_{G_1^*+G_2z}, M_{G_2^*+G_1z}, M_z)$, it follows that G_1 and G_2 satisfy Equation (1.3). □

Remark 4.4. In [7] (Theorem 5.10), it was shown that every pure tetrablock isometry (A, B, P) on $\mathcal{H}$ is unitarily equivalent to $(M_{\tau_1^*+\tau_2 z}, M_{\tau_2^*+\tau_1 z}, M_z)$ on $H^2_{\mathcal{E}}(\mathbb{D})$ for some τ_1, τ_2 in $\mathcal{B}(\mathcal{E})$. Corollary 4.3 shows that the space $\mathcal{E}$ can be taken to be $\mathcal{D}_{P^*}$ and the operators τ_1, τ_2 can be taken to be the fundamental operators of (A^*, B^*, P^*).

5. A complete set of unitary invariants

Given two contractions P and P' on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively, we say that the characteristic functions of P and P' coincide if there are unitary operators $u : \mathcal{D}_P \to \mathcal{D}_{P'}$ and $u_* : \mathcal{D}_{P^*} \to \mathcal{D}_{P'^*}$ such that the following diagram commutes for all $z \in \mathbb{D}$,

$$\begin{array}{ccc} \mathcal{D}_P & \xrightarrow{\Theta_P(z)} & \mathcal{D}_{P^*} \\ u\big\downarrow & & \big\downarrow u_* \\ \mathcal{D}_{P'} & \xrightarrow[\Theta_{P'}(z)]{} & \mathcal{D}_{P'^*}. \end{array}$$

In [13], Sz.-Nagy and Foias proved that the characteristic function of a c.n.u. contraction is a complete unitary invariant. In other words,

Theorem 5.1. *Two completely nonunitary contractions are unitarily equivalent if and only if their characteristic functions coincide.*

In this section we give a complete set of unitary invariants for a pure tetrablock contraction.

Proposition 5.2. *If two tetrablock contractions* (A, B, P) *and* (A', B', P') *defined on* $\mathcal{H}$ *and* $\mathcal{H}'$ *respectively are unitarily equivalent then so are their fundamental operators.*

Proof. Let $U : \mathcal{H} \to \mathcal{H}'$ be a unitary such that $UA = A'U, UB = B'U$ and $UP = P'U$. Then we have

$$UD_P^2 = U(I - P^*P) = U - P'^*PU = D_{P'}^2 U,$$

which gives $UD_P = D_{P'}U$. Let $\tilde{U} = U|_{\mathcal{D}_P}$. Then note that $\tilde{U} \in \mathcal{B}(\mathcal{D}_P, \mathcal{D}_{P'})$ and $\tilde{U}D_P = D_{P'}\tilde{U}$. Let F_1, F_2 and F_1', F_2' be fundamental operators of

(A,B,P) and (A',B',P') respectively. Then

$$\begin{aligned} D_{P'}\tilde{U}F_1\tilde{U}^*D_{P'} = \tilde{U}D_PF_1D_P\tilde{U}^* &= \tilde{U}(A-B^*P)D_P\tilde{U}^* \\ &= A'-B'^*P' = D_{P'}F_1'D_{P'}. \end{aligned}$$

Therefore we have $\tilde{U}F_1\tilde{U}^* = F_1'$. Similarly one can prove $\tilde{U}F_2\tilde{U}^* = F_2'$. □

The next result is a sort of converse to the previous proposition for pure tetrablock contractions.

Proposition 5.3. *Let (A,B,P) and (A',B',P') be two pure tetrablock contractions defined on $\mathcal{H}$ and $\mathcal{H}'$ respectively. Suppose that the characteristic functions of P and P' coincide and the fundamental operators (G_1,G_2) of (A^*,B^*,P^*) and (G_1',G_2') of (A'^*,B'^*,P'^*) are unitarily equivalent by the same unitary that is involved in the coincidence of the characteristic functions of P and P'. Then (A,B,P) and (A',B',P') are unitarily equivalent.*

Proof. Let $u:\mathcal{D}_P\to\mathcal{D}_{P'}$ and $u_*:\mathcal{D}_{P^*}\to\mathcal{D}_{P'^*}$ be unitary operators such that

$$u_*G_1 = G_1'u_*,\ u_*G_2 = G_2'u_* \text{ and } u_*\Theta_P(z) = \Theta_{P'}(z)u$$

hold for all $z\in\mathbb{D}$. The unitary operator $u_*:\mathcal{D}_{P^*}\to\mathcal{D}_{P'^*}$ induces another unitary operator $U_*:H^2(\mathbb{D})\otimes\mathcal{D}_{P^*}\to H^2(\mathbb{D})\otimes\mathcal{D}_{P'^*}$ defined by

$$U_*(z^n\otimes\xi) = (z^n\otimes u_*\xi)$$

for all $\xi\in\mathcal{D}_{P^*}$ and $n\geq 0$. Note that

$$U_*(M_{\Theta_P}f(z)) = u_*\Theta_P(z)f(z) = \Theta_{P'}(z)uf(z) = M_{\Theta_{P'}}(uf(z)),$$

for all $f\in H^2(\mathbb{D})\otimes\mathcal{D}_P$ and $z\in\mathbb{D}$. Hence U_* takes $\operatorname{Ran} M_{\Theta_P}$ onto $\operatorname{Ran} M_{\Theta_{P'}}$. Since U_* is unitary, we have

$$U_*(\mathcal{H}_P) = U_*((\operatorname{Ran} M_{\Theta_P})^\perp) = (U_*\operatorname{Ran} M_{\Theta_P})^\perp = (\operatorname{Ran} M_{\Theta_{P'}})^\perp = \mathcal{H}_{P'}.$$

By definition of U_* we have

$$\begin{aligned} U_*(I\otimes G_1^* + M_z\otimes G_2)^* &= (I\otimes u_*)(I\otimes G_1 + M_z^*\otimes G_2^*) \\ &= I\otimes u_*G_1 + M_z^*\otimes u_*G_2^* \\ &= I\otimes G_1'u_* + M_z^*\otimes G_2'^*u_* \\ &= (I\otimes G_1' + M_z^*\otimes G_2'^*)(I\otimes u_*) \\ &= (I\otimes G_1'^* + M_z\otimes G_2')^*U_*. \end{aligned}$$

Similar calculation gives us

$$U_*(I\otimes G_2^* + M_z\otimes G_1)^* = (I\otimes G_2'^* + M_z\otimes G_1')^*U_*.$$

Therefore $\mathcal{H}_{P'} = U_*(\mathcal{H}_P)$ is a co-invariant subspace of $(I\otimes G_1'^* + M_z\otimes G_2')$ and $(I\otimes G_2'^* + M_z\otimes G_1')$. Hence

$$P_{\mathcal{H}_P}(I\otimes G_1^* + M_z\otimes G_2)|_{\mathcal{H}_P} \cong P_{\mathcal{H}_{P'}}(I\otimes G_1'^* + M_z\otimes G_2')|_{\mathcal{H}_{P'}}$$

and

$$P_{\mathcal{H}_P}(I\otimes G_2^*+M_z\otimes G_1)|_{\mathcal{H}_P}\cong P_{\mathcal{H}_{P'}}(I\otimes G_2'^*+M_z\otimes G_1')|_{\mathcal{H}_{P'}}$$

and the unitary operator which unitarizes them is $U_*|_{\mathcal{H}_P}:\mathcal{H}_P\to\mathcal{H}_{P'}$.

And also by definition of U_* we have

$$\begin{aligned}U_*(M_z\otimes I_{\mathcal{D}_{P^*}})=(I\otimes u_*)(M_z\otimes I_{\mathcal{D}_{P^*}})&=(M_z\otimes I_{\mathcal{D}_{P'^*}})(I\otimes u_*)\\&=(M_z\otimes I_{\mathcal{D}_{P'^*}})U_*.\end{aligned}$$

So $P_{\mathcal{H}_P}(M_z\otimes I_{\mathcal{D}_{P^*}})|_{\mathcal{H}_P}\cong P_{\mathcal{H}_{P'}}(M_z\otimes I_{\mathcal{D}_{P'^*}})|_{\mathcal{H}_{P'}}$ and the same unitary $U_*|_{\mathcal{H}_P}:\mathcal{H}_P\to\mathcal{H}_{P'}$ unitarizes them. Therefore $(A,B,P)\cong(A',B',P')$. □

Combining the last two propositions and Theorem 5.1, we get the following theorem which is the main result of this section.

Theorem 5.4. *Let (A,B,P) and (A',B',P') be two pure tetrablock contractions defined on $\mathcal{H}$ and $\mathcal{H}'$ respectively. Suppose (G_1,G_2) and (G_1',G_2') are fundamental operators of (A^*,B^*,P^*) and (A'^*,B'^*,P'^*) respectively. Then (A,B,P) is unitarily equivalent to (A',B',P') if and only if the characteristic functions of P and P' coincide and (G_1,G_2) is unitarily equivalent to (G_1',G_2') by the same unitary that is involved in the coincidence of the characteristic functions of P and P'.*

6. An example

6.1. Fundamental operators. Consider the Hilbert space

$$H^2(\mathbb{D}^2)=\left\{f:\mathbb{D}^2\to\mathbb{C}:f(z_1,z_2)=\sum_{i,j=0}^{\infty}a_{ij}z_1^iz_2^j\text{ with }\sum_{i,j=0}^{\infty}|a_{ij}|^2<\infty\right\}$$

with the inner product

$$\left\langle\sum_{i,j=0}^{\infty}a_{ij}z_1^iz_2^j,\sum_{i,j=0}^{\infty}b_{ij}z_1^iz_2^j\right\rangle=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}\overline{b_{ij}}.$$

Consider the commuting triple of operators $(M_{z_1},M_{z_2},M_{z_1z_2})$ on $H^2(\mathbb{D}^2)$. It can be easily checked by Theorem 5.4 in [7], that $(M_{z_1},M_{z_2},M_{z_1z_2})$ is a tetrablock unitary on $L^2(\mathbb{T}^2)$. Note that $(M_{z_1},M_{z_2},M_{z_1z_2})$ on $H^2(\mathbb{D}^2)$ is the restriction of the tetrablock unitary $(M_{z_1},M_{z_2},M_{z_1z_2})$ on $L^2(\mathbb{T}^2)$ to the common invariant subspace $H^2(\mathbb{D}^2)$ (naturally embedded) of $L^2(\mathbb{T}^2)$. Hence by definition, $(M_{z_1},M_{z_2},M_{z_1z_2})$ on $H^2(\mathbb{D}^2)$ is a tetrablock isometry. In this section we calculate the fundamental operators of the tetrablock co-isometry $(M_{z_1}^*,M_{z_2}^*,M_{z_1z_2}^*)$ on $H^2(\mathbb{D}^2)$. For notational convenience, we denote M_{z_1},M_{z_2} and $M_{z_1z_2}$ by A,B and P respectively.

Note that every element $f \in H^2(\mathbb{D}^2)$ has the form $\sum_{i,j=0}^{\infty} a_{ij} z_1^i z_2^j$ where $a_{ij} \in \mathbb{C}$, for all $i, j \geq 0$. So we can write f in the matrix form

$$((a_{ij}))_{i,j=0}^{\infty} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where (ij)-th entry in the matrix, denotes the co-efficient of $z_1^i z_2^j$ in the series $\sum_{i,j=0}^{\infty} a_{ij} z_1^i z_2^j$. We shall write the matrix form instead of writing the series. In this notation,

$$A\left(((a_{ij}))_{i,j=0}^{\infty}\right) = \left(a_{(i-1)j}\right), \tag{6.1}$$

$$B\left(((a_{ij}))_{i,j=0}^{\infty}\right) = \left(a_{i(j-1)}\right), \tag{6.2}$$

$$P\left(((a_{ij}))_{i,j=0}^{\infty}\right) = \left(a_{(i-1)(j-1)}\right), \tag{6.3}$$

with the convention that a_{ij} is zero if either i or j is negative.

Lemma 6.1. *The adjoints of the operators A, B and P are as follows:*

$$A^*\left(((a_{ij}))_{i,j=0}^{\infty}\right) = \left(a_{(i+1)j}\right),$$
$$B^*\left(((a_{ij}))_{i,j=0}^{\infty}\right) = \left(a_{i(j+1)}\right),$$
$$P^*\left(((a_{ij}))_{i,j=0}^{\infty}\right) = \left(a_{(i+1)(j+1)}\right).$$

Proof. This is a matter of easy inner product calculations. □

Lemma 6.2. *The defect space of P^* in the matrix form is*

$$\mathcal{D}_{P^*} = \left\{ \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : |a_{00}|^2 + \sum_{j=1}^{\infty} |a_{0j}|^2 + \sum_{j=1}^{\infty} |a_{j0}|^2 < \infty \right\}.$$

The defect space in the function form is $\overline{\text{span}}\{1, z_1^i, z_2^j : i, j \geq 1\}$. The defect operator for P^* is

$$D_{P^*} \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. Since P is an isometry, D_{P^*} is the projection onto the orthogonal complement of $\text{Ran}(P)$. The rest follows from the formula for P in (6.3). □

Definition 6.3. Define $G_1, G_2 : \mathcal{D}_{P^*} \to \mathcal{D}_{P^*}$ by

$$G_1 \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ a_{30} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
$$G_2 \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_{01} & a_{02} & a_{03} & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

for all $a_{j0}, a_{0j} \in \mathbb{C}, j = 0, 1, 2, \dots$ with $\sum_{j=0}^{\infty} |a_{0j}|^2 + \sum_{j=1}^{\infty} |a_{j0}|^2 < \infty$.

Lemma 6.4. *The operators G_1 and G_2 defined in Definition 6.3 are the fundamental operators of (A^*, B^*, P^*).*

Proof. We must show that G_1 and G_2 satisfy the fundamental equations. Using Lemma 6.1, we get

$$(A^* - BP^*) \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$= \begin{pmatrix} a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ a_{30} & a_{31} & a_{32} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} - B \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$= \begin{pmatrix} a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ a_{30} & a_{31} & a_{32} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} 0 & a_{11} & a_{12} & \dots \\ 0 & a_{21} & a_{22} & \dots \\ 0 & a_{31} & a_{32} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$= \begin{pmatrix} a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ a_{30} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$
\begin{aligned}
&(B^* - AP^*)\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} a_{01} & a_{02} & a_{03} & \dots \\ a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} - A\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} a_{01} & a_{02} & a_{03} & \dots \\ a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \dots \\ a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} a_{01} & a_{02} & a_{03} & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\end{aligned}
$$

Using Lemma 6.2 and Definition 6.3, we get

$$
\begin{aligned}
D_{P^*}G_1D_{P^*}\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} &= D_{P^*}G_1\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ a_{30} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \\
D_{P^*}G_2D_{P^*}\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} &= D_{P^*}G_2\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} a_{01} & a_{02} & a_{03} & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\end{aligned}
$$

Therefore, G_1 and G_2 are the fundamental operators of (A^*, B^*, P^*). $\square$

6.2. Explicit unitary equivalence. From Corollary 4.3, we now know that if (A, B, P) is a pure tetrablock isometry, then (A, B, P) is unitarily equivalent to $(M_{G_1^*+G_2z}, M_{G_2^*+G_1z}, M_z)$, where G_1 and G_2 are the fundamental operators of (A^*, B^*, P^*). The operator $M_{z_1z_2}$ on $H^2(\mathbb{D}^2)$ is a pure contraction as can be checked from the formula of P^* in Lemma 6.1. In the final theorem of this section, we find the unitary operator which implements the unitary equivalence of the pure tetrablock isometry (A, B, P) on $H^2(\mathbb{D}^2)$.

Theorem 6.5. *The operator* $U : H^2(\mathbb{D}^2) \to H^2_{\mathcal{D}_{P^*}}(\mathbb{D})$ *defined by*

$$Uf(z) = D_{P^*}(I - zP^*)^{-1}f, \text{ for all } f \in H^2(\mathbb{D}^2) \text{ and } z \in \mathbb{D} \tag{6.4}$$

is a unitary operator and satisfies

$$U^*(M_{G_1^*+G_2z}, M_{G_2^*+G_1z}, M_z)U = (A, B, P).$$

Proof. We first prove that U is one-one. Expanding the series in (6.4), we get

$$Uf(z) = D_{P^*}f + zD_{P^*}P^*f + z^2D_{P^*}{P^*}^2f + \cdots . \tag{6.5}$$

Therefore

$$\begin{aligned}\|Uf\|^2_{H^2_{\mathcal{D}_{P^*}}(\mathbb{D})} &= \|D_{P^*}f\|^2_{\mathcal{D}_{P^*}} + \|D_{P^*}P^*f\|^2_{\mathcal{D}_{P^*}} + \|D_{P^*}{P^*}^2f\|^2_{\mathcal{D}_{P^*}} + \cdots \\ &= \|f\|^2 - \lim_{n\to\infty}\|P^{*n}f\|^2 = \|f\|^2_{H^2(\mathbb{D}^2)}. \text{ [since } P \text{ is pure]}\end{aligned}$$

From the explicit series form of U (Equation 6.5), we see that U in matrix form is the following.

$$\begin{aligned}U\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} &= \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ &\quad + z\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & 0 & 0 & \dots \\ a_{31} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ &\quad + z^2\begin{pmatrix} a_{22} & a_{23} & a_{24} & \dots \\ a_{32} & 0 & 0 & \dots \\ a_{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots .\end{aligned}$$

From this representation, it is easy to see that U is onto $H^2_{\mathcal{D}_{P^*}}(\mathbb{D})$. It can be easily checked by definition of G_1 and G_2 (Definition 6.3) that

$$G_1^* \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ a_{00} & 0 & 0 & 0 & \dots \\ a_{10} & 0 & 0 & 0 & \dots \\ a_{20} & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$G_2^* \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & a_{00} & a_{01} & a_{02} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To prove $U^*(M_{G_1^*+G_2z}, M_{G_2^*+G_1z}, M_z)U = (A, B, P)$, we proceed by proving $U^*M_zU = P$ first. Note that

$$M_zU \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = z \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & 0 & 0 & \dots \\ a_{20} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z^2 \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & 0 & 0 & \dots \\ a_{31} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z^3 \begin{pmatrix} a_{22} & a_{23} & a_{24} & \dots \\ a_{32} & 0 & 0 & \dots \\ a_{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots .$$

Therefore

$$U^*M_zU \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & a_{00} & a_{01} & a_{02} & \dots \\ 0 & a_{10} & a_{11} & a_{12} & \dots \\ 0 & a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = P \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now to prove $M_{z_1} = U^*M_{G_1^*+zG_2}U$, we first calculate $M_{G_1^*+zG_2}U$. Using the definition of G_1, G_2 and U, it is a matter of straightforward computation to obtain

$$M_{G_1^*+zG_2}U\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ a_{00} & 0 & 0 & \dots \\ a_{10} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z\begin{pmatrix} a_{01} & a_{02} & a_{03} & \dots \\ a_{11} & 0 & 0 & \dots \\ a_{21} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + z^2\begin{pmatrix} a_{12} & a_{13} & a_{14} & \dots \\ a_{22} & 0 & 0 & \dots \\ a_{32} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \cdots .$$

Therefore

$$U^*M_{G_1^*+zG_2}U\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ a_{00} & a_{01} & a_{02} & a_{03} & \dots \\ a_{10} & a_{11} & a_{12} & a_{13} & \dots \\ a_{20} & a_{21} & a_{22} & a_{23} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = M_{z_1}\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The proof of $M_{z_2} = U^*M_{G_2^*+zG_1}U$ is similar. □

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(Haripada Sau) Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
sau10@math.iisc.ernet.in

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