

Doubly slice knots with low crossing number

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ABSTRACT. A knot in S^3 is doubly slice if it is the cross-section of an unknotted two-sphere in S^4 . For low-crossing knots, the most complete work to date gives a classification of doubly slice knots through 9 crossings. We extend that work through 12 crossings, resolving all but four cases among the 2,977 prime knots in that range. The techniques involved in this analysis include considerations of the Alexander module and signature functions as well as calculations of the twisted Alexander polynomials for higher-order branched covers. We give explicit illustrations of the double slicing for each of the 20 knots shown to be smoothly doubly slice. We place the study of doubly slice knots in a larger context by introducing the *double slice genus* of a knot.

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1. Introduction

In 1962, Fox included the following question in his list of problems in knot theory [8].

Question 1.1. *Which slice knots and weakly slice links can appear as the cross-sections of the unknotted S^2 in S^4 .*

Such a knot is called *doubly slice*. Many of the techniques that have been successful in the study of slice knots and knot concordance over the last 50 years have applications to the study of doubly slice knots and double null concordance of knots. Nevertheless, doubly slice knots remain far less understood than their slice counterparts.

The goal of this note is to address Fox's question for prime knots with 12 or fewer crossings. A precedent for this work was set in 1971 when Sumners showed that for knots with nine or fewer crossings, there is only one prime doubly slice knot, namely, the knot 9_{46} [31].

There are 158 known prime slice knots with 12 or fewer crossings, and it is unknown whether the knot 11_{n34} is slice. Of these 159 knots, we show that at least 20, but no more than 24, are smoothly doubly slice.

Theorem 1.2. *The following knots are smoothly doubly slice.*

$$\begin{array}{ccccccc} 9_{46} & 10_{99} & 10_{123} & 10_{155} & 11_{n42} & 11_{n49} & 11_{n74} \\ 12_{a0427} & 12_{a1105} & 12_{n0268} & 12_{n0309} & 12_{n0313} & 12_{n0397} & 12_{n0414} \\ 12_{n0430} & 12_{n0605} & 12_{n0636} & 12_{n0706} & 12_{n0817} & 12_{n0838}. \end{array}$$

Furthermore, with the possible exception of the following four knots, no other prime knots of 12 or fewer crossings are smoothly doubly slice:

$$11_{n34} \quad 11_{n73} \quad 12_{a1019} \quad 12_{a1202}.$$

Our contributions to this computation include the following:

- the first application of twisted Alexander polynomials to obstruct double sliceness,
- the first low-crossing examples of slice knots with nonvanishing signature function,
- explicit constructions of unknotted embeddings of S^2 into S^4 with equatorial cross-section isotopic to each of the 20 knots on the list.

The knot 9_{46} was first smoothly double sliced by Terasaka and Hosokawa [32], with a later construction given by Sumners [31], while 11_{n42} was shown to be doubly slice in [4]. The double slicing of 10_{123} included below was shown to the second author by Donald, who has contributed to the study of double slice knots by studying the problem of embedding 3-manifolds into S^4 [7], where he gives a double slicing of 11_{n74} .

We show below that the Conway knot 11_{n34} is topologically (locally flat) doubly slice (see Section 3), but it is unknown whether it can be smoothly sliced or double sliced.

1.1. A brief history of doubly slice knots.

The study of slice knots is naturally placed in the context of the concordance group \mathcal{C} and the homomorphism $\phi: \mathcal{C} \rightarrow \mathcal{G}$, where \mathcal{G} is the algebraic concordance group, defined and classified by Levine [21, 22]. There are analogous groups \mathcal{C}_{ds} and \mathcal{G}_{ds} defined in the context of doubly slice knots; however, Levine's classification of \mathcal{G} does not carry over to \mathcal{G}_{ds} , and there are other complications that make \mathcal{C}_{ds} and \mathcal{G}_{ds} difficult to study.

It is known that the kernel of the canonical map $\mathcal{G}_{ds} \rightarrow \mathcal{G}$ is infinitely generated [6], but beyond that, the structure of \mathcal{G}_{ds} remains a mystery. (See, however, [2, 29, 30]. Recently, Orson has made significant progress in further understanding \mathcal{G}_{ds} , [26, 27].) Furthermore, it can be shown using Casson–Gordon invariants that there are algebraically doubly slice knots that are not topologically doubly slice [12]. Friedl developed further metabelian invariants that can be used to obstruct double sliceness [11].

As in the study of slice knots, there is an important distinction between the smooth and topologically locally flat categories. However, this distinction does not feature prominently in our work here; we find no low-crossing examples of knots that are topologically doubly slice but not smoothly doubly slice, even though such knots have been shown to exist [25]. Other interesting constructions in the study of doubly slice knots include the fibered examples of Aitchison and Silver [1] and the extension of the Cochran–Teichner–Orr filtration to topologically doubly slice knots by Kim [18].

1.2. Organization.

In Sections 2 and 3, we discuss obstructions to double slicing knots coming from the algebraic and topological categories, respectively. In Section 4, we discuss some techniques that can be used to construct double slicings of knots in either the topological or smooth categories. In Section 5, we place the study of doubly slice knots in context by considering knots as cross-sections of unknotted surfaces in S^4 .

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2. Algebraic obstructions to double slicing knots

In this section, we will present three algebraic obstructions to double slicing a knot. These are applied to obtain an initial list of prime knots with at most twelve crossings that could potentially be doubly slice.

2.1. Hyperbolic torsion coefficients.

A knot K in S^3 is said to be *algebraically doubly slice* if there exists a Seifert matrix A_K for K that has the form

$$A_K = \begin{bmatrix} 0 & B_1 \\ B_2 & 0 \end{bmatrix},$$

where B_1 and B_2 are square matrices of equal dimension. Matrices of this form are called *hyperbolic* and have been studied by Levine [23] and others [6, 30]. If K is (smoothly or topologically) doubly slice, then K is algebraically double slice [31].

Let A_K be a hyperbolic Seifert matrix for K . Then,

$$A_K + A_K^T = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where $B = B_1 + B_2^T$. The matrix $B \oplus B$ is a presentation matrix for $H_1(\Sigma_2(K))$. It follows that $H_1(\Sigma_2(K))$ splits as a direct sum $G \oplus G$, where G is presented by the matrix B . Thus, we have our first obstruction.

Proposition 2.1. *Let K be a knot in S^3 . If K is algebraically doubly slice, then, for some finite group G , $H_1(\Sigma_2(K)) = G \oplus G$.*

Of the 2,977 prime knots with at most 12 crossings, 62 knots satisfy Proposition 2.1. Furthermore, if K is algebraically doubly slice, then K is algebraically slice. Among these 62 knots, there are 36 that are algebraically slice. These knots form our short-list of candidates to be algebraically doubly slice and are shown below:

9 ₄₁	9 ₄₆	10 ₉₉	10 ₁₂₃	10 ₁₅₃	10 ₁₅₅	11 _{n34}	11 _{n42}
11 _{n49}	11 _{n73}	11 _{n74}	11 _{n116}	12 _{a0427}	12 _{a1019}	12 _{a1105}	12 _{a1202}
12 _{n0019}	12 _{n0210}	12 _{n0214}	12 _{n0257}	12 _{n0268}	12 _{n0309}	12 _{n0313}	12 _{n0318}
12 _{n0397}	12 _{n0414}	12 _{n0430}	12 _{n0440}	12 _{n0582}	12 _{n0605}	12 _{n0636}	12 _{n0706}
12 _{n0813}	12 _{n0817}	12 _{n0838}	12 _{n0876} .				

2.2. The signature function.

Let K be a knot in S^3 with Seifert matrix A_K . Let ω be a unit complex number, and consider the matrix

$$(1 - \omega)A_K + (1 - \bar{\omega})A_K^T.$$

Denote by $\sigma_\omega(K)$ the signature of this matrix. Note that this matrix will be nonsingular provided that $\Delta_K(\omega) \neq 0$, where $\Delta_K(t)$ is the Alexander polynomial of K . In any event, $\sigma_K(\omega)$ is a well-defined knot invariant for any unit complex number ω . See [13] for details. It is well-known that

$$|\sigma_K(\omega)| \leq 2g_4(K)$$

whenever $\Delta_K(\sigma) \neq 0$. Thus, if K is algebraically slice, then $\sigma_\omega(K) = 0$ away from the roots of the Alexander polynomial. Moreover, we have the following result from [23].

Proposition 2.2. *Let K be a knot in S^3 . If K is algebraically doubly slice, then $\sigma_\omega(K) = 0$ for any unit complex number ω .*

In fact, we can consider these signature invariants as a function

$$\sigma(K) : S^1 \rightarrow \mathbb{Z},$$

defined by $\sigma(K)(\omega) = \sigma_\omega(K)$, called the *signature function*. If a knot K satisfies Proposition 2.2, we say that the signature function for K *vanishes*.

Example 2.3. *Let $K = 12_{n0582}$. Then, $\Delta_K(t) = (t^2 - t + 1)^2$, and the roots of $\Delta_K(t)$ are contained on the unit circle. Since K is slice, we know that $\sigma_K(\omega) = 0$ away from these roots. However, if we consider the roots, ζ and $\bar{\zeta}$, where ζ is a sixth root of unity, we can compute that*

$$\sigma_\zeta(K) = \sigma_{\bar{\zeta}}(K) = -1.$$

(Note that this calculation depends on a Seifert matrix A_K , but any choice will do and we do not include the details here.) It follows from Proposition 2.2 that K cannot be algebraically doubly slice.

Example 2.4. *Let $K = 12_{n0813}$. Then,*

$$\Delta_K(t) = (t - 2)(2t - 1)(t^2 - t + 1)^2.$$

Two of the roots of $\Delta_K(t)$ are primitive sixth roots of unity; the other two roots do not lie on the unit circle, so no information can be gained by considering them. If we consider the roots of unity, we find that

$$\sigma_\zeta(K) = \sigma_{\bar{\zeta}}(K) = +1.$$

(Again, we have used some matrix A_K for this calculation.) It follows from Proposition 2.2 that K cannot be algebraically doubly slice.

Thus, we remove 12_{n0582} and 12_{n0813} from our list of potentially algebraically doubly slice knots.

2.3. The Alexander module.

Continuing, let K be a knot in S^3 and let $X_\infty(K)$ denote the infinite cyclic cover of $S^3 \setminus K$. The group $H_1(X_\infty(K))$ can be regarded as a Λ -module, where $\Lambda = \mathbb{Z}[t, t^{-1}]$. This Λ -module is called the *Alexander module* and is presented by the matrix $V_K = A_K - tA_K^T$.

Sumners obstructed 9_{41} from being doubly slice by carefully analyzing the module structure of $H_1(X_\infty(K))$. We follow a similar approach to analyze two more knots.

We begin by switching to coefficients in the finite field with p elements, \mathbb{Z}_p . In this case, $H_1(X_\infty(K), \mathbb{Z}_p)$ is a module over a PID, $\Lambda_p = \mathbb{Z}_p[t, t^{-1}]$. With this, we have the following result.

Proposition 2.5. *K is doubly slice, then as a Λ_p -module,*

$$H_1(X_\infty(K), \mathbb{Z}_p) \cong \bigoplus_i (\Lambda_p / \langle f_i(t) \rangle \oplus \Lambda_p / \langle f_i(t^{-1}) \rangle)$$

for some set of polynomials $f_i(t) \in \Lambda_p$.

Example 2.6. Let $K = 11_{n116}$, which has $\Delta_K(t) = (1+t-t^2)(-1+t+t^2)$. Using the Seifert form V_K taken from *KnotInfo* [5] and working with \mathbb{Z}_2 -coefficients, we find that as a Λ_2 -module,

$$H_1(X_\infty(K), \mathbb{Z}_2) \cong \Lambda_2 / \langle (1+t+t^2)^2 \rangle.$$

This does not decompose as a nontrivial direct sum of modules, so

$$K = 11_{n116}$$

cannot be doubly slice.

Example 2.7. Let $K = 12_{n0876}$, which has

$$\Delta_K(t) = (-2 + 4t - 2t^2 + t^3)(-1 + 2t - 4t^2 + 2t^3).$$

Again using the Seifert form V_K taken from *KnotInfo*, but now working with \mathbb{Z}_3 -coefficients, we compute that as a Λ_3 -module,

$$H_1(X_\infty(K), \mathbb{Z}_3) \cong \Lambda_3 / \langle (1+t)^2 \rangle \oplus \Lambda_3 / \langle (1+t^2)^2 \rangle.$$

This does not decompose further, so 12_{n0876} cannot be doubly slice.

2.4. Algebraic conclusions.

In conclusion, consideration of the torsion invariants reduced our search for doubly slice knots to a set of 36 knots. An analysis of the signature function removed another two, and an examination of Alexander modules eliminate three more, including the one found by Sumners. Of the remaining 31 knots, we will use the techniques described in Section 4 to show that one is topologically doubly slice and 20 are smoothly doubly slice. It follows that these 21 knots are algebraically doubly slice, leaving us with only 10 knots that may or may not be algebraically doubly slice.

Question 2.8. Are any of the following knots algebraically doubly slice?

$$\begin{array}{ccccc} 10_{153} & 11_{n73} & 12_{a1019} & 12_{a1202} & 12_{n0019} \\ 12_{n0210} & 12_{n0214} & 12_{n0257} & 12_{n0318} & 12_{n0440} \end{array}$$

3. Topological obstructions to double slicing knots

We now move from abelian to metabelian invariants. We begin by quickly recalling the twisted polynomial. Let $M_q(K)$ be the q -fold cyclic cover of $S^3 \setminus K$, let $\Sigma_q(K)$ be the branched cyclic cover, and let $\rho: H_1(\Sigma_q(K)) \rightarrow \mathbb{Z}_p$ be a homomorphism, where q is a prime power and p is an odd prime. Let $\Gamma_p = \mathbb{Q}(\zeta_p)[t, t^{-1}]$, where ζ_p is a primitive p^{th} -root of unity. As described in [20], there is an associated *twisted Alexander polynomial* $\Delta_{K,\rho}(t) \in \Gamma_p$. This polynomial is well-defined up to multiplication by a unit in Γ_p . Given $f(t) \in \Gamma_p$, let $\overline{f(t)}$ denote the result of complex conjugation of the coefficients of $f(t)$.

A result of [20] states the following.

Theorem 3.1. *If K is topologically slice, then there is a subgroup*

$$H \subset H_1(\Sigma_q(K))$$

satisfying the following properties.

- (1) $|H|^2 = |H_1(\Sigma_q(K))|$.
- (2) *The subgroup H is invariant under the action of the deck transformation of $\Sigma_q(K)$.*
- (3) *For all $\rho: H_1(\Sigma_q(K)) \rightarrow \mathbb{Z}_p$ satisfying $\rho(H) = 0$, one has*

$$\Delta_{K,\rho}(t) = af(t)\overline{f(t^{-1})}$$

for some unit $a \in \Gamma_p$.

If K is doubly slice, then it satisfies strengthened conditions.

Theorem 3.2. *If K is topologically doubly slice, then there exists a decomposition*

$$H_1(\Sigma_q(K)) \cong H_1 \oplus H_2$$

satisfying the following properties.

- (1) $H_1 \cong H_2$.
- (2) *The subgroups H_1 and H_2 are invariant under the action of the deck transformation of $\Sigma_q(K)$.*
- (3) *For all $\rho: H_1(\Sigma_q(K)) \rightarrow \mathbb{Z}_p$ for which $\rho(H_1) = 0$ or $\rho(H_2) = 0$, one has that*

$$\Delta_{K,\rho}(t) = af(t)\overline{f(t^{-1})}$$

for some unit $a \in \Gamma_p$.

Proof. The proof is very similar to that of Theorem 3.1 in [20], so we just summarize it here.

In Theorem 3.1, the subgroup H can be taken as the kernel of the inclusion $\Sigma_q(K) \rightarrow \overline{W}_q(D)$, where $\overline{W}_q(D)$ is the q -fold branched cover of B^4 over a slice disk D of K . In the case that K is doubly slice, the q -fold branched cover $\Sigma_q(K)$ embeds in S^4 , since S^4 is the the q -fold branched cover of S^4 over the (unknotted) double slicing 2-sphere for K . It follows that $\Sigma_q(K)$ splits S^4 into manifolds Y_1 and Y_2 .

The subgroups H_1 and H_2 can be taken as the kernels of the two inclusions $H_1(\Sigma_q(K)) \rightarrow Y_1$ and $H_1(\Sigma_q(K)) \rightarrow Y_2$. The direct sum decomposition arises from the Meyer-Vietoris Theorem; the fact that $H_1 \cong H_2$ follows from duality, as first noticed by Hantzsche [16].

The rest of the argument follows identically to that in [20]. □

Equipped with Theorem 3.2, we are ready to prove our second result.

Theorem 3.3. *The following knots are not topologically doubly slice:*

$$\begin{array}{cccc} 10_{153} & 12_{n0019} & 12_{n0210} & 12_{n0214} \\ 12_{n0257} & 12_{n0318} & 12_{n0440} & \end{array}$$

Proof. The proof is nearly identical in each case, so we describe only one case in detail.

Let $K = 10_{153}$. Then $H_1(\Sigma_3(K)) \cong (\mathbb{Z}_7)^2$, and the action of the deck transformation splits the homology as $E_2 \oplus E_4$. Here E_2 is the 2-eigenspace of the action of the deck transformation on $H_1(\Sigma_3(K))$ and E_4 is the 4-eigenspace. Notice that $2^3 = 4^3 = 1 \pmod{7}$.

Let $\rho_2 : (\mathbb{Z}_7)^2 \rightarrow E_2$ denote projection onto E_2 , so $\rho_2|_{E_4} \equiv 0$, and let $\Delta_{K,\rho_2}(t)$ denote the associated twisted Alexander polynomial. Then, we have

$$\Delta_{K,\rho_2}(t) = (-t^2 + (\zeta^4 + \zeta^2 + \zeta + 1)t + 1)(-t^2 + (\zeta^4 + \zeta^2 + \zeta)t + 1),$$

where ζ is a 7th-root of unity. If we let

$$f(t) = (-t^2 + (\zeta^4 + \zeta^2 + \zeta + 1)t + 1),$$

then $\Delta_{K,\rho_2}(t) = -t^2 f(t) \overline{f(t^{-1})}$. To see this, use the fact that

$$1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6 = 0.$$

On the other hand, if one considers the other projection

$$\rho_4 : H_1(\Sigma_3(K)) \rightarrow E_4,$$

so that $\rho_4|_{E_2} \equiv 0$, one finds that the associated twisted polynomial is given by

$$\Delta_{K,\rho_4}(t) = t^4 + 3t^2 + 1.$$

The following lemma states that $t^4 + 3t^2 + 1$ is irreducible in Γ_7 . It follows from Theorem 3.2 that K cannot be topologically doubly slice, since the twisted polynomials associated to this metabolizing representation do not factor as norms.

Lemma 3.4. *The polynomial $p(t) = t^4 + 3t^2 + 1$ is irreducible in Γ_7 .*

Proof. If $\alpha \in \mathbb{Q}(\zeta_7)$ is a root of $p(t)$, then so is α^{-1} . Thus, if $p(t)$ has a linear factor, it has two distinct linear factors, and hence it has a quadratic factor. So, suppose that $p(t)$ factors into two quadratic polynomials. One can assume the factorization is of the form

$$p(t) = (t^2 + at + b)(t^2 + a't + b').$$

By examining coefficients, the factorization further simplifies to be of the form

$$p(t) = (t^2 + at + b)(t^2 - at + b),$$

where $b = \pm 1$ and $a^2 = 2b - 3$. If $b = 1$, then $a^2 = -1$. If $b = -1$, then $a^2 = -5$. Thus, the proof is completed by showing that $\mathbb{Q}(\zeta_7)$ contains neither $\sqrt{-1}$ nor $\sqrt{-5}$.

The Galois group of $\mathbb{Q}(\zeta_p)$ is cyclic, isomorphic to \mathbb{Z}_{p-1} , and thus contains a unique index two subgroup. It follows that $\mathbb{Q}(\zeta_p)$ contains a unique quadratic extension of \mathbb{Q} . A standard result in number theory (see [24]) states that this field is $\mathbb{Q}(\sqrt{p})$ or $\mathbb{Q}(\sqrt{-p})$, depending on whether p is congruent to

1 or 3 modulo 4, respectively. This quickly yields the desired contradiction; for instance, it is clear that $\mathbb{Q}(\sqrt{-5}) \not\subseteq \mathbb{Q}(\sqrt{-7})$. \square

Knot	Cover Homology	Irreducible Twisted Polynomial
10 ₁₅₃	$\Sigma_3(K) \cong (\mathbb{Z}_7)^2$	$\Delta_{K,\rho_4}(t) = t^4 + 3t^2 + 1$
12 _{n0019}	$\Sigma_3(K) \cong (\mathbb{Z}_{13})^2$ $\zeta^{13} = 1$	$\Delta_{K,\rho_9}(t) = t^4 + 2t^2 + 1$ $+t^3(\zeta^{11} + \zeta^9 + \zeta^8 + \zeta^7 + 2\zeta^6 + 2\zeta^5 + \zeta^3 + 2\zeta^2 + \zeta + 1)$ $+t(\zeta^{11} - \zeta^9 + \zeta^8 + \zeta^7 - \zeta^3 - \zeta)$
12 _{n0214}	$\Sigma_3(K) \cong (\mathbb{Z}_7)^2$ $\zeta^7 = 1$	$\Delta_{K,\rho_2}(t) = -29t^4 + (31 + 8\zeta + 8\zeta^2 + 8\zeta^4)$ $+t^3(-27 + 37\zeta + 37\zeta^2 + 37\zeta^4)$ $+t(48 + 47\zeta + 47\zeta^2 + 47\zeta^4)$ $+t^2(17 + 68\zeta + 68\zeta^2 + 68\zeta^4)$
12 _{n0257}	$\Sigma_3(K) \cong (\mathbb{Z}_{13})^2$ $\zeta^{13} = 1$	$\Delta_{K,\rho_9}(t) = -13t^4 + 13$ $+t^3(37 + 48\zeta + 21\zeta^2 + 48\zeta^3 + 21\zeta^5 + 21\zeta^6 + 14\zeta^7 + 14\zeta^8 + 48\zeta^9 + 14\zeta^{11})$ $+t^2(39 + 78\zeta + 13\zeta^2 + 78\zeta^3 + 13\zeta^5 + 13\zeta^6 + 65\zeta^7 + 65\zeta^8 + 78\zeta^9 + 65\zeta^{11})$ $+t(11 + 48\zeta + 34\zeta^2 + 48\zeta^3 + 34\zeta^5 + 34\zeta^6 + 27\zeta^7 + 27\zeta^8 + 48\zeta^9 + 27\zeta^{11})$
12 _{n0318}	$\Sigma_3(K) \cong (\mathbb{Z}_7)^2$ $\zeta^7 = 1$	$\Delta_{K,\rho_2}(t) = 1 + 3t^2 + t^4$ $+t(3 - \zeta - \zeta^2 - \zeta^4)$ $+t^3(4 + \zeta + \zeta^2 + \zeta^4)$
12 _{n0440}	$\Sigma_3(K)$ $\cong (\mathbb{Z}_2)^4 \oplus (\mathbb{Z}_7)^2$	$\Delta_{K,\rho_2}(t) = t^4 - 3t^3 + 6t^2 - 3t + 1$

TABLE 1. Twisted Alexander polynomial calculations.

It follows from Theorem 3.2 that K cannot be topologically doubly slice, since the twisted polynomials associated to this metabolizing representation does not factor as a norm.

The general proof of Theorem 3.3 proceeds by checking that each of the relevant twisted Alexander polynomials does not factor as a norm. The pertinent information needed to verify the result for the other knots is described in Table 1. The Maple program developed in conjunction with [17] was used to find the twisted polynomials and Maple could also be used to check the factoring conditions.

The knot 12_{n0210} was shown to not to be topologically slice in [17] using twisted polynomials, and hence it is not topologically doubly slice. \square

Question 3.5. *Are any of the knots in the following list topologically doubly slice?*

$$11_{n34} \quad 11_{n73} \quad 12_{a1019} \quad 12_{a1202}.$$

As we will see in the next section, this is the same list of knots that arises in the smooth setting.

4. Double slicing knots

In this section, we discuss some techniques that can be used to show that a knot is doubly slice. We will address the issue of double sliceness in both the smooth and the locally flat settings.

4.1. Band systems.

In [7], Donald showed that if a knot can be sliced by two different sequences of band moves, and if the bands are related in a certain way, then combining the two ribbon disks yields an unknotted 2-sphere. In this section we present a concise treatment of a special case of his result.

Let L be a link in S^3 and let b be the image of a 2-disk embedded in S^3 such that $L \cap b$ consists of two disjoint arcs in ∂b . We refer to such a b as a *band* and denote by $L * b$ the link formed as the closure of $(L \cup \partial b) \setminus (L \cap b)$. Notice that $(L * b) * b = L$; also, if b and c are disjoint, then $(L * b) * c = (L * c) * b$, so we can write both as $L * b * c$.

The reader should be familiar with the fact that the *band move* $L \rightarrow L * b$ yields a cobordism from L to $L * b$ in $S^3 \times [0, 1]$. A sequence of n such cobordisms from a knot K to the unlink of $n + 1$ components yields a ribbon disk in B^4 formed as the union of the cobordism and disjoint disks bounded by the unlink. Two such sequences yield an embedded sphere formed as the union of the ribbon disks in $S^4 = B^4 \cup B^4$. If the sequences arise from single bands b and c , we denote the knotted 2-sphere (K, b, c) . We have the following reinterpretations of two special cases of Donald's double slicing criterion [7].

Theorem 4.1. *If K is a knot and b and c are disjoint bands for which $K * b$ is an unlink, $K * c$ is an unlink, and $K * b * c$ is an unknot, then (K, b, c) is unknotted.*

Proof. Write $U_2 = K * b$ and $U'_2 = K * c$. Both are unlinks. Write

$$U_1 = K * b * c,$$

which is an unknot. The surface (K, b, c) corresponds to the sequence

$$U_2 \rightarrow U_2 * b = K \rightarrow K * c = U'_2.$$

Changing the order of the bands, this can be rewritten as

$$U_2 \rightarrow U_2 * c \rightarrow U_2 * c * b.$$

Since $U_2 = K * b$, we can express this as

$$U_2 \rightarrow K * b * c \rightarrow K * b * c * b.$$

Using the facts that $K * b * c = U_1$ and

$$K * b * c * b = K * b * b * c = K * c,$$

we finally rewrite the sequence as $U_2 \rightarrow U_1 \rightarrow U'_2$.

According to Scharlemann [28], a ribbon disk for the unknot with two minima is trivial. Thus, (K, b, c) is the union of two trivial disks; hence it is the unknot. \square

Note that Scharlemann's theorem is used above to show that certain slicing disks for the unknot are trivial. In each of the examples we consider, one can quickly show that the relevant slice disks for the unknot are trivial by observing that they are built using trivial band sums of the unlink; in particular, for our examples, one need not use the depth of Scharlemann's theorem.

More generally, if v and ω are two disjoint sets of n bands for a knot K such that $K * v$ and $K * \omega$ are unlinks of $n + 1$ components, then we let (K, v, ω) denote the sphere obtained by gluing the corresponding ribbon disks along their common boundary, K . Note that ω can be viewed as a set of bands for the unlink $U_{n+1} = K * v$, while v can be viewed as a set of bands for the unlink $U'_{n+1} = K * \omega$.

Proposition 4.2. *Suppose that v and ω are disjoint sets of n bands for a knot K such that the following properties hold:*

- (1) $K * v$ is an unlink of $n + 1$ components and ω is isotopic to a trivial set of bands for $K * v$.
- (2) $K * \omega$ is an unlink of $n + 1$ components and v is isotopic to a trivial set of bands for $K * \omega$.

Then the sphere (K, v, ω) is unknotted.

Proof. The sphere (K, v, ω) is built by capping off both ends of the following cobordism with sets of trivial disks:

$$U_{n+1} = K * v \rightarrow (K * v) * v \rightarrow (K * v) * v * \omega = K * \omega = U'_{n+1}.$$

Reversing the order of band attachments yields the cobordism:

$$U_{n+1} = K * v \rightarrow (K * v) * \omega \rightarrow (K * v) * \omega * v = K * \omega = U'_{n+1}.$$

The first half of this cobordism,

$$U_{n+1} = K * v \rightarrow (K * v) * \omega,$$

consists of adding trivial bands to an unlink to form an unknot. The corresponding surface, capped off, is a trivial disk in the 4-ball.

Reversing the other half of the cobordism yields the sequence:

$$U'_{n+1} = K * \omega \rightarrow K * \omega * v.$$

Again, this cobordism is built by adding n trivial bands to an unlink. Hence, the capped off cobordism is a trivial disk. The union of two trivial disks forms the unknot, completing the proof. \square

4.2. Superslice knots.

A knot K is called *superslice* if there is a slice disk D for K such that the double of D along K is an unknotted 2-sphere in S^4 .

Suppose that K is obtained by attaching a band v to an unlink of two components. See Figure 2 for the pertinent three examples. Let D_1 and D_2 denote the standard pair of disks bounded by the two-component unlink. In this case, the union $D = D_1 \cup v \cup D_2$ is an obvious ribbon disk for K . This disk is immersed in S^3 with ribbon singularities, but if we push the interiors of D_1 and D_2 into B^4 , we obtain an embedded disk, still called D , with two minima and one saddle with respect to the standard radial Morse function. We can assume that D is properly embedded by pushing the entire interior into B^4 , but pushing the interiors of D_1 and D_2 in farther.

Let \mathcal{K} be the 2-knot obtained by doubling the disk D . That is, glue two copies of (B^4, D) together along their common (S^3, K) boundary (via the identity map) to get (S^4, \mathcal{K}) . By construction, we see that \mathcal{K} is formed by taking two unknotted 2-spheres S_1 and S_2 in S^4 and attaching a tube Υ that connects them. Here, S_i is the double of D_i and Υ is the double of v .

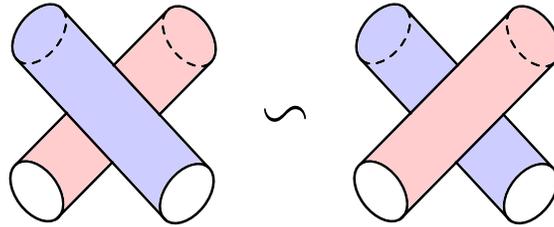


FIGURE 1. A local picture of a 2-knot isotopy that passes one tube through another.

Suppose that locally, we see two pieces of Υ as in Figure 1; there is an isotopy that passes these two pieces through each other, as shown. This isotopy corresponds to passing pieces of v past each other. This changes the isotopy class of the band v , giving a new band v' and a new ribbon knot K' , which is obtained by attaching v' to the original unlink. Because this change resulted from an isotopy of \mathcal{K} , we see that both K and K' are cross-sections of \mathcal{K} . If K' is unknotted, then \mathcal{K} is unknotted, as in the proof of Theorem 4.1, and we can conclude that K is doubly slice. We summarize this with the following criterion.

Proposition 4.3. *Let K be a knot that is obtained by attaching a single band v to an unlink of two components. Let K' be the result of passing the band v through itself as discussed above. If K' is the unknot, then K is smoothly superslice. In particular, if the band is relatively homotopic to a trivial band in the complement of a neighborhood of the unlink, then K is smoothly superslice.*

Figure 2 shows three examples of ribbon knots that satisfy the above criterion and can therefore be seen to be smoothly superslice.

Corollary 4.4. *The isotopy class of \mathcal{K} depends only on the homotopy class of the core of v .*

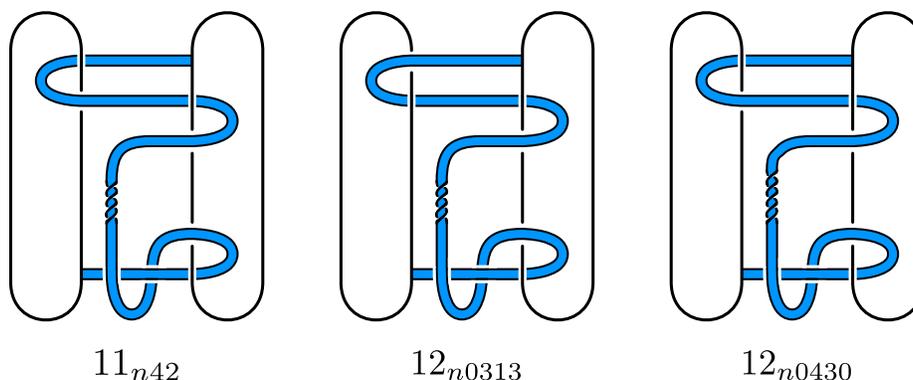


FIGURE 2. The above knots are smoothly superslice.
See Subsection 4.2.

4.3. Freedman and the locally flat setting.

Let K be a knot in S^3 , and let $\Delta_K(t)$ denote the Alexander polynomial of K . It is a well-known consequence of the work of Freedman and Quinn that any knot K with $\Delta_K(t) = 1$ bounds a topologically locally flat disk in B^4 [9, 10]. In fact, a stronger, yet less well-known, fact is true. (See [25] for more detail.)

Theorem 4.5. *Let K be a knot in S^3 . If $\Delta_K = 1$, then K is topologically superslice.*

There are four knots, up to 12 crossing, with trivial Alexander polynomial. The first is the Conway knot 11_{n34} , and the other three are shown in Figure 2. Theorem 4.5 shows that 11_{n34} is topologically doubly slice. Interestingly, it turns out that each of the other three knots is smoothly superslice; by Proposition 4.3 the double of the ribbon disk is an unknotted 2-sphere in S^4 . Thus we are led to the following problem, which at the moment seems inaccessible.

Problem 4.6. *Find a smoothly slice knot K with $\Delta_K(t) = 1$ that is not smoothly superslice.*

Superslice knots were first studied by Gordon–Sumners [15], who showed that the Whitehead double of any slice knot is superslice and that for any superslice knot K , $\Delta_K(t) = 1$. Superslice knots were also studied in relation to the Property R Conjecture [3, 14, 19].

We remark that many infinite families of superslice knots can be created by taking any properly embedded arc in the complement of an unlink that is homotopic, but not isotopic, to the trivial arc connecting the two components and banding along the arc with some framing. Changing the framing produces infinitely many knots in each family which can be distinguished from each other by their Jones polynomials. For example, any of the three knots shown in Figure 2 gives rise to such a family by adding twists to the band in each case.

4.4. Proof of Theorem 1.2.

We are now equipped to prove our main result.

Theorem 1.2. *The following knots are smoothly doubly slice.*

$$\begin{array}{ccccccc} 9_{46} & 10_{99} & 10_{123} & 10_{155} & 11_{n42} & 11_{n49} & 11_{n74} \\ 12_{a0427} & 12_{a1105} & 12_{n0268} & 12_{n0309} & 12_{n0313} & 12_{n0397} & 12_{n0414} \\ 12_{n0430} & 12_{n0605} & 12_{n0636} & 12_{n0706} & 12_{n0817} & 12_{n0838}. \end{array}$$

Furthermore, with the possible exception of the following four knots, no other prime knots of 12 or fewer crossings are smoothly doubly slice:

$$11_{n34} \quad 11_{n73} \quad 12_{a1019} \quad 12_{a1202}.$$

Proof. The Kinoshita–Terasaka knot 11_{n42} was shown to be smoothly superslice in [4]. Figure 2 shows ribbon disks for 12_{n0313} and 12_{n0430} . It is easy to see that each knot satisfies the hypotheses of Proposition 4.3; therefore, each of these knots is smoothly superslice, hence smoothly doubly slice.

The remaining 17 knots are shown in Figures 3, 4, and 5. With the exception of 12_{n0636} , these knots all satisfy Theorem 4.1. The pair of two-band systems for 12_{n0636} consisting of $\nu = \{a, b\}$ and $\omega = \{c, d\}$ illustrated in Figure 5 satisfy the condition of Proposition 4.2, and thus 12_{n0636} is also doubly slice. \square

Portions of Theorem 1.2 were previously known: 9_{46} was first smoothly double sliced by Terasaka and Hosokawa [32], with a later construction given by Sumners [31]; 11_{n42} was double sliced by Carter, Kamada, and Saito [4]; and 10_{123} (private communication) and 11_{n74} (in [7]) were double sliced by Donald.

Question 4.7. *Are any of the following knots smoothly doubly slice?*

$$11_{n34} \quad 11_{n73} \quad 12_{a1019} \quad 12_{a1202}$$

Recall that 11_{n34} is topologically doubly slice. Other than this, Question 4.7 applies equally well in the topological setting and covers all possibilities. This completes our analysis.

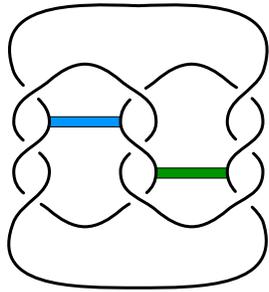
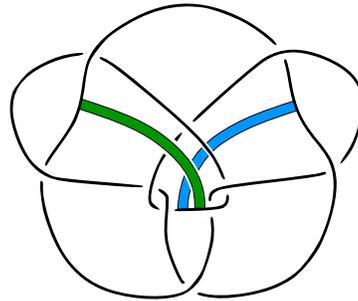
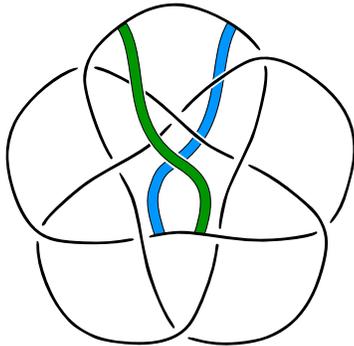
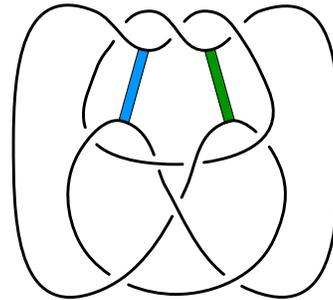
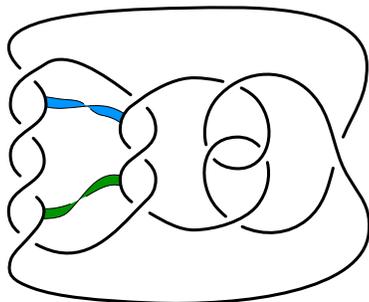
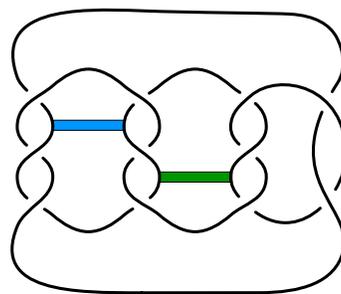
 9_{46}  10_{a99}  10_{a123}  10_{n155}  11_{n49}  11_{n74}

FIGURE 3. The above knots are smoothly doubly slice. See Subsection 4.1.

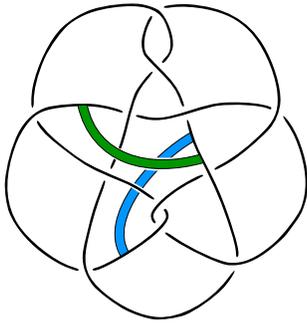
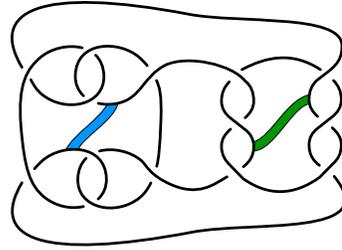
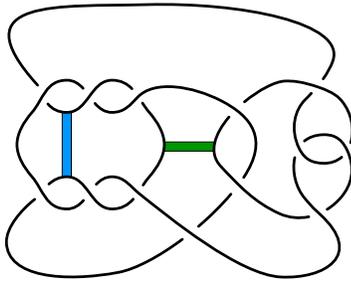
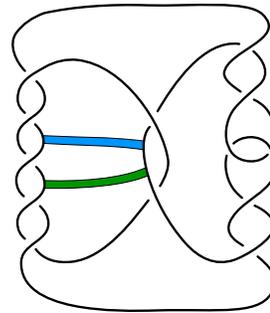
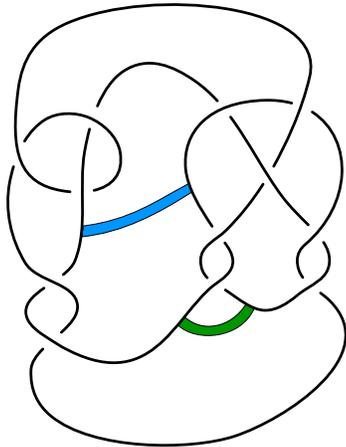
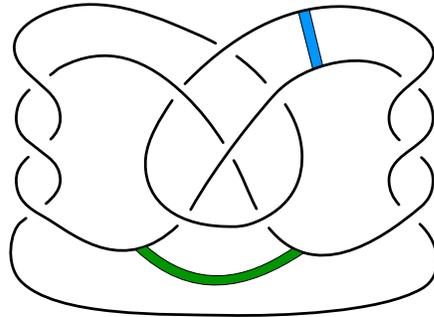
 12_{a1105}  12_{a0427}  12_{n0268}  12_{n0309}  12_{n0397}  12_{n0414}

FIGURE 4. The above knots are smoothly doubly slice. See Subsection 4.1.

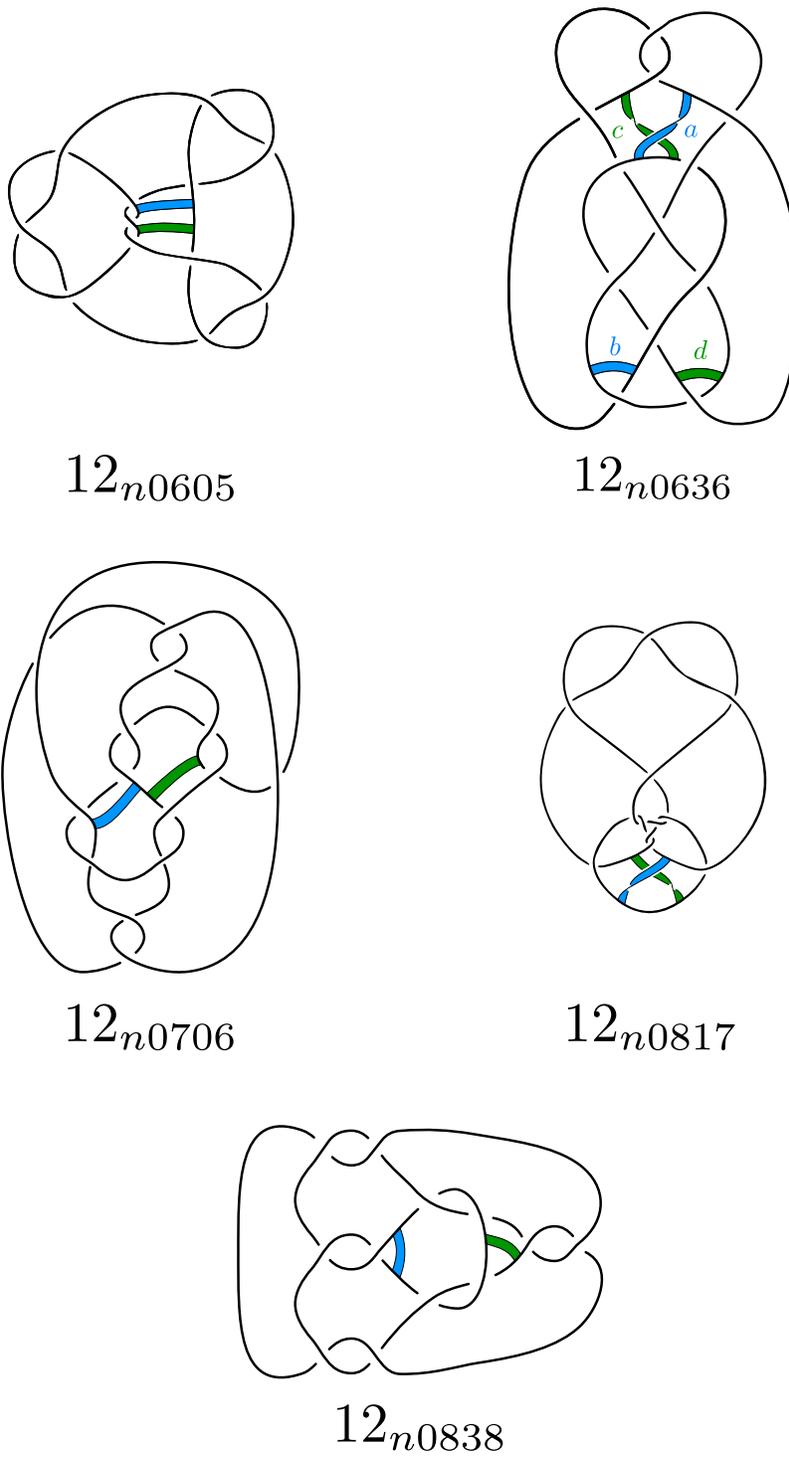


FIGURE 5. The above knots are smoothly doubly slice. See Subsection 4.1.

5. The double slice genus of knots

The study of doubly slice knots can be placed in the broader context of the relationship between knots in the 3-sphere and surfaces in the 4-sphere. In this section, we will briefly describe this more general setting.

Let \mathcal{S} be an orientable surface in S^4 . We say that \mathcal{S} is *unknotted* if \mathcal{S} bounds a handlebody H in S^4 . Let \mathcal{S} be an unknotted surface in S^4 , and suppose that \mathcal{S} transversely intersects the standard S^3 in a knot K . We say that K *divides* \mathcal{S} .

Let K be a knot in S^3 and let F be a Seifert surface for K with $g(F) = g$. We think of $F \subset S^3 \subset S^4$, where S^3 lies as the equator of S^4 . Let $H = F \times [-1, 1]$, with $H \cap S^3 = F$; the surface F is the intersection of a handlebody $H \subset S^4$ with S^3 . Let $\mathcal{S} = \partial H$. Then, \mathcal{S} is an unknotted surface in S^4 (by definition) and $K = \mathcal{S} \cap S^3$. It follows that every knot K in S^3 divides an unknotted surface in S^4 .

Therefore, we define

$$g_{ds}(K) = \min\{g(\mathcal{S}) \mid \mathcal{S} \subset S^4, \mathcal{S} \text{ unknotted, and } \mathcal{S} \cap S^3 = K\}.$$

We call $g_{ds}(K)$ the *double slice genus* of K . Note that $g_{ds}(K) = 0$ if and only if K is doubly slice. Furthermore, we saw above that $g_{ds}(K) \leq 2g_3(K)$. Similarly, it is clear that $2g_4(K) \leq g_{ds}(K)$.

The restriction $2g_4(K) \leq g_{ds}(K) \leq 2g_3(K)$ is already enough to determine the double slice genus for a third of the knots up to nine crossings. A more detailed analysis will be the subject of future study by the authors.

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