New York Journal of Mathematics

New York J. Math. **18** (2012) 657–665.

On dual-valued operators on Banach algebras

María J. Aleandro and Carlos C. Peña

Abstract. Let $\mathcal{U}$ be a regular Banach algebra and let $D : \mathcal{U} \to \mathcal{U}^*$ be a bounded linear operator, where $\mathcal{U}^*$ is the topological dual space of $\mathcal{U}$. We seek conditions under which the transpose of D becomes a bounded derivation on $\mathcal{U}^{**}$. We focus our attention on the class $\mathcal{D}(\mathcal{U})$ of bounded derivations $D : \mathcal{U} \to \mathcal{U}^*$ so that $\langle a, D(a)\rangle = 0$ for all $a \in \mathcal{U}$. We consider this matter in the setting of Beurling algebras on the additive group of integers. We show that $\mathcal{U}$ is a weakly amenable Banach algebra if and only if $\mathcal{D}(\mathcal{U}) \neq \{0\}$.

Contents

1. Introduction

Throughout this article $\mathcal{U}$ will be a Banach algebra. By $\Box$ and $\Diamond$ we will denote the first and second Arens products on $\mathcal{U}^{**}$ (cf. [1]). The Banach algebra $\mathcal{U}$ is said to be *regular* when these products coincide, in which case we will simply write $\Box = \Diamond = \bullet$. If $\mathcal{U}$ is regular it is readily seen that $\mathcal{U}^*$ becomes a Banach $\mathcal{U}^{**}$-bimodule. As usual, $\mathcal{B}(\mathcal{U},\mathcal{U}^*)$ will denote the space of bounded operators between $\mathcal{U}$ and $\mathcal{U}^*$ and $\mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$ will be the space of bounded derivations between $\mathcal{U}^{**}$ and $\mathcal{U}^*$. As is well known, when endowed with the uniform norm $\mathcal{B}(\mathcal{U},\mathcal{U}^*)$ and $\mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$ are Banach spaces. By $\mathcal{D}(\mathcal{U})$ we will denote the class of $\mathcal{D}$-derivations consisting of bounded derivations $D : \mathcal{U} \to \mathcal{U}^*$ such that $\langle a, D(a)\rangle = 0$ if $a \in \mathcal{U}$. Clearly any inner derivation from $\mathcal{U}$ into $\mathcal{U}^*$ is a $\mathcal{D}$-derivation. For problems related to these special classes of derivations, their characterization and examples in the context of Banach algebras of continuous functions or projective Banach algebras, we recommend [3]. In Proposition 1 we will characterize

Received May 17, 2011, and in revised form on August 22, 2012.

2010 *Mathematics Subject Classification.* 46H35, 47D30.

Key words and phrases. Arens products, amenable and weakly amenable Banach algebras, dual Banach algebras, Beurling algebras.

ISSN 1076-9803/2012

those operators $D \in \mathcal{B}(\mathcal{U},\mathcal{U}^*)$ whose dual belongs to $\mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$ under the hypothesis that $\mathcal{U}$ is a regular Banach algebra. Further, the corresponding problem if $D \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$ will be considered in Proposition 2. In Theorem 6 we will provide conditions under which $D \in \mathcal{D}(\mathcal{U})$ if $D^* \in \mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$. In Proposition 7 it will be shown that any $D \in \mathcal{D}(\mathcal{U})$ is (w,w) continuous. This matter and examples in the setting of Beurling algebras on $\mathbb{Z}$ will be considered in Theorem 8. For further information and background on the subject of this paper, we recommend [11], §1.4, p. 46. In addition, important articles concerning the regularity of Banach algebras are [8], [12] and [13]. Conditions under which the second transpose of a $\mathcal{U}^*$-valued bounded derivation on $\mathcal{U}$ becomes a bounded derivation on $\mathcal{U}^{**}$ endowed with the first Arens product were investigated in [7] and [2].

2. Transposes and bounded derivations between $\mathcal{U}$ and $\mathcal{U}^*$

Proposition 1. If $\mathcal{U}$ is a regular Banach algebra and if $D \in \mathcal{B}(\mathcal{U},\mathcal{U}^*)$, then the following assertions are equivalent:

(i) $D^* \in \mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$.

(ii) If $a \in \mathcal{U}$ and if $\Phi, \Psi \in \mathcal{U}^{**}$, then

$$\langle aD^*(\Phi), \Psi\rangle = \langle \Psi D(a) - D^*(\Psi)\, a, \Phi\rangle .$$

(iii) If $a \in \mathcal{U}$ and if $\Phi, \Psi \in \mathcal{U}^{**}$, then

$$\langle D^*(\Psi)\, a, \Phi\rangle = \langle D(a)\Phi - aD^*(\Phi), \Psi\rangle .$$

Proof. (i)⇒(ii). Let $\Phi, \Psi \in \mathcal{U}^{**}$ and $a \in \mathcal{U}$. Then

$$\begin{aligned}\langle \Psi D(a), \Phi\rangle &= \langle D(a), \Phi \bullet \Psi\rangle \\ &= \langle a, D^*(\Phi \bullet \Psi)\rangle \\ &= \langle a, D^*(\Phi)\Psi + \Phi D^*(\Psi)\rangle \\ &= \langle aD^*(\Phi), \Psi\rangle + \langle D^*(\Psi)a, \Phi\rangle .\end{aligned}$$

(ii)⇒(iii). Given $\Phi, \Psi \in \mathcal{U}^{**}$, $a \in \mathcal{U}$, it will suffice to see that

$$\langle \Psi D(a), \Phi\rangle - \langle aD^*(\Phi), \Psi\rangle = \langle D(a)\Phi - aD^*(\Phi), \Psi\rangle . \tag{1}$$

But (1) is an immediate consequence of the regularity of $\mathcal{U}$.

(iii)⇒(i). If $a \in \mathcal{U}$ and $\Phi, \Psi \in \mathcal{U}^{**}$ we have

$$\begin{aligned}\langle a, D^*(\Phi \bullet \Psi)\rangle &= \langle D(a), \Phi \bullet \Psi\rangle \\ &= \langle D(a)\Phi, \Psi\rangle \\ &= \langle D^*(\Psi)\, a, \Phi\rangle + \langle aD^*(\Phi), \Psi\rangle \\ &= \langle a, \Phi D^*(\Psi) + D^*(\Phi)\, \Psi\rangle .\end{aligned}$$

Since a is arbitrary the claim holds. □

Proposition 2. Let $\mathcal{U}$ be a regular Banach algebra and let $k_{\mathcal{U}^*} : \mathcal{U}^* \hookrightarrow \mathcal{U}^{***}$ be the natural embedding of $\mathcal{U}^*$ into $\mathcal{U}^{***}$. Given $D \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$, the following assertions are equivalent:

(i) $D^* \in \mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$.
(ii) If $a \in \mathcal{U}$ and if $\Phi \in \mathcal{U}^{**}$, then $k_{\mathcal{U}^*}(aD^*(\Phi)) + aD^{**}(\Phi) = 0$.
(iii) If $a \in \mathcal{U}$ and if $\Phi \in \mathcal{U}^{**}$, then $D^{**}(a\Phi) + k_{\mathcal{U}^*}(D^*(a\Phi)) = 0$.

Proof. (i)$\Rightarrow$(ii). Let $D^* \in \mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$, $a \in \mathcal{U}$. Given $\Phi, \Psi \in \mathcal{U}^{**}$, consider bounded nets $\{b_i\}_{i\in I}$, $\{c_j\}_{j\in J}$ in $\mathcal{U}$ such that $\Phi = w^*\text{-}\lim_{i\in I} k_{\mathcal{U}}(b_i)$ and $\Psi = w^*\text{-}\lim_{j\in J} k_{\mathcal{U}}(c_j)$, where $k_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{U}^{**}$ denotes the usual isometric embedding of $\mathcal{U}$ into its second dual space $\mathcal{U}^{**}$ by means of evaluations. Hence

$$\langle D^*(\Psi)a, \Phi\rangle = \lim_{i\in I}\langle b_i, D^*(\Psi)a\rangle = \lim_{i\in I}\langle D(ab_i), \Psi\rangle = \lim_{i\in I}\lim_{j\in J}\langle c_j, D(ab_i)\rangle .$$

Further,

$$\begin{aligned}(2)\qquad \langle \Psi D(a) - D^*(\Psi)a, \Phi\rangle &= \langle D(a), \Phi \bullet \Psi\rangle - \langle a, \Phi D^*(\Psi)\rangle \\ &= \lim_{i\in I}\lim_{j\in J}\left(\langle b_i c_j, D(a)\rangle - \langle c_j, D(ab_i)\rangle\right) \\ &= -\lim_{i\in I}\lim_{j\in J}\langle c_j, aD(b_i)\rangle \\ &= -\lim_{i\in I}\langle aD(b_i), \Psi\rangle \\ &= -\langle D^*(\Psi a), \Phi\rangle\end{aligned}$$

and the conclusion follows from Proposition 1 and (2).

(ii)$\Rightarrow$(iii). If $a \in \mathcal{U}$ and $\Phi, \Psi \in \mathcal{U}^{**}$ we write

$$(3)\quad \langle D^*(\Psi)a, \Phi\rangle = \langle D^*(\Psi a) + \Psi D(a), \Phi\rangle = \langle \Psi D(a), \Phi\rangle - \langle aD^*(\Phi), \Psi\rangle .$$

Moreover, $\langle \Psi D(a), \Phi\rangle = \langle D(a)\Phi, \Psi\rangle$ because $\mathcal{U}$ is regular. Hence, by (3) we obtain

$$\langle D^*(\Psi)a, \Phi\rangle = \langle D(a)\Phi - aD^*(\Phi), \Psi\rangle = -\langle D^*(a\Phi), \Psi\rangle .$$

(iii)$\Rightarrow$(i). If $a \in \mathcal{U}$ and $\Phi, \Psi \in \mathcal{U}^{**}$ we write

$$\begin{aligned}\langle a, D^*(\Phi \bullet \Psi)\rangle &= \langle D(a)\Phi, \Psi\rangle \\ &= \langle aD^*(\Phi) - D^*(a\Phi), \Psi\rangle \\ &= \langle aD^*(\Phi), \Psi\rangle + \langle D^*(\Psi)a, \Phi\rangle \\ &= \langle a, D^*(\Phi)\Psi + \Phi D^*(\Psi)\rangle .\end{aligned}$$

□

Corollary 3. *Let $\mathcal{U}$ be a regular Banach algebra. Given $D \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$ such that $D^* \in \mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$, then*

$$\mathcal{U}D^{**}(\mathcal{U}^{**}) \cup D^{**}(\mathcal{U}^{**})\mathcal{U} \hookrightarrow \mathcal{U}^*.$$

Theorem 4 (cf. [3, Theorem 2.1])**.** *Let $\mathcal{U}$ be a general Banach algebra such that $\mathcal{U}^2$ is dense in $\mathcal{U}$, where*

$$\mathcal{U}^2 = \operatorname{span}\{xy : x, y \in \mathcal{U}\} .$$

Then for $D \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$, the following assertions are equivalent:

(i) $D \in \mathcal{D}(\mathcal{U})$.

(ii) $\langle x, D(y)\rangle + \langle y, D(x)\rangle = 0$ *for all* $x, y \in \mathcal{U}$.
(iii) $D^* \circ k_{\mathcal{U}} \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$.
(iv) $D + D^* \circ k_{\mathcal{U}} = 0_{\mathcal{U},\mathcal{U}^*}$.

Corollary 5. *Let* $\mathcal{U}$ *be a general Banach algebra such that* $\mathcal{U}^2$ *is dense in* $\mathcal{U}$. *If* $D \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$, *then* $D \in \mathcal{D}(\mathcal{U})$ *if and only if for all* $a, b, c \in \mathcal{U}$ *the following identity*

$$\langle ab, D(c)\rangle + \langle ca, D(b)\rangle + \langle bc, D(a)\rangle = 0 \tag{4}$$

holds.

Proof. $(\Rightarrow)$ For $a, b, c \in \mathcal{U}$ and $D \in \mathcal{D}(\mathcal{U})$

$$\begin{aligned}\langle ab, D(c)\rangle + \langle ca, D(b)\rangle + \langle bc, D(a)\rangle &= \langle ab, D(c)\rangle + \langle ca, D(b)\rangle - \langle a, D(bc)\rangle \\ &= 0.\end{aligned}$$

$(\Leftarrow)$ If $a, b \in \mathcal{U}$ let $\{b_n\}$ and $\{c_n\}$ be sequences in $\mathcal{U}$ such that $b = \lim_{n\to\infty}(b_n c_n)$, then

$$\begin{aligned}\langle a, D(b)\rangle + \langle b, D(a)\rangle &= \lim_{n\to\infty}\{\langle a, D(b_n c_n)\rangle + \langle b_n c_n, D(a)\rangle\} \\ &= \lim_{n\to\infty}\{\langle a, D(b_n)c_n + b_n D(c_n)\rangle + \langle b_n c_n, D(a)\rangle\} \\ &= \lim_{n\to\infty}\{\langle c_n a, D(b_n)\rangle + \langle ab_n, D(c_n)\rangle + \langle b_n c_n, D(a)\rangle\} \\ &= 0.\end{aligned}$$

□

Theorem 6. *Let* $\mathcal{U}$ *be a regular Banach algebra, and let* $D \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$.
(i) *If* $\mathcal{U}^2$ *is dense in* $\mathcal{U}$ *and* $D^* \in \mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$ *then* $D \in \mathcal{D}(\mathcal{U})$.
(ii) *Suppose* $D \in \mathcal{D}(\mathcal{U})$ *has the property that*

$$\lim_{i\in I}\lim_{j\in J}\langle c_j, aD(b_i)\rangle = \lim_{j\in J}\lim_{i\in I}\langle c_j, aD(b_i)\rangle \tag{5}$$

for every pair of bounded sequences in $\mathcal{U}$, $\{b_i\}_{i\in I}$, $\{c_j\}_{j\in J}$, *and every* $a \in \mathcal{U}$ *for which both iterated limits exist. Then* $D^* \in \mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$.

Proof. (i) By Proposition 2 if $D^* \in \mathcal{Z}^1(\mathcal{U}^{**},\mathcal{U}^*)$, the equality (4) holds for all $a, b, c \in \mathcal{U}$. Thus the conclusion follows from Corollary 5.

(ii) If $a, b \in \mathcal{U}$, then $aD^{**}(k_{\mathcal{U}}(b)) = k_{\mathcal{U}^*}(aD(b))$. So, by Theorem 4 we get

$$\begin{aligned}0 &= k_{\mathcal{U}^*}(aD(b)) - aD^{**}(k_{\mathcal{U}}(b)) \\ &= k_{\mathcal{U}^*}(aD^*(k_{\mathcal{U}}(-b))) + aD^{**}(k_{\mathcal{U}}(-b)).\end{aligned}$$

If $\Phi \in \mathcal{U}^{**}$ let $\{b_i\}_{i\in I}$ be a bounded net in $\mathcal{U}$ such that $\Phi = w^*\text{-}\lim_{i\in I} k_{\mathcal{U}}(b_i)$. Define $\zeta \in \mathcal{U}^*$ by $\langle c, \zeta\rangle \triangleq \langle D^*(k_{\mathcal{U}}(c)\,a), \Phi\rangle$. Thus $\zeta = w^*\text{-}\lim_{i\in I} aD(b_i)$ and $k_{\mathcal{U}^*}(\zeta) = aD^{**}(\Phi)$. For, let $\Psi \in \mathcal{U}^{**}$ such that $\Psi = w^*\text{-}\lim_{j\in J} k_{\mathcal{U}}(c_j)$ in $\mathcal{U}^{**}$ for some bounded net $\{c_j\}_{j\in J}$ in $\mathcal{U}$. So, by (5) we have

$$\langle \Psi, aD^{**}(\Phi)\rangle = \lim_{i\in I}\lim_{j\in J}\langle c_j, aD(b_i)\rangle = \lim_{j\in J}\lim_{i\in I}\langle c_j, aD(b_i)\rangle = \langle \zeta, \Psi\rangle.$$

Consequently,

$$\begin{aligned}\langle \Psi, k_{\mathcal{U}^*}(aD^*(\Phi)) + aD^{**}(\Phi)\rangle &= \langle \Psi, k_{\mathcal{U}^*}(aD^*(\Phi)+\zeta)\rangle \\ &= \langle aD^*(\Phi)+\zeta, \Psi\rangle \\ &= \lim_{j\in J}\langle c_j, aD^*(\Phi)+\zeta\rangle \\ &= \lim_{j\in J}\left[\langle D(c_j a), \Phi\rangle + \langle \zeta, k_{\mathcal{U}}(c_j)\rangle\right] \\ &= \lim_{j\in J}\lim_{i\in I}\left[\langle b_i, D(c_j a)\rangle + \langle aD(b_i), k_{\mathcal{U}}(c_j)\rangle\right] \\ &= \lim_{j\in J}\lim_{i\in I}\langle c_j, a\left(D^*(k_{\mathcal{U}}(b_i)) + D(b_i)\right)\rangle \\ &= 0.\end{aligned}$$

Since Ψ was arbitrary, $k_{\mathcal{U}^*}(aD^*(\Phi)) + aD^{**}(\Phi) = 0$ and the conclusion follows from Proposition 2. □

Proposition 7. If $D \in \mathcal{D}(\mathcal{U})$ then D^* is (w,w)-continuous.

Proof. If $D \in \mathcal{D}$, let $\{\Phi_i\}_{i\in I}$ be a net in $\mathcal{U}^{**}$ such that $w\text{-}\lim_{i\in I} D^*(\Phi_i) \neq 0_{\mathcal{U}^*}$. There exists $\Theta \in \mathcal{U}^{**}$ and a subnet $\{\Phi_i\}_{i\in I_1}$ of $\{\Phi_i\}_{i\in I}$ such that

$$|\langle D^*(\Phi_i), \Theta\rangle| \geq 1 \ if \ i \in I_1.$$

Let $\{a_j\}_{j\in J}$ be a bounded net in $\mathcal{U}$ such that

$$\Theta = w^* - \lim_{j\in J} k_{\mathcal{U}}(a_j).$$

Since $\{k_{\mathcal{U}^*}(D(a_j))\}_{j\in J}$ is a bounded net in $\mathcal{U}^{***}$ by the Banach–Alaoglu theorem there is a subnet $\{a_j\}_{j\in J_1}$ such that the limit $w^*\text{-}\lim_{j\in J_1} k_{\mathcal{U}^*}(D(a_j))$ defines an element M in $\mathcal{U}^{***}$. As $D^{**} \in (w^*, w^*)$,

$$D^{**}(\Theta) = w^* - \lim_{j\in J_1} D^{**}(k_{\mathcal{U}}(a_j)).$$

In particular, by Theorem 4 we deduce that $D^{**} \circ k_U = k_{\mathcal{U}^*} \circ D$. Hence, if $i \in I_1$ we obtain

$$\begin{aligned}1 &\leq |\langle D^*(\Phi_i), \Theta\rangle| \\ &= |\langle \Phi_i, D^{**}(\Theta)\rangle| \\ &= \lim_{j\in J_1} |\langle \Phi_i, D^{**}(k_{\mathcal{U}}(a_j))\rangle| \\ &= \lim_{j\in J_1} |\langle \Phi_i, k_{\mathcal{U}^*}(D(a_j))\rangle| \\ &= |\langle \Phi_i, M\rangle|,\end{aligned}$$

i.e., $w\text{-}\lim_{i\in I} \Phi_i \neq 0_{\mathcal{U}^{**}}$. □

3. An application to Beurling algebras on the group $(\mathbb{Z},+)$

Given a function $w:\mathbb{Z}\to\mathbb{R}^+$ let $\mathcal{U}\triangleq\ell^1(\mathbb{Z},w)$ be the space of complex sequences $\{a_m\}_{m\in\mathbb{Z}}$ such that $\|a\|_{1,w}\triangleq\sum_{m\in\mathbb{Z}}|a_m|\,w(m)$ is finite. With the natural vector space operations $\left(\mathcal{U},\|\circ\|_{1,w}\right)$ is a Banach space. Further, let us suppose that w is a weight function, i.e., $w(m+n)\le w(m)\,w(n)$ for all $m,n\in\mathbb{Z}$ and $w(0)=1$. Then, for $a,b\in\mathcal{U}$ the convolution product

$$a*b\triangleq\left\{\sum_{m\in\mathbb{Z}}a_mb_{n-m}\right\}_{n\in\mathbb{Z}}$$

is well defined and $\mathcal{U}$ becomes a Banach algebra. These algebras are called *Beurling algebras* on the additive group $\mathbb{Z}$ (cf. [6], [9]). The topological dual $\mathcal{U}^*$ consists of all functions $\lambda:\mathbb{Z}\to\mathbb{C}$ such that

$$\|\lambda\|_{\infty,w^{-1}}\triangleq\sup\left\{|\lambda(m)|\,w(m)^{-1}:m\in\mathbb{Z}\right\}$$

is finite. Indeed, $\mathcal{U}$ is a dual Banach algebra whose predual can be identified with the the closed subspace $\mathrm{c}_0\left(\mathbb{Z},w^{-1}\right)$ consisting of those sequences $\lambda\in\ell^\infty\left(\mathbb{Z},w^{-1}\right)$ such that λw^{-1} vanishes at infinity. Since the additive group of integers is discrete and countable there are weights w on $\mathbb{Z}$ such that $\ell^1(\mathbb{Z},w)$ is regular. Further, $\mathcal{U}$ is regular if

$$\inf_{i\le j}\frac{w(m_i+n_j)}{w(m_i)\,w(n_j)}=0$$

for all sequences of distinct elements of $\mathbb{Z}$ (see [5]). For instance, $\mathcal{U}$ is not regular if $w(m)=1$ or $w(m)=\exp(|m|)$, and it is regular if $w(m)=1+|m|$ for all $m\in\mathbb{Z}$.

Theorem 8. *Let $D\in\mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$.*

(i) *There is a unique complex sequence $\{\lambda_m\}_{m\in\mathbb{Z}}$ such that*

$$\|D\|=\sup_{m\in\mathbb{Z}}\left\{\frac{|m|}{w(m)}\sup_{p\in\mathbb{Z}}\frac{|\lambda_{m+p-1}|}{w(p)}\right\},\tag{6}$$

and if $a\in\mathcal{U}$ we have

$$D(a)=\left\{\sum_{m\in\mathbb{Z}}m\lambda_{m+p-1}a_m\right\}_{p\in\mathbb{Z}}.\tag{7}$$

(ii) *If we write $D_0(a)\triangleq\{-ma_{-m}\}_{m\in\mathbb{Z}}$ for $a\in\mathcal{U}$ then $D_0\in\mathcal{D}(\mathcal{U})$ and any other element of $\mathcal{D}(\mathcal{U})$ is a constant multiple of D_0.*

(iii) *$\mathcal{D}(\mathcal{U})\neq\{0\}$ if and only if $\mathcal{U}$ is a non-weakly amenable Banach algebra.*

(iv) *If $D\in\mathcal{D}(\mathcal{U})$ then $D(\mathcal{U})\subseteq\mathrm{c}_0\left(\mathbb{Z},w^{-1}\right)$.*

(v) *If $D\in\mathcal{D}(\mathcal{U})$ then $D^*+D\circ k^*_{\mathrm{c}_0(\mathbb{Z},w^{-1})}=0_{\ell^\infty(\mathbb{Z},w^{-1})^*,\ell^\infty(\mathbb{Z},w^{-1})}$.*

(vi) *If $D\in\mathcal{D}(\mathcal{U})$ then $D\circ k^*_{\mathrm{c}_0(\mathbb{Z},w^{-1})}=k^*_{\ell^1(\mathbb{Z},w)}\circ D^{**}$.*

Proof. (i) If $m \in \mathbb{Z}$, let e_m be the characteristic function of $\{m\}$ considered as an element of $\mathcal{U}$ and let $D(e_m) = \{\lambda_{m,p}\}_{p\in\mathbb{Z}}$ in $\ell^\infty(\mathbb{Z}, w^{-1})$. Since D satisfies the Leibnitz rule, the following identities $\lambda_{m+p,q} = \lambda_{m,p+q} + \lambda_{p,m+q}$ hold for all $m, p, q \in \mathbb{Z}$. Let us write $\lambda_m \triangleq \lambda_{1,m}$ for $m \in \mathbb{Z}$. It is readily seen that $\lambda_{m,p} = m\lambda_{m+p-1}$ if $m, p \in \mathbb{Z}$. Hence (7) holds since for each $p \in \mathbb{Z}$ the linear form $\mu \to \langle e_p, \mu\rangle$ belongs to $\ell^\infty(\mathbb{Z}, w^{-1})^*$. Now,

$$\begin{aligned}\sup_{m\in\mathbb{Z}} \left\| D\left(\frac{e_m}{w(m)}\right)\right\|_{\infty,w^{-1}} &= \sup_{m\in\mathbb{Z}} \frac{1}{w(m)} \sup_{p\in\mathbb{Z}} \frac{|\lambda_{m,p}|}{w(p)} \\ &= \sup_{m\in\mathbb{Z}} \frac{|m|}{w(m)} \sup_{p\in\mathbb{Z}} \frac{|\lambda_{m+p-1}|}{w(p)} \le \|D\|.\end{aligned}$$

We can assume that $D \neq 0$. If $0 < t < \|D\|$ there exist $m, p \in \mathbb{Z}$ such that $|m\lambda_{m+p-1}|/w(m)w(p) > t$. Otherwise, we can choose $u, v \in [\mathcal{U}]_1$ such that

$$t < |\langle v, D(u)\rangle| \le \sum_{p\in\mathbb{Z}} |v_p| \sum_{m\in\mathbb{Z}} |m\lambda_{m+p-1}u_m| \le t\,\|u\|_{1,w}\,\|v\|_{1,w} \le t,$$

which is absurd. Thus (6) follows.

(ii) It is straightforward to see that $D_0 \in \mathcal{D}(\mathcal{U})$. Moreover, with the above notation let $D \in \mathcal{D}(\mathcal{U})$ and $m, p \in \mathbb{Z}$. By Theorem 4(ii) we see that

$$0 = \langle e_m, D(e_p)\rangle + \langle e_p, D(e_m)\rangle = (m+p)\,\lambda_{m+p-1}.$$

Hence $\lambda_{m,p} = \lambda_{m+p-1} = 0$ if $m + p \neq 0$ while $\lambda_{m,-m} = m\lambda_{-1}$. Consequently $D(e_m) = \lambda_{-1} m e_{-m}$ and $D = \lambda_{-1}D_0$.

(iii) Observe that $\mathcal{U}$ is not weakly amenable if and only if

$$\sup_{m\in\mathbb{Z}} \frac{|m|}{w(m)w(-m)} < +\infty \tag{8}$$

(cf. [10], Corollary 4.8). Further, by (6),

$$\|D_0\| = \sup_{m\in\mathbb{Z}} \frac{|m|}{w(m)w(-m)} \tag{9}$$

and the conclusion now follows.

(iv) If $a \in \mathcal{U}$ and $m \in \mathbb{Z}$ by (9) we have

$$\frac{|-ma_{-m}|}{w(m)} = \frac{|m|}{w(m)w(-m)}\,|a_{-m}|\,w(-m) \le \|D_0\|\,|a_{-m}|\,w(-m),$$

i.e., $\lim_{m\to\infty}(-ma_{-m})/w(m) = 0$.

(v) Let $\mathfrak{K}$ be the subset of elements $F \in \ell^\infty(\mathbb{Z})^*$ whose induced finitely additive set function $\mu_F(E) \triangleq \langle \chi_E, F\rangle$ defined for all $E \in \mathcal{P}(\mathbb{Z})$ vanishes on finite subsets of $\mathbb{Z}$. Certainly

$$\ell^\infty(\mathbb{Z})^* = k_{\ell^1(\mathbb{Z})}\left[\ell^1(\mathbb{Z})\right] \oplus \mathfrak{K}$$

(cf. [4, Theorem 3.2]). Further, since $\mathrm{Id}_{\ell^1(\mathbb{Z},w)} = k^*_{c_0(\mathbb{Z},w^{-1})} \circ k_{\ell^1(\mathbb{Z},w)}$ then

$$\ell^\infty(\mathbb{Z}, w^{-1})^* = k_{\ell^1(\mathbb{Z},w)}\left[\ell^1(\mathbb{Z}, w)\right] \oplus \ker\left[k^*_{c_0(\mathbb{Z},w^{-1})}\right]. \tag{10}$$

Let $A_w : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z}, w)$ be the isometric isomorphism such that

$$A_w(x) \triangleq \{x(m)/w(m)\}_{m\in\mathbb{Z}}$$

if $x \in \ell^1(\mathbb{Z})$. Then

$$A_w^{**}(\mathfrak{K}) = \ker\left[k^*_{c_0(\mathbb{Z},w^{-1})}\right]. \tag{11}$$

For, let be given $F \in \mathfrak{K}$ and $\lambda \in c_0(\mathbb{Z}, w^{-1})$. Then

$$\begin{aligned}\left\langle \lambda, k^*_{c_0(\mathbb{Z},w^{-1})}\left(A_w^{**}(F)\right)\right\rangle &= \left\langle A_w^*\left(k_{c_0(\mathbb{Z},w^{-1})}(\lambda)\right), F\right\rangle \\ &= \left\langle \{\lambda(m)/w(m)\}_{m\in\mathbb{Z}}, F\right\rangle \\ &= \int_{\mathbb{Z}} \frac{\lambda}{w} d\mu_F.\end{aligned} \tag{12}$$

But $\{e_m\}_{m\in\mathbb{Z}}$ can be considered as a Schauder basis of $c_0(\mathbb{Z}, w^{-1})$. Moreover, using (12) we can write

$$\begin{aligned}\left\langle \lambda, k^*_{c_0(\mathbb{Z},w^{-1})}\left(A_w^{**}(F)\right)\right\rangle &= \left\langle \sum_{m\in\mathbb{Z}} \lambda(m)\, e_m, k^*_{c_0(\mathbb{Z},w^{-1})}\left(A_w^{**}(F)\right)\right\rangle \\ &= \sum_{m\in\mathbb{Z}} \lambda(m) \left\langle e_m, k^*_{c_0(\mathbb{Z},w^{-1})}\left(A_w^{**}(F)\right)\right\rangle \\ &= \sum_{m\in\mathbb{Z}} \lambda(m) \int_{\mathbb{Z}} \frac{e_m}{w} d\mu_F \\ &= 0.\end{aligned} \tag{13}$$

Since λ was arbitrary then $k^*_{c_0(\mathbb{Z},w^{-1})}\left(A_w^{**}(F)\right) = 0_{\ell^1(\mathbb{Z},w)}$. On the other hand, given $\Phi \in \ker\left[k^*_{c_0(\mathbb{Z},w^{-1})}\right]$ we set $F \triangleq \left(A_w^{-1}\right)^{**}(\Phi)$. If $m \in \mathbb{Z}$, let $\chi^\infty_{\{m\}}$ be the characteristic function of $\{m\}$ considered as an element of $\ell^\infty(\mathbb{Z})$. Given $a \in \ell^1(\mathbb{Z}, w)$ we see that

$$\begin{aligned}\left\langle \chi^\infty_{\{m\}}, F\right\rangle &= \left\langle \left(A_w^{-1}\right)^*\left(\chi^\infty_{\{m\}}\right), \Phi\right\rangle \\ &= \left\langle w(m)\, k_{c_0(\mathbb{Z},w^{-1})}(e_m), \Phi\right\rangle \\ &= w(m)\left\langle e_m, k^*_{c_0(\mathbb{Z},w^{-1})}(\Phi)\right\rangle \\ &= 0.\end{aligned}$$

Therefore, $F \in \mathfrak{K}$ and (8) holds. If $\Phi \in \mathcal{U}^{**}$, then by (10) and (11), there are unique elements $a \in \mathcal{U}$ and $F \in \mathfrak{K}$ such that $\Phi = k_{\mathcal{U}}(a) + A_w^{**}(F)$. Finally, it is easy to verify that $a = k^*_{c_0(\mathbb{Z},w^{-1})}(\Phi)$ and given $b \in \mathcal{U}$ we have

$$\begin{aligned}\left\langle b, D_0^*(\Phi)\right\rangle &= \left\langle b, -D_0(a)\right\rangle + \left\langle A_w^{**}(F), k_{c_0(\mathbb{Z},w^{-1})}(D_0(b))\right\rangle \\ &= \left\langle b, -\left(D_0 \circ k^*_{c_0(\mathbb{Z},w^{-1})}\right)(\Phi)\right\rangle.\end{aligned}$$

(vi) It suffices to apply Theorem 4 and (v). □

References

[1] Arens, Richard. Operations induced in function classes. *Monatsh. Math.* **55**, (1951), 1–19. MR0044109 (13,372b), Zbl 0042.35601.

[2] Barootkoob, S.; Vishki, H. R. Ebrahimi. Lifting derivations and n-weak amenability of the second dual of a Banach algebra. *Bull. Aust. Math. Soc.* **83** (2011), no. 1, 122–129. MR2765419 (2012a:46084), Zbl 05864254, arXiv:1007.1649, doi: 10.1017/S0004972710001838.

[3] Barrenechea, A. L.; Peña, C. C. On bounded dual-valued derivations on certain Banach algebras. *Publ. Inst. Math. (Beograd) (N.S.)* **86**(100) (2009), 107–114. MR2567770 (2010m:46076), Zbl 05656373, doi: 10.2298/PIM0900107B.

[4] Civin, Paul; Yood, Bertram. The second conjugate space of a Banach algebra as an algebra. *Pacific J. Math.* **11** (1961), 847–870. MR0143056 (26 #622), Zbl 0119.10903.

[5] Craw, I. G.; Young, N. J. Regularity of multiplication in weighted group and semigroup algebras. *Quart. J. Math. Oxford Ser.* (2) **25** (1974), 351–358. MR0365029 (51 #1282) Zbl 0304.46027.

[6] Dales, H. G.; Lau, A. T.-M. The second duals of Beurling algebras. *Mem. Amer. Math. Soc.* **177** (2005), no. 836, vi+191 pp. MR2155972 (2006k:43002), Zbl 1075.43003.

[7] Dales, H. G.; Rodríguez-Palacios, A.; Velasco, M. V. The second transpose of a derivation. *J. London Math. Soc.* (2) **64** (2001), no. 3, 707–721. MR1865558 (2003e:46077), Zbl 1023.46051, doi: 10.1112/S0024610701002496.

[8] Duncan, J.; Hosseiniun, S. A. R. The second dual of a Banach algebra. *Proc. Roy. Soc. Edinburgh, Sect. A* **84**, (1979), no. 3–4, 309–325. MR0559675 (81f:46057), Zbl 0427.46028, doi: 10.1017/S0308210500017170.

[9] Dzinotyiweyi, Heneri A. M. Weighted function algebras on groups and semigroups. *Bull. Austral. Math. Soc.* **33** (1986), no. 2, 307–318. MR0832532 (87h:43005), Zbl 0571.43006, doi: 10.1017/S0004972700003178.

[10] Grønbæk, Niels. A characterization of weakly amenable Banach algebras. *Studia Math.* **94** (1989), no. 2, 149–162. MR1025743 (92a:46055), Zbl 0704.46030.

[11] Palmer, Theodore W. Banach algebras and the general theory of $*$-algebras. Vol. I. Algebras and Banach algebras. Encyclopedia of Mathematics and its Applications, 49. *Cambridge University Press, Cambridge*, 1994. xii+794 pp. ISBN: 0-521-36637-2. MR1270014 (95c:46002), Zbl 1176.46052.

[12] Pym, John S. The convolution of functionals on spaces of bounded functions. *Proc. London Math. Soc.* (3) **15** (1965), 84–104. MR0173152 (30 #3367), Zbl 0135.35503, doi: 10.1112/plms/s3-15.1.84.

[13] Young, N. J. Periodicity of functionals and representations of normed algebras on reflexive spaces. *Proc. Edinburgh Math. Soc.* (2) **20**, (1976/77), no. 2, 99–120. MR0435849 (55 #8800), Zbl 0331.46042, doi: 10.1017/S0013091500010610.

CONICET - UNCPBA. FCExactas, Dpto. de Matemáticas, NUCOMPA.
`aleandro@exa.unicen.edu.ar`

UNCPBA. FCExactas, Dpto. de Matemáticas, NUCOMPA.
`ccpenia@exa.unicen.edu.ar`

This paper is available via `http://nyjm.albany.edu/j/2012/18-35.html`.