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Fundamental groupoids of *k*-graphs

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Abstract. k-graphs are higher-rank analogues of directed graphs which were first developed to provide combinatorial models for operator algebras of Cuntz–Krieger type. Here we develop a theory of the fundamental groupoid of a k-graph, and relate it to the fundamental groupoid of an associated graph called the 1-skeleton. We also explore the failure, in general, of k-graphs to faithfully embed into their fundamental groupoids.

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1. Introduction

k-graphs are combinatorial structures which are k-dimensional analogues of (directed) graphs. They were introduced by Kumjian and the first author [7] to help understand work of Robertson and Steger on higher-rank analogues of the Cuntz–Krieger algebras [12]. Strictly speaking, k-graphs are generalizations of path categories: modulo conventions as regards composition, the 1-graphs are precisely the path categories of ordinary graphs. As the theory of k-graphs has developed, the depth of analogy with graphs has been remarkable — it seems that almost every aspect of graphs has a valid and interesting analogue for the more general k-graphs.

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One of the most useful invariants of a graph is its fundamental groupoid, which classifies the coverings of the graph, and thereby gives a purely combinatorial approach to covering space theory. In a subsequent paper [11] we will develop a theory of coverings for k-graphs; in preparation for this we develop in this paper an elementary theory of a *fundamental groupoid* $\mathcal{G}(\Lambda)$ of a k-graph Λ.

One problem with our fundamental groupoid is that there is no good analogue of the usual reduced-word description of elements of $\mathcal{G}(\Lambda)$. Indeed, it is not even true that the k-graph itself embeds in $\mathcal{G}(\Lambda)$. Our main objective, therefore, is to realize our fundamental groupoid as a well-controlled quotient of the fundamental groupoid of an associated graph called the 1-*skeleton* of Λ. This allows us to perform calculations in a situation where reduced-word arguments are available.

Our approach to realizing the groupoid of a k-graph as a quotient of the groupoid of its 1-skeleton is to first realize the k-graph itself as a quotient of the associated 1-graph. To see how this could be possible, recall that a k-graph Λ is a category with a "degree functor" $d\colon \Lambda \to \mathbb{N}^k$ satisfying a certain factorization property (see Section 2 for the precise definition). The elements of Λ whose degree is a standard basis vector can be regarded as the edges of the 1-skeleton, and the various factorizations of an arbitrary element of Λ into edge-paths give the desired equivalence relation on the associated 1-graph.

Because every small category is isomorphic to a quotient of a path category, it will be clear from the proofs that all our results carry over to arbitrary small categories; however, we eschew such a generalization since we have no useful applications.

After we completed this paper, we learned of the existence of [1, 6]. [1, Appendix] develops the elementary theory of the fundamental group of a small category and proves results similar to some of ours. Bridson and Haefliger concentrate on the fundamental *group* — indeed, they stop just short of defining the fundamental groupoid. [6] develops, in the specific context of k-graphs, the fundamental groupoid and the existence of the universal covering. We also wish to thank Kumjian for bringing [1] to our attention.

We begin in Section 2 by stating our conventions for k-graphs and also for quotients of path categories of graphs. In Section 3 we introduce our notion of fundamental groupoids of k-graphs, using a universal construction from category theory, namely categories of fractions. In Section 4, we characterize k-graphs as quotients of the path categories of their 1-skeletons, and in Section 5 we perform the same job for the associated fundamental groupoids.

In Section 6, we give a geometric interpretation of our fundamental groupoid $\mathcal{G}(\Lambda)$, showing that at least the fundamental group agrees with the usual one for a topological space constructed from Λ. [1] mentions that in general the fundamental group of a small category is isomorphic to that of the classifying space, but in the case of k-graphs our geometric realization seems more elementary.

In the last two sections we consider the question of when the canonical functor $i\colon \Lambda \to \mathcal{G}(\Lambda)$ is injective, and how we might get around the problem. In Section 7, we give several very simple examples which illustrate that i is often noninjective. In Section 8, we discuss the possibility of replacing Λ by its image $i(\Lambda)$, in which we actually have $i(\Lambda) \to \mathcal{G}(i(\Lambda)) = \mathcal{G}(\Lambda)$ injective. Unfortunately we have been unable to prove that $i(\Lambda)$ has the unique factorization property.

2. Preliminaries

Small categories. Let $\mathcal{C}$ be a small category. We regard $\mathcal{C}$ more as an algebraic structure rather than a set of objects and morphisms. For this we identify the objects with certain idempotent elements of $\mathcal{C}$. Among our conventions, we write:

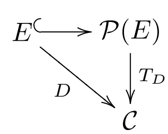
- $\mathcal{C}^0$ for the set of objects in $\mathcal{C}$;
- $s(a)$ for the domain of $a \in \mathcal{C}$, and call it the *source*;
- $r(a)$ for the codomain of $a \in \mathcal{C}$, and call it the *range*;
- composition as juxtaposition: $ab = a \circ b$ whenever $s(a) = r(b)$;
- $u\mathcal{C} = \{a \in \mathcal{C} \mid r(a) = u\}$, $\mathcal{C}v = \{a \in \mathcal{C} \mid s(a) = v\}$, and $u\mathcal{C}v = \mathrm{Hom}(v, u)$ (for $u, v \in \mathcal{C}^0$).

In fact, in general we often write composition of maps as juxtaposition, especially when we are chasing around commutative diagrams.

Path categories. In order to deal effectively with quotients of 1-graphs, we regard these quotients as presentations of categories with generators and relations. More precisely, a 1-graph may be regarded as the free category generated by the edges and vertices, and the relations will be any set of ordered pairs of paths generating the equivalence relation defining the quotient.

For a large part of our development we mainly adopt the conventions and results of Schubert's book [13], particularly regarding the use of graphs in (small) category theory — see Chapter 6 of the Schubert book. Alternative sources are [9, 2, 5, 10] (particularly Chapter III of the latter). However, we occasionally prefer notation and terminology from the graph algebra literature (see [8], for example).

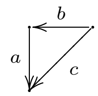
A *graph* is a pair $E = (E^0, E^1)$ of sets equipped with two maps $s, r \colon E^1 \to E^0$; E^0 comprises the *vertices* and E^1 the *edges*, and the *source* and *range* of an edge e are $s(e)$ and $r(e)$, respectively. A *diagram of type* E in a category $\mathcal{C}$ is a graph morphism from E to the underlying graph of $\mathcal{C}$ (obtained by forgetting the composition); for simplicity we'll regard the diagram as a map from E to $\mathcal{C}$. E embeds a category $\mathcal{P}(E)$ having the universal property that for every diagram D of type E in a category $\mathcal{C}$ there exists a unique functor T_D making the diagram

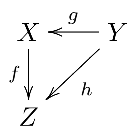


commute. $\mathcal{P}(E)$ is the *path category* of E, and the embedding $E \hookrightarrow \mathcal{P}(E)$ is the *canonical diagram of type* E.

Warning. To be consistent with the k-graph literature (e.g., [7]) and Schubert [13], we write paths of edges in E in the order appropriate for composition, i.e., if $e_1, \ldots, e_n \in E^1$ with $s(e_i) = r(e_{i+1})$ for $i = 1, \ldots, n-1$, then $e_1 \cdots e_n$ is a path in $\mathcal{P}(E)$; this is the opposite of the convention in much of the literature of graphs (e.g., [4]) and graph (operator) algebras (e.g., [8]).

$E \mapsto \mathcal{P}(E)$ is functorial from graphs to small categories. In fact, it is a left adjoint for the underlying-graph functor. A *relation* for E is a pair (α, β) of paths in $\mathcal{P}(E)$, where $s(\alpha) = s(\beta)$ and $r(\alpha) = r(\beta)$. We say a diagram $D \colon E \to \mathcal{C}$ *satisfies* a relation (α, β) if $D(\alpha) = D(\beta)$.

Example 2.1. A diagram of type $E =$  in a category $\mathcal{C}$ has the form

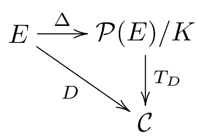


and satisfies the relation (ab, c) if and only if it commutes in the usual sense.

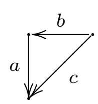
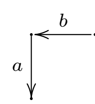
If K is a set of relations for E, we say a diagram of type E *satisfies* K if it satisfies all relations in K. The quotient of $\mathcal{P}(E)$ by the smallest equivalence relation containing K is a category which we denote by $\mathcal{P}(E)/K$ (by self-explanatory abuse of notation) and call the *relative path category of E with relations K*. The quotient map $\mathcal{P}(E) \to \mathcal{P}(E)/K$ is injective on objects, so we can identify the objects of $\mathcal{P}(E)/K$ with the vertices of E. We call the composition

$$\Delta\colon E \hookrightarrow \mathcal{P}(E) \to \mathcal{P}(E)/K$$

the *canonical diagram of type E satisfying K*; it has the universal property that for every diagram $D\colon E \to \mathcal{C}$ satisfying K there exists a unique functor T_D making the diagram



commute.

Example 2.2. If $E =$ , $K = \{(ab, c)\}$, and $F =$ , then $\mathcal{P}(E)/K \cong \mathcal{P}(F)$.

***k*-graphs.** We adopt the conventions of [7]: a *k-graph* is a category Λ equipped with a *degree functor* $d\colon \Lambda \to \mathbb{N}^k$ satisfying the following *factorization property* (also known as *unique factorization*): for all $\alpha \in \Lambda$ and $n, l \in \mathbb{N}^k$ such that $d(\alpha) = n + l$ there exist unique $\beta, \gamma \in \Lambda$ such that $d(\beta) = n$, $d(\gamma) = l$, and $\alpha = \beta\gamma$.

If Λ is a k-graph, $n \in \mathbb{N}^k$, and $u, v \in \Lambda^0$, we write $\Lambda^n = d^{-1}(n)$, $u\Lambda^n = u\Lambda \cap \Lambda^n$, and $\Lambda^n v = \Lambda v \cap \Lambda^n$. A *morphism* of k-graphs is a degree-preserving functor.

3. Fundamental groupoids

Let Λ be a k-graph. As a category, Λ has very few invertible elements: just the vertices. We shall associate a groupoid to Λ, doing as little damage to Λ as possible, and making all the elements of Λ invertible.

A standard construction in category theory (see [13, Section 19.1], for example) takes any subset Σ of Λ and produces a new category $\Lambda[\Sigma^{-1}]$, called a *category of fractions*, and a functor $i\colon \Lambda \to \Lambda[\Sigma^{-1}]$ such that $i(a)$ is invertible for all $a \in \Sigma$, and having the universal property that if $T : \Lambda \to \mathcal{C}$ is a functor with $T(a)$ invertible for all $a \in \Sigma$ then there is a unique functor T' making the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{i} & \Lambda[\Sigma^{-1}] \\ & {\scriptstyle T}\searrow & \downarrow{\scriptstyle T'} \\ & & \mathcal{C} \end{array}$$

commute. Since Λ is small, we can, and shall, take $\Sigma = \Lambda$, and we write

$$\mathcal{G}(\Lambda) = \Lambda[\Lambda^{-1}].$$

Since $i(\alpha)$ is invertible for all $\alpha \in \Lambda$, there is a subcategory of $\mathcal{G}(\Lambda)$ which is a groupoid and contains $i(\Lambda)$; by universality this subcategory in fact coincides with $\mathcal{G}(\Lambda)$. Thus $\mathcal{G}(\Lambda)$ is a groupoid, and is generated as a groupoid by $i(\Lambda)$. Moreover, the universal property may be rephrased as follows: for any functor T from Λ into a groupoid $\mathcal{H}$ there exists a unique groupoid morphism T' making the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{i} & \mathcal{G}(\Lambda) \\ & {\scriptstyle T}\searrow & \downarrow{\scriptstyle T'} \\ & & \mathcal{H} \end{array}$$

commute. The pair $(\mathcal{G}(\Lambda), i)$ is unique up to isomorphism.

Definition 3.1. With the above notation, $\mathcal{G}(\Lambda)$ is the *fundamental groupoid* of Λ, and i is the *canonical functor*.

Λ and $\mathcal{G}(\Lambda)$ have the same vertices (objects), and the restriction $i|\Lambda^0$ is the identity map. Since $\mathcal{G}(\Lambda)$ is a groupoid, its hom-set $v\mathcal{G}(\Lambda)v$ at any vertex $v \in \Lambda^0$ is a group.

Definition 3.2. The *fundamental group* of Λ at a vertex v is $\pi(\Lambda, v) := v\mathcal{G}(\Lambda)v$.

By construction, for any morphism $T\colon \Lambda \to \Omega$ between k-graphs, there is a unique morphism T_* making the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{i} & \mathcal{G}(\Lambda) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T_*} \\ \Omega & \xrightarrow[i]{} & \mathcal{G}(\Omega) \end{array}$$

commute, because $iT\colon \Lambda \to \mathcal{G}(\Omega)$ is a functor from Λ to a groupoid. It follows quickly from this that $\Lambda \mapsto \mathcal{G}(\Lambda)$ is functorial from k-graphs to groupoids. In fact, this functor extends readily to small categories, and then it becomes a left adjoint for the inclusion functor from groupoids to small categories.

Since Λ generates $\mathcal{G}(\Lambda)$ as a groupoid, it is easy to see, for example, that Λ is connected if and only if its fundamental groupoid $\mathcal{G}(\Lambda)$ is (a small category $\mathcal{C}$ is *connected* if the equivalence relation on $\mathcal{C}^0$ generated by $\{(u, v) | u\mathcal{C}v \neq \emptyset\}$ is $\mathcal{C}^0 \times \mathcal{C}^0$).

4. Presentation of k-graphs

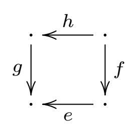
In this section we shall give a presentation of any k-graph Λ as a relative path category.

Notation and Terminology. We need an unambiguous notation for the standard basis vectors in $\mathbb{N}^k$ — the usual e_i is not good for us because it conflicts with typical notation for edges of a graph — we use $n_1, \dots, n_k$ for the standard basis vectors.

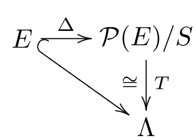
Definition 4.1. The 1-*skeleton* of Λ is the graph E with $E^0 = \Lambda^0$ and $E^1 = \bigcup_{i=1}^k \Lambda^{n_i}$, and range and source maps inherited from Λ.

Definition 4.2. If e and f are composable edges in E with orthogonal degrees, the factorization property of the degree functor gives unique edges g and h such that ef and gh are both *edge-path factorizations* of the same element of Λ, with the degrees interchanged: $d(e) = d(h)$ and $d(f) = d(g)$. The ordered pair (ef, gh) is a relation on E which we call a *commuting square*. We let S denote the set of all commuting squares of E.

We typically visualize a commuting square (ef, gh) as



Proposition 4.3. *Let Λ be a k-graph, with 1-skeleton E and commuting squares S. Then there is a unique isomorphism T making the diagram*



commute.

Proof. The inclusion $E \hookrightarrow \Lambda$ is a diagram satisfying S, so there is a unique *functor* T making the diagram commute. In fact, T is the unique factorization through the relative path category $\mathcal{P}(E)/S$ of the unique functor $R\colon \mathcal{P}(E) \to \Lambda$ extending the diagram $E \to \Lambda$. Thus T is surjective.

For $\lambda, \mu \in \mathcal{P}(E)$, we have $R(\lambda) = R(\mu)$ if and only if λ and μ are edge-path factorizations of the same element of Λ. This gives an equivalence relation on $\mathcal{P}(E)$. It follows from the factorization property of the degree functor and a routine induction argument that S generates this equivalence relation, and the result follows. □

5. Presentation of fundamental groupoids

Let Λ be a k-graph. In this section we shall give a presentation of the fundamental groupoid $\mathcal{G}(\Lambda)$ as a relative path category. The value of this is that while $\mathcal{G}(\Lambda)$ is created specifically to have universal properties, we have more tools to effectively compute with, and prove things about, relative path categories.

Schubert would construct the fundamental groupoid of a small category such as Λ as a relative path category of a certain augmented version of the underlying graph of Λ, obtained by adjoining inverse edges with appropriate relations. In our case, Proposition 4.3 already gives a presentation of the k-graph Λ as $\mathcal{P}(E)/S$; it seems natural to want to work directly with the 1-skeleton E. In Proposition 5.5 below, we obtain the fundamental groupoid of a k-graph Λ as a relative path category of an augmented version of the 1-skeleton E, namely:

Definition 5.1. For each edge $e \in E^1$ introduce a new edge e^{-1} with source and range interchanged: $s(e^{-1}) = r(e)$ and $r(e^{-1}) = s(e)$, and put $E^{-1} = \{e^{-1} \mid e \in E^1\}$. Next let $E^+ = E \cup E^{-1}$. More precisely, the graph E^+ has edges $E^1 \cup E^{-1}$ and vertices E^0. We call E^+ the *augmented graph* of E, and E^{-1} the *inverse edges*.

It is notationally convenient to write $(e^{-1})^{-1}$ to mean e, because then we can write e^{-1} for any $e \in E^1 \cup E^{-1}$.

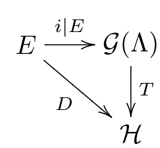
Definition 5.2. Let C denote the set of relations for E^+ of the form $(e^{-1}e, s(e))$ for $e \in E^1 \cup E^{-1}$. We call C the set of *cancellation relations for E*.

Observation 5.3. *Every diagram D of type E in a groupoid $\mathcal{H}$ extends uniquely to a diagram D^+ of type E^+ in $\mathcal{H}$ satisfying C: just put*

$$D^+(e^{-1}) = D(e)^{-1} \qquad \text{for } e \in E^1.$$

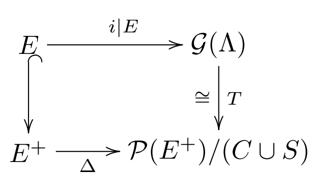
We also need to know that the canonical functor $i\colon \Lambda \to \mathcal{G}(\Lambda)$ gives rise to yet another universal property:

Observation 5.4. *Let Λ be a k-graph with 1-skeleton E and canonical functor $i\colon \Lambda \to \mathcal{G}(\Lambda)$. Then for any diagram D of type E in a groupoid $\mathcal{H}$ there exists a unique morphism T making the diagram*



commute.

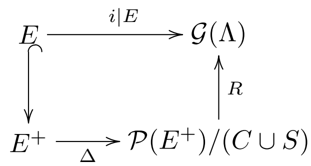
Theorem 5.5. *Let Λ be a k-graph, with 1-skeleton E, commuting squares S, augmented graph E^+, and cancellation relations C. Let $\Delta\colon E^+ \to \mathcal{P}(E^+)/(C \cup S)$ be the canonical diagram of type E^+ satisfying $C \cup S$. Then there is a unique isomorphism T making the diagram*



commute.

Proof. Note that by universality there is a unique *morphism* T making the diagram commute, since $\Delta|E$ is a diagram of type E, satisfying S, in the groupoid $\mathcal{P}(E^+)/(C \cup S)$; we must show T is an isomorphism. Because T is bijective on units, and $\Delta(E)$ generates $\mathcal{P}(E^+)/(C \cup S)$ as a groupoid, T is surjective.

It suffices to show that there exists a morphism R making the diagram

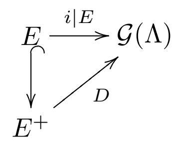


commute, for if we have such an R then

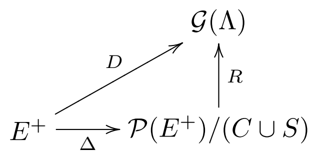
$$RTi|E = R\Delta|E = i|E,$$

so $RT = \mathrm{id}$ by universality, hence T is injective.

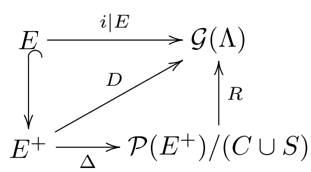
Since $i|E$ is a diagram of type E in a groupoid, by Observation 5.3 it extends uniquely to a diagram D of type E^+ satisfying C, so that the diagram



commutes. Because $i|E$ satisfies S, the diagram D satisfies $C \cup S$. Thus there exists a unique morphism R making the diagram



commute. Combining diagrams, because both triangles of the diagram



commute, so does the outer rectangle, as desired. □

Let's look a little more closely at the case $k = 1$: there are no commuting squares, and a 1-graph Λ is not only isomorphic to, but in fact coincides with, the path category $\mathcal{P}(E)$ of its 1-skeleton E. The above theorem gives us an isomorphism

$$\mathcal{G}(\Lambda) \cong \mathcal{P}(E^+)/C.$$

The left-hand side is by definition the fundamental groupoid of the path category $\mathcal{P}(E)$, and is constructed without direct reference to E itself. On the other hand, the right-hand side is what we expect the fundamental groupoid of a graph to be: words in edges and their inverses, modulo cancellation of subwords like ee^{-1} or $e^{-1}e$. This latter deserves to be associated formally with the graph:

Definition 5.6. Let E be a graph, with augmented graph E^+ and cancellation relations C. We define the *fundamental groupoid of E* to be

$$\pi(E) := \mathcal{P}(E^+)/C.$$

$E \mapsto \pi(E)$ is a left adjoint for the underlying graph functor on groupoids. We can use this to give another interpretation of Theorem 5.5: let Λ be an arbitrary k-graph, with 1-skeleton E, commuting squares S, augmented graph E^+, and cancellation relations C. Relative path categories can be formed iteratively ("in stages"):

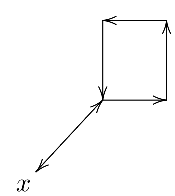
$$\mathcal{G}(\Lambda) \cong \mathcal{P}(E^+)/(C \cup S) \cong \big(\mathcal{P}(E^+)/C\big)/S = \pi(E)/S,$$

where we continue our abuse of notation by using "$/S$" to mean we take the quotient by the equivalence relation generated by S. Note that the commuting squares S pass unaffected into the relative path category $\mathcal{P}(E^+)/C$.

6. Geometric interpretation

The fundamental groupoid $\mathcal{G}(\Lambda)$ of a k-graph Λ is a purely combinatorial object — even in the case $k = 1$ it is *not* isomorphic to the (classical) fundamental groupoid of the topological realization of the graph, because $\mathcal{G}(\Lambda)$ does not have enough units. However, it is almost obvious that the (combinatorial) fundamental groupoid of a graph is isomorphic to the *reduction* of the topological fundamental groupoid to the subset of the unit space consisting of the vertices of the graph. If the graph is connected, then the fundamental *groups* of the 1-graph and of the topological realization *are* isomorphic. Something like this persists for $k > 1$. We only sketch the construction — it's primarily folklore from algebraic topology. For simplicity, assume Λ is connected. Let E be the 1-skeleton of Λ, with commuting squares S. Then let $|E|$ be the usual geometric realization as a topological space, namely as a 1-cell complex where a 1-cell is attached for each edge of E. Since the k-graph Λ is connected, so is the topological space $|E|$.

Next, for each commuting square $(\alpha, \beta) \in S$, attach a 2-cell to $|E|$ along the path $\alpha\beta^{-1}$ (using the obvious identification of elements of $\pi(E)$ with certain continuous paths in $|E|$). Doing this for all commuting squares, we get a 2-cell complex X. Now fix a vertex $x \in \Lambda^0$. It is a standard fact from algebraic topology that the fundamental group $\pi_1(X, x)$ of X at the point x is isomorphic to the quotient of the fundamental group $\pi_1(|E|, x)$ by the normal subgroup generated by loops of the form



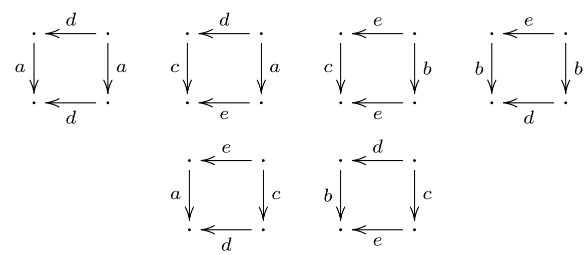
i.e., follow any path from x to a corner of one of the attached 2-cells, then go around the boundary of this 2-cell, and finally "retrace your steps" back to x. The boundary of the 2-cell is of the form $\alpha\beta^{-1}$ for some commuting square $(\alpha, \beta) \in S$. It follows from Theorem 5.5, then, that this quotient group is also isomorphic to the (combinatorial) fundamental group $\pi(\Lambda, x)$. Thus $\pi(\Lambda, x)$ is isomorphic to the fundamental group of the 2-cell complex X. Also, the fundamental groupoid $\mathcal{G}(\Lambda)$ is isomorphic to an appropriate reduction of the fundamental groupoid of X.

We could proceed to attach a 3-cell for each "commuting cube" in Λ, and so on up to dimension k, but this would have no effect on the fundamental group of the cell complex.

7. Failure of embedding

Since the canonical functor $i\colon \Lambda \to \mathcal{G}(\Lambda)$ *is* injective for 1-graphs (folklore), and k-graphs are higher-dimensional generalizations, it's natural to ask: Is i injective for every k-graph Λ? It seems that the standard injectivity result in the category literature involves a hypothesis which Schubert [13] would call "Λ admits a calculus of left fractions" (some other writers would say "Λ is localizing"), which requires that for any $\alpha, \beta \in \Lambda$ with $s(\alpha) = s(\beta)$ there exist $\gamma, \delta \in \Lambda$ such that $\gamma\alpha = \delta\beta$. This is no help for Λ, since a k-graph does not typically admit a calculus of left fractions. In fact, that's a good thing, because the answer to the injectivity question for k-graphs is no! We shall give counterexamples in this section. In fact, our counterexamples will show that injectivity fails as soon as $k \geq 2$.

Example 7.1. The following 2-graph is the simplest example we could find where the canonical functor $i\colon \Lambda \to \mathcal{G}(\Lambda)$ is noninjective. It has a single vertex, 2 edges of degree $(1,0)$, and 3 edges of degree $(0,1)$, with commuting squares



Then in the fundamental group G we have:

$$a = dad^{-1} = de^{-1}c = dbe^{-1} = b.$$

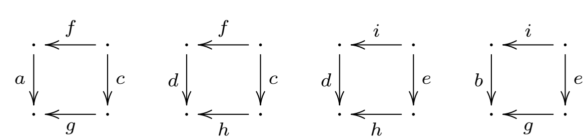
In fact, it follows that in G we have $a = b = c$ and $d = e$.

Another interpretation of this example: we have a group G with generators a, b, c, d, e and relations

$$ad = da, \quad cd = ea, \quad ce = eb, \quad be = db, \quad ae = dc, \quad bd = ec,$$

and the conclusion is that the generators b, c, e, and the last two relations, are redundant and the group G is isomorphic to $\mathbb{Z}^2$ (with free commuting generators a, d).

Example 7.2. In the above example, we only used the top four commuting squares, and we tried to minimize the number of edges, which required us to use only 1 vertex. Here is an example with 4 vertices:



where again $i(a) = i(b)$ in the fundamental groupoid. To embed these diagrams in a 2-graph, we have to add more diagrams to ensure that they give a bijection between the vertical-horizontal edge-paths and the horizontal-vertical ones (see [7, Section 6]).

We could further generalize the previous examples by replacing the edges $a, b, \dots$ by arbitrary paths $\alpha, \beta, \dots$. Then the squares are interpreted as commuting diagrams in the k-graph — and for emphasis we point out that there is no reason why k must be 2, except that when $k > 2$ the existence of a k-graph with the desired properties becomes somewhat more delicate: by [3, Theorem 2.1 and Remark 2.3] we must also check consistency of the commuting cubes ("associativity").

From a k-graph Λ we can form k "component" 1-graphs

$$\Lambda_i := d^{-1}(\mathbb{N}n_i), \quad i = 1, \dots, k,$$

where n_i is the ith standard basis vector. The above examples show that in general even the component 1-graphs do not embed faithfully in the k-graph fundamental groupoid $\mathcal{G}(\Lambda)$.

Each component 1-graph Λ_i *does* embed faithfully in *its* fundamental groupoid $\mathcal{G}(\Lambda_i)$. Thus, we see that in general the k-graph fundamental groupoid $\mathcal{G}(\Lambda)$ is a kind of "twisted product" of nonfaithful copies of the component 1-graph fundamental groupoids $\mathcal{G}(\Lambda_i)$.

8. Lambda bar

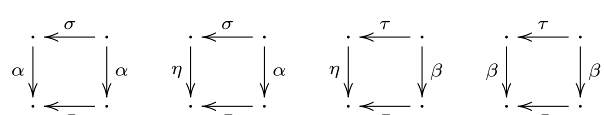
Let Λ be a k-graph, $d \colon \Lambda \to \mathbb{N}^k$ the degree functor, and $i : \Lambda \to \mathcal{G}(\Lambda)$ the canonical functor into the fundamental groupoid. The universal property of i tells us that there is a functor $d' : \mathcal{G}(\Lambda) \to \mathbb{Z}^k$ such that $d' \circ i = d$. Thus the restriction $\overline{d}$ of d' to the image $\overline{\Lambda} := i(\Lambda)$ is a functor with values in $\mathbb{N}^k$. We would very much like to know:

Question 8.1. Is $(\overline{\Lambda}, \overline{d})$ a k-graph?

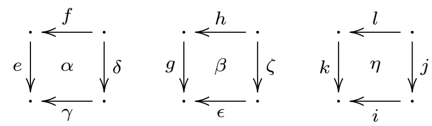
The point is that the canonical functor $\overline{i}$ of $\overline{\Lambda}$ into $\mathcal{G}(\overline{\Lambda}) = \mathcal{G}(\Lambda)$ is injective, so that if the answer is affirmative, we could for many purposes replace Λ by a k-graph which has the same fundamental groupoid and has i injective. Fortunately, our general theory does not depend upon the answer to Question 8.1. Nevertheless, we explore this question a little further: How could we search for a counterexample? It suffices to find Λ containing elements α and β and factorizations $\alpha = \gamma\delta$ and $\beta = \epsilon\zeta$ with

$$i(\alpha) = i(\beta), \quad d(\gamma) = d(\epsilon), \quad \text{and} \quad i(\gamma) \neq i(\epsilon).$$

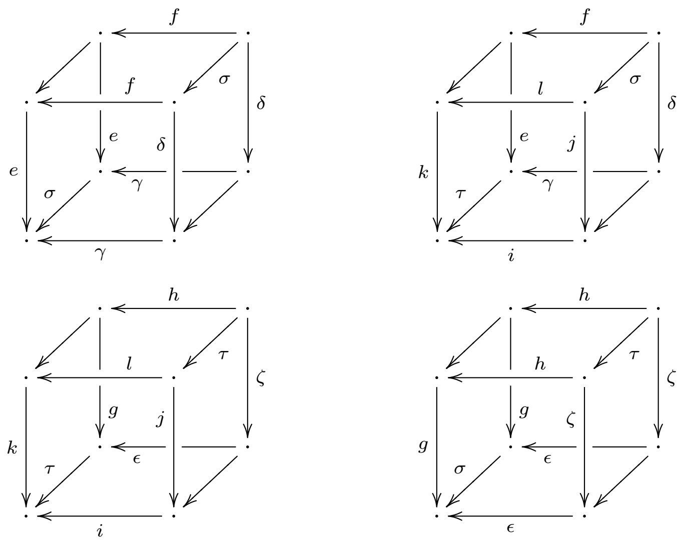
With an eye to keeping things as simple as possible, suppose γ and ϵ are edges with degree $(1, 0, 0)$, and δ and ζ have degree $(0, 1, 0)$, so that α and β have degree $(1, 1, 0)$ (we'll need the third coordinate shortly). The techniques we used to find the examples in the preceding section show that if Λ has commutative diagrams of the form



then $i(\alpha) = i(\beta)$. Still striving for simplicity, let's consider the possibility that σ and τ are edges, say with degree $(0, 0, 1)$ (a little thought reveals that if σ is an edge with degree the same as either $d(\gamma)$ or $d(\delta)$ then $\alpha = \beta$, which is not what we want). If we label the commuting squares for α, β, and η as



then the 4 relations give commuting cubes



To continue this search for a counterexample would require answers to the following questions, which we have so far been unable to supply:

(i) Can the above cubes be completed to a 3-graph?
(ii) Is it true that the relations do not imply $i(\gamma) = i(\epsilon)$?

References

[1] M. Bridson and A. Haefliger, *Metric spaces of nonpositive curvature*, Grundlehren der Mathematischen Wissenschaften, **319**, Springer-Verlag, Berlin, 1999, MR 1744486 (2000k:53038), Zbl 0988.53001.

[2] R. Brown, *Elements of modern topology*, McGraw–Hill, New York–Toronto–London, 1968, MR 0227979 (37 #3563), Zbl 0159.52201.

[3] N.J. Fowler and A. Sims, *Product systems over right-angled Artin semigroups*, Trans. Amer. Math. Soc. **354** (2002) 1487–1509, MR 1873016 (2002j:18006), Zbl 1017.20032.

[4] J.L. Gross and T.W. Tucker, *Topological graph theory*, Wiley, New York, 1987, MR 0898434 (88h:05034), Zbl 0621.05013.

[5] P.J. Higgins, *Notes on categories and groupoids*, Van Nostrand Rienhold Mathematical Studies, **32**, Van Nostrand Reinhold, London–New York–Melbourne, 1971, MR 0327946 (48 #6288), Zbl 0226.20054.

[6] A. Kumjian, *The universal cover of a higher rank graph*, in progress.

[7] A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000) 1–20, MR 1745529 (2001b:46102), Zbl 0946.46044.

[8] A. Kumjian, D. Pask and I. Raeburn, *Cuntz–Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998) 161–174, MR 1626528 (99i:46049), Zbl 0917.46056.

[9] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, **5**, Springer-Verlag, New York–Berlin, 1971, MR 0354798 (50 #7275), Zbl 0232.18001.

[10] R.C. Lyndon and P.E. Schupp, *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **89**, Springer-Verlag, Berlin–New York, 1977, MR 0577064 (58 #28182), Zbl 0368.20023.
[11] D. Pask, J. Quigg and I. Raeburn, *Coverings of k-graphs*, preprint, 2003.
[12] G. Robertson and T. Steger, *C^*-algebras arising from group actions on the boundary of a triangle building*, Proc. London Math. Soc. **72** (1996) 613–637, MR 1376771 (98b:46088), Zbl 0869.46035.
[13] H. Schubert, *Categories*, Springer-Verlag, New York–Heidelberg, 1972, MR 0349793 (50 #2286), Zbl 0253.18002.

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