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Recursion in Curve Geometry

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Abstract. Recursion schemes are familiar in the theory of soliton equations, e.g., in the discussion of infinite hierarchies of conservation laws for such equations. Here we develop a variety of special topics related to curves and curve evolution in two and three-dimensional Euclidean space, with *recursion* as a unifying theme. The interplay between curve geometry and soliton theory is highlighted.

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1. Introduction

Among soliton equations, the *filament model* (FM), $\gamma_t = \gamma_s \times \gamma_{ss}$, is particularly simple in form, and easy to interpret geometrically. FM describes a curve $\gamma(s,t)$ evolving in three-dimensional space E^3, and arose as a model of thin vortex tubes in ideal three-dimensional fluids. (In this context, FM is generally known as the *localized induction equation* or the *Betchov Da Rios equation*—see [Ri] for historical

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background.) As we intend to illustrate, the structure of FM lends insight and a rich set of examples to the study of curves; geometry repays the debt, providing a setting for an elementary demonstration of some of the basic "miracles" of soliton theory, in which many computations related to FM gain simple geometric meaning.

For soliton equations, the associated infinite hierarchies of commuting Hamiltonian flows, conserved variational integrals, and explicitly computable "soliton solutions" are closely related, basic elements of integrable structure. In particular, the *FM recursion scheme*, Equation 1, yields a sequence of differential operators, $X_0,\ X_1,\ \ldots,\ X_n, \ldots$, such that the above Hamiltonian flows are defined by the PDE's $\gamma_t = X_n[\gamma]$, and the stationary equations, $0 = X_n[\gamma]$, describe (initial conditions for) the soliton solutions to FM. While recursion schemes are typically "derived" in soliton theory from (presumably) more fundamental principles, Equation 1 is adopted here as *starting point*; the latter is simpler-looking than better-known recursion schemes in soliton theory, leads more transparently to closed form solution, and yields formulas which may be directly and systematically applied to several interesting topics in curve geometry.

Nevertheless, we begin §2 with a brief motivation of the FM recursion scheme itself, via the condition of *unit speed parametrization*, $\langle \gamma_s, \gamma_s \rangle = 1$. We proceed to develop basic results on the solution to the recursion scheme (see Theorem 2), representing the general solution as a formal series of vectorfields (or vector-valued operators), $X = \sum_{n=0}^{\infty} \lambda^n X_n$, starting with $X_0 = -\gamma_s$, and depending on a sequence of 'constants of integration'. As it turns out, the *length* of X is independent of s, and normalization by the assumption of unit length—extending the unit speed condition on γ itself—uniquely determines a special solution Y to the recursion scheme. This normalization device, which conveniently fixes all constants of integration (without reference to boundary conditions or any analytic machinery), is used repeatedly throughout the paper, beginning with the description of *planar* and *binormal* FM subhierarchies along planar curves (see Corollary 3).

In §3 we consider *statics* of curves belonging to *the soliton class*, $\Gamma = \{\gamma :$ some X_n vanishes along $\gamma\}$, beginning with formulas for *first integrals*, *Killing fields*, and expression of Euclidean coordinates of $\gamma \in \Gamma$ *by quadrature*, in terms of FM fields X_n (see Theorem 4). We also observe that Y *converges* for such curves, suggesting more geometrical interpretations of Y—e.g., as a canonical extension $Y[T]$ of closed spherical curves T to spherical mappings of a cylinder. Next, we demonstate the exceptionally good fit between the soliton class Γ and Frenet theory—using both *standard* and *natural* Frenet systems. The latter introduce into the picture a second parameter, σ, which ultimately (in §5) will be identified with the *spectral parameter* in the standard sense of soliton theory. The lower order examples (beginning with *lines*, *helices*, *elastic rods* and *buckled rings*) illustrate how Γ provides integrable geometric variational problems and (finite dimensional) Hamiltonian systems—indeed, integrable physical models. Here we present basic results on the soliton class, partly with a view towards the broader potential of Γ as a significant class of curves; briefly, Γ is large enough to represent arbitrary geometrical and topological complexity, yet highly structured and admitting a variety of explicit constructions.

In §4, we take up curve *dynamics*, especially the PDE's $\gamma_t = X_n[\gamma]$ of the FM hierarchy. There are brief discussions of *non-stretching motions* in general, of the Hamiltonian nature of FM and the *FM constants of motion*, and of the *congruence solutions* (special soliton solutions) associated to the soliton class Γ. The relationships to the *non-linear Schrödinger* (NLS) and *modified Korteweg-de Vries* (mKdV) equations are derived as a corollary to the *variation of natural curvatures formula* (Theorem 14), which gives the *FM recursion operator* a role in the geometry of curves. We then proceed to consider equations which preserve *planar*, *spherical*, and *constant torsion curves*, relating all these to the (real) mKdV hierarchy, and the last to *pseudospherical surfaces* and the *sine-Gordon equation*. A closer look at the constant torsion-preserving flows leads to a slight genereralization of the FM recursion scheme, in which the parameters λ and σ may be allowed to interact; a specialization yields a description of the FM vectorfields, in terms of *covariant constancy* of a series X^σ.

Finally, Section 5 makes the bridge between the special topics on curves and the more widely known formalism of soliton theory. First we recast the natural Frenet system for curves in R^3 in terms of the standard spectral problem for the non-linear Schrödinger equation in the $SU(2)$ setting. Then we recall the technique of *differentiation with respect to the spectral parameter* (due to Sym and Pohlmeyer [Sym]), which produces unit speed curves from a set of *eigenfunctions*. After briefly deriving the NLS hierarchy from the *zero curvature condition*, we explain the equivalence between the FM and NLS recursion schemes—in a word, the two are related like "body" and "space" coordinates. The simple conclusion deserves amplification, for several reasons. First, another geometric interpretation of the spectral parameter (as the inverse of a spherical radius—see [D-S]) has been proposed; however, it does not admit the same clean translation between the linear systems underlying FM and NLS. Second, in the context of curve geometry, natural frames are generally considered only with $\sigma = 0$—these appear to suffice for many purposes, but the discussion here suggests valuable information may be lost by so specializing too quickly.

Which brings us back to the main technique, the common thread of the paper; for the *spectral parameter* and *recursion* are two faces of a coin—continuous and discrete aspects of an underlying symmetry, a key degree of freedom in a highly structured system. The spectral parameter and the recursion are the *slip* and the *rattle* by which the inner workings of the mechanism are heard.

2. The FM recursion scheme

In the Frenet theory of curves, the notion of *arclength-parametrization* is essential. Though one can compute expressions for *curvature* κ and *torsion* τ of a curve γ using a more general parametrization, these quantities give very limited information about γ, unless referred to a unit speed parameter. Ironically, in elementary mathematics, arclength-parametrization is mostly an abstraction—one rarely encounters it in the flesh! Happily, soliton theory ultimately provides a large supply of arclength-parametrized curves; especially, ways to deform a given such curve to obtain many others.

Turning things around, we wish to motivate the FM recursion scheme by borrowing a lemma of non-stretching curve dynamics (see Section 4.1):

Lemma 1. *The curve-speed* $v = \|\frac{\partial\gamma}{\partial u}\| = \frac{\partial s}{\partial u} \neq 0$ *of an evolving regular curve* $\gamma(u,t)$ *is preserved—*$v(u,t)$ *is independent of* t*—if and only if* $W = \frac{\partial\gamma}{\partial t}$ *satisfies the* *condition $\langle T, \partial W\rangle = 0$. Here, $T = \frac{\partial\gamma}{\partial s}$ is the unit tangent vector, and $\partial = \nabla_T$ is the covariant derivative along γ.*

Proof. The lemma is valid, as stated, in a Riemannian manifold. In the present (Euclidean) context, we simply use partial derivative notation: $(v^2)_t = \frac{\partial}{\partial t}\langle\gamma_u, \gamma_u\rangle = 2\langle\gamma_u, \gamma_{ut}\rangle = 2\langle\gamma_u, W_u\rangle = 2v^2\langle T, \partial W\rangle$. □

To paraphrase, W is a *locally arclength-preserving* (LAP) vectorfield along γ if and only if W satisfies $JX = \partial W$, for some vectorfield X; here, $J = T\times$ is the operator which takes cross product with the unit tangent.

The most obvious way to satisfy this condition is to let W be the unit tangent vector itself, $W = T$. Note that the corresponding motion of γ is just *slipping* of γ along itself (shifting of parameter), without change of shape or position. Of course, we would like to describe more interesting non-stretching motions. To do so, we introduce the *FM recursion scheme*,

$$JX_n = \partial X_{n-1} \tag{1}$$

Here, the recursion starts with $X_0 = -\gamma_s = -T$. Assuming we can determine $X_1, X_2, \ldots,$ we should thus have a sequence of increasingly complicated non-stretching motions (note X_n depends on n derivatives of $T = \gamma_s$).

We now show how to compute the X_n from Equation 1. Since $J^2 = -Id$ on normal vectorfields, (1) implies

$$X_n = f_n T - J\partial X_{n-1}, \tag{2}$$

for some f_n. As it turns out, there are *two* ways to compute f_n in terms of $X_1, X_2, \ldots, X_{n-1}$. This is a key fact.

First, replacing n by $(n+1)$ in Equation 1, we obtain further information about X_n; namely, $\langle T, \partial X_n\rangle = 0$, so $\partial f_n = \partial\langle T, X_n\rangle = -\langle\partial X_0, X_n\rangle$, i.e.,

$$\partial f_n = \langle X_1, JX_n\rangle \tag{3}$$

Since the normal part of X_n is already "known", antidifferentiation of (3) yields f_n, uniquely, up to an arbitrary constant of integration. By this approach, one could compute X_1, X_2, X_3, explicitly, with the help of "good luck": at each step, the required antiderivative, f_n, turns out to be computable in closed form.

For the second approach, it's convenient to consider formal power series, $X = \sum_{n=0}^{\infty}\lambda^n X_n$, and to make use of the natural extensions to such series of the vector operations ∂, J, $\langle\,,\rangle$, etc. For instance, we can write $JX = \sum_{n=1}^{\infty}\lambda^n JX_n$, and $\lambda\partial X = \sum_{n=0}^{\infty}\lambda^{n+1}\partial X_n = \sum_{n=1}^{\infty}\lambda^n\partial X_{n-1}$. Evidently, (1) can be rewritten as

$$JX = \lambda\partial X \tag{4}$$

This invites the product rule: $\lambda\partial\langle X, X\rangle = 2\langle\lambda\partial X, X\rangle = 2\langle JX, X\rangle = 0$, by skew-adjointness of J. In other words,

$$\langle X, X\rangle = p(\lambda), \tag{5}$$

where $p(\lambda) = 1 + \sum_{n=1}^{\infty} C_n\lambda^n$ is a series in λ, with coefficients C_n which do not depend on s. Thus, for fixed real λ, X describes a spherical curve (assuming convergence). Note that the λ^n term of (5) is $\sum_{k=0}^{n}\langle X_k, X_{n-k}\rangle = C_n$; hence, for

$n = 2, 3, \ldots,$

$$2f_n = -C_n + \sum_{k=1}^{n-1} \langle X_k, X_{n-k} \rangle \tag{6}$$

This equation is clearly the preferred way to compute f_n; in fact, comparison with Equation 3 explains the "perfect derivative phenomenon" by way of the following interesting identity (whose significance is explained in §4.1):

$$\partial \sum_{k=1}^{n-1} \langle X_k, X_{n-k} \rangle = 2\langle X_1, JX_n \rangle \tag{7}$$

We will often use the convenient normalization $p(\lambda) = 1$—all C_n are zero—and denote by $Y = \sum_{n=0}^{\infty} \lambda^n Y_n$ the resulting series (which corresponds to the "obvious" choices of antiderivatives in the first approach). For convenient reference, we list the first few terms before summarizing the main conclusions of this section:

$$\begin{aligned} Y_0 &= -\gamma_s, \\ Y_1 &= \gamma_s \times \gamma_{ss}, \\ Y_2 &= \frac{3}{2}\langle \gamma_{ss}, \gamma_{ss} \rangle \gamma_s + \gamma_{sss}, \\ Y_3 &= \langle \gamma_s \times \gamma_{ss}, \gamma_{sss} \rangle \gamma_s - \gamma_s \times \gamma_{ssss} - \frac{3}{2}\langle \gamma_{ss}, \gamma_{ss} \rangle \gamma_s \times \gamma_{ss} \end{aligned}$$

Theorem 2. *Let $X = \sum_{n=0}^{\infty} \lambda^n X_n$ satisfy $JX_n = \partial X_{n-1}$, with $X_0 = -\gamma_s$. Then $\langle X, X \rangle = p(\lambda)$ does not depend on s. Further,*

a) *The normalized solution, $Y = \sum_{n=0}^{\infty} \lambda^n Y_n$, is given inductively by*

$$Y_n = \left(\frac{1}{2} \sum_{k=1}^{n-1} \langle Y_k, Y_{n-k} \rangle\right) T - J\partial Y_{n-1}, \tag{8}$$

which uniquely defines $Y_n[\gamma]$ as an $(n+1)^{st}$-order differential operator on regular curves γ.

b) *In the general case, $p(\lambda) = 1 + \sum_{n=1}^{\infty} C_n \lambda^n$, X may be written*

$$X = \sum_{n=0}^{\infty} \lambda^n \sum_{k=0}^{n} A_{n-k} Y_k = \left(\sum_{i=0}^{\infty} A_i \lambda^i\right)\left(\sum_{j=0}^{\infty} \lambda^j Y_j\right) = \sqrt{p(\lambda)}\; Y \tag{9}$$

c) *For $1 \leq m < n$, the following derivative identity holds:*

$$\partial \sum_{k=1}^{n-m} \langle X_{m+k-1}, X_{n-k} \rangle = 2\langle X_m, JX_n \rangle \tag{10}$$

Proof. Equation 8 just combines Equations (2) and (6), with $C_n = 0$. Note that, in terms of Euclidean coordinates, $\gamma = (x^1, x^2, x^3)$, each component of Y_n is a polynomial in the $3(n+1)$ quantities, $\partial^j x^i$, $i = 1, 2, 3$, $j = 1, \ldots, (n+1)$; this *locality result* is an immediate but fundamental consequence of Equation 8.

For part b), note that Equations (2) and (3) imply that the general solution to Equation 4 has the form $X = \sum_{n=0}^{\infty} \lambda^n X_n = \sum_{n=0}^{\infty} \lambda^n \sum_{k=0}^{n} A_{n-k} Y_k$. The remaining formulas for X now follow by formal multiplication and the normalization of Y. In particular, $p(\lambda) = (\sum_{k=0}^{\infty} A_k \lambda^k)^2$, i.e., the "integration constants" $A_1, A_2, \ldots,$ are related to the C_m by $C_m = \sum_{k=0}^{m} A_k A_{m-k}$ (with $A_0 = 1$). We remark that, in the expansion of $\langle X, X\rangle = p(\lambda)$, only the $\langle Y_0, Y_0\rangle$ terms contribute; the remaining terms must cancel for the result to be independent of s.

Part c) directly generalizes Equation 7, and can be proved as follows. For $m = 0, 1, \dots$, let $X^{(m)}$ denote the *shifted* series, $X^{(m)} = \sum_{n=0}^{\infty} \lambda^n X_{n+m}$. Noting $J(X^{(m)} - X_m) = \lambda\partial X^{(m)}$, one obtains $\lambda\partial\langle X^{(m)}, X^{(m)}\rangle = 2\langle X_m, JX^{(m)}\rangle$. The λ^{n-m}-term yields Equation 10. □

Corollary 3. *Along a planar curve γ, the even fields $Y_{2n}[\gamma]$ are planar, while the odd fields $Y_{2n+1}[\gamma]$ are "binormal" (perpendicular to the plane of γ). Further, the planar subhierarchy, $Y_{2n}[\gamma]$, may be computed inductively by: $Y_{2n+2} = J^2\partial^2 Y_{2n} + f_{2n+2}T$, where $2f_{2n+2} = \sum_{k=1}^{n}\langle Y_{2k}, Y_{2(n+1-k)}\rangle + \sum_{k=0}^{n}\langle \partial Y_{2k}, \partial Y_{2(n-k)}\rangle$, $n = 1, 2, \dots$.*

Proof. Assuming γ is planar, we use induction to prove Y_{2j-1} is binormal and Y_{2j} is planar, for $j = 1, 2, \dots$. Assume this holds for $1 \le j \le n$ (obviously valid when $n = 1$). Then $f_{2n+1} = \frac{1}{2}\sum_{k=1}^{2n}\langle Y_k, Y_{2n+1-k}\rangle = 0$, since each term is the dot product of a planar field with a binormal field. Therefore, $Y_{2n+1} = f_{2n+1}T - J\partial Y_{2n} = -J\partial Y_{2n}$ is binormal, and $Y_{2n+2} = f_{2n+2}T - J\partial Y_{2n+1}$ is planar, and the induction argument is concluded. Further, we can write $Y_{2n+2} = f_{2n+2}T + J\partial J\partial Y_{2n} = f_{2n+2}T + J^2\partial^2 Y_{2n}$, since $(\partial T)\times(\partial Y_{2n}) = 0$. The sum $f_{2n+2} = \frac{1}{2}\sum_{k=1}^{2n+1}\langle Y_k, Y_{2n+2-k}\rangle$ may be split into terms with even and odd indices; applying Equation 1 to the odd (binormal) terms, $\langle Y_{2k+1}, Y_{2(n-k)+1}\rangle = \langle JY_{2k+1}, JY_{2(n-k)+1}\rangle$, yields the given formula. □

We remark that the even and odd parts, $X^e = \frac{1}{2}(X_\lambda + X_{-\lambda})$ and $X^o = \frac{1}{2}(X_\lambda - X_{-\lambda})$, of $X = X_\lambda$ have constant formal dot product, $\langle X^e, X^o\rangle = \frac{1}{4}(p(\lambda) - p(-\lambda))$, vanishing for $p(\lambda)$ even; along planar γ, X^e is then planar and X^o binormal. The corollary will be extended to constant torsion curves, via introduction of the *spectral parameter* (in §4.3, where *all* the main formulas of this section will be generalized).

The FM recursion scheme was considered in earlier work with Ron Perline ([L-P 1]), in terms of a recursion operator (see §4.2). We subsequently found the closed form inductive solution (Equation 8), in collaboration with Annalisa Calini and David Singer. In the present paper, we have adopted a formal power series approach (as in Equations (4), (5), (9)); systematic use of this formalism not only clarifies some technical issues (especially those related to "constants of integration"), but also invites geometric interpretation of the recursion scheme and its solution Y.

3. Statics of soliton curves

3.1. **The soliton class Γ.** The n^{th} *soliton class*, $\Gamma_n = \{\gamma : 0 = X_n\}$, is defined by an n^{th}-order ODE for $T = \gamma_s$, depending on n arbitrary constants: $0 = X_n = \sum_{k=0}^{n} A_{n-k}Y_k$, $A_0 = 1$. For instance, $\Gamma_1 = \{$*straight lines*$\}$, $\Gamma_2 = \{$*helices*$\}$, $\Gamma_3 = \{$*Kirchhoff elastic rods*$\}$, and the closed planar curves in Γ_4 describe *buckled rings* (see Examples 7–10, below). The stationary problems $0 = X_n$ can be formulated also as *geometric variational problems*; e.g., elastic rods are critical for linear combinations of *length*, *total torsion*, and *total squared curvature* (see [L-S 4]). The first two parts of the following theorem provide basic computational tools for soliton curves, while part c) gives geometric meaning to the formal series $Y = Y[\gamma]$.

Theorem 4. *Let $\gamma \in \Gamma_n$ ($\gamma \notin \Gamma_{n-1}$) satisfy $0 = X_n = \sum_{k=0}^{n} A_{n-k}Y_k$. Then*

a) *The following $m-1$ first integrals are also satisfied:*

$$\sum_{k=1}^{n-m} \langle X_{m+k-1}, X_{n-k}\rangle = constant, \qquad m = 1,\ 2,\ \ldots,\ (n-1), \tag{11}$$

b) *The vectorfields X_{n-1} and X_{n-2} are the restrictions to γ of Killing fields on E^3. In fact, X_{n-1} is a translation (constant) field, and X_{n-2} is a screw field; the two fields commute, hence, associate to γ a system of cylindrical coordinates, r, θ, z. As functions along γ, these coordinates satisfy*

$$r^2 = \alpha^{-2}\|X_{n-2}\|^2 - \beta^2, \quad z_s = \alpha^{-1} f_{n-1}, \quad r^2\theta_s = \alpha^{-1}(\beta f_{n-1} - f_{n-2}), \tag{12}$$

where $\alpha = \|X_{n-1}\|$ and $\beta = \alpha^{-2}\langle X_{n-1}, X_{n-2}\rangle$ are constants.

c) *$Y = Y[\gamma]$ converges. In fact, $X = X[\gamma]$ may be assumed to terminate, $p(\lambda) = \langle X, X\rangle$ is a non-vanishing polynomial, and $T(s;\lambda) = -Y = -X/\sqrt{p(\lambda)}$ defines a homotopy of curves in the unit sphere, deforming the tangent indicatrix, $T(s;0) = T(s)$, of γ to the point*(s) *$T(s;\pm\infty) = -(\pm 1)^{n-1}\alpha^{-1}X_{n-1}$, as $\lambda \to \pm\infty$.*

Proof. Part a) follows at once from part c) of Theorem 2. Part b) is established by the following sequence of elementary observations.

i) $0 = JX_n = \partial X_{n-1}$, so $X_{n-1} = constant \neq 0$ i.e., X_{n-1} is the restriction to γ of a *translation* (constant) vectorfield on E^3. Thus, we may set $X_{n-1} = \alpha\partial_z$, where α is the constant $\alpha = \|X_{n-1}\|$.

ii) $\partial(\gamma \times X_{n-1}) = JX_{n-1} = \partial X_{n-2}$, so $X_{n-2} = \gamma \times X_{n-1} + V$, for some constant vector V. In fact, by translating coordinates ($\gamma \mapsto \gamma - \alpha^{-2}X_{n-1} \times V$), we can write $X_{n-2} = \gamma \times X_{n-1} + \beta X_{n-1}$, with $\beta = \alpha^{-2}\langle X_{n-1}, V\rangle = constant$, hence also $\langle X_{n-1}, X_{n-2}\rangle = \alpha^2\beta = constant$. Note that X_{n-2} is the restriction to γ of a *screw field* (translation field plus rotation field) on E^3, with axis ∂_z. Thus we may write $X_{n-2} = \alpha(\beta\partial_z - \partial_\theta)$.

iii) The equation $\|X_{n-2}\|^2 = \alpha^2(\beta^2 + r^2)$ may be regarded as a formula for $r(s)$, the first cylindrical coordinate along γ. Similarly, writing $T = r_s\partial_r + \theta_s\partial_\theta + z_s\partial_z$, we obtain the formulas $\alpha z_s = \alpha\langle T, \partial_z\rangle = \langle T, X_{n-1}\rangle = f_{n-1}$, and $f_{n-2} = \langle T, X_{n-2}\rangle = \beta f_{n-1} - \alpha r^2\theta_s$. Thus, $z(s)$ and $\theta(s)$ are given by quadrature, in terms of $\|X_{n-2}\|^2$, f_{n-1}, f_{n-2}, α, and β.

To prove part c), note that the differential operators $X_1, X_2, \ldots$ are uniquely specified by constants $A_0 = 1,\ A_1, A_2, \ldots,$ according to $X = \sum_{r=0}^{\infty} \lambda^r X_r = \sum_{r=0}^{\infty} \lambda^r \sum_{k=0}^{r} A_{r-k}Y_k$. The theorem assumes $A_1, A_2, \ldots, A_n$ are such that $X_n[\gamma] = 0$. An induction argument shows that $A_{m+1} = \langle T, \sum_{k=0}^{m} A_{m-k}Y_{k+1}\rangle$, $m \geq n$, defines constants $A_{n+1}, A_{n+2}, \ldots$ such that X evaluates to the terminating series $X[\gamma] = \sum_{r=0}^{n-1} \lambda^r X_r[\gamma]$. (The fact that the remaining constants are not taken to be zero points out why the interpretation of the A_k as "constants of integration" requires one to be careful.) Note that $p(\lambda)$ is non-vanishing, since otherwise $X[\gamma] = 0$ for some λ, implying $\gamma \in \Gamma_{n-1}$. The remaining statements now follow easily from Theorem 2, and $X_{n-1} = constant$. □

Note antidifferentiation in part c) yields a regular homotopy, $\gamma(s;\lambda) = \int -Y\,ds$, deforming γ to a straight line as $\lambda \to \pm\infty$. We consider this *canonical straightening process* for soliton curves in [La], where explicit examples are worked out and topological and geometrical behavior are considered. This is an example of a *parametrized family construction*—a recurring theme.

3.2. **FM and Frenet theory.** Next, we recall the *Frenet equations* of classical curve theory: $T_s = \kappa N, \quad N_s = -\kappa T + \tau B, \quad B_s = -\tau N$; here, the *curvature* $\kappa(s) \neq 0$ and *torsion* $\tau(s)$ describe the *shape* of γ, and the tangent $T(s)$, *normal* $N(s)$, and *binormal* $B(s)$ form an (adapted) orthonormal frame along γ. Using these equations, we can write the X_n in the form $X_n = a_n T + b_n N + c_n B$, where $a_n = f_n,\ b_n,\ c_n$ are expressed as polynomials in $\partial^i \kappa, \quad i = 0, 1, \ldots, (n-1)$, and $\partial^j \tau, \quad j = 0, 1, \ldots, (n-2)$. In view of Theorem 4, we therefore have:

Corollary 5. *The Frenet equations for a curve γ in Γ are integrable by quadrature*; $\gamma(s) = (r(s)\cos\theta(s), r(s)\sin\theta(s), z(s))$, *where r and z_s are polynomial in $\kappa(s), \tau(s)$, and derivatives of these functions, while θ_s is rational in the same.*

We will be making even more frequent use of *natural Frenet systems*:

$$T_s = u_1 U_1 + u_2 U_2, \quad (U_1)_s = -u_1 T + \sigma U_2, \quad (U_2)_s = -u_2 T - \sigma U_1, \tag{13}$$

where σ is a constant. The relationship to the classical Frenet system can be written $u_1 + iu_2 = \kappa e^{i\theta}$, and $U_1 + iU_2 = (N + iB)e^{i\theta}$, where $\theta = \int^s \tau(u) - \sigma du$; also, $\kappa^2 = u_1^2 + u_2^2$ and $\tau = u^{-2}(u_1(u_2)_s - u_2(u_1)_s) + \sigma$. While κ, τ and $\{T, N, B\}$ are uniquely defined along a regular space curve γ (with $\kappa \neq 0$), the curvatures u_1, u_2 and frame vectors U_1, U_2 are determined (given σ) only up to multiplication by a complex unit, $e^{i\theta_0}$ – this freedom corresponds to the choice of antiderivative in the above formulas. Bishop [Bi] pointed out the virtues of natural frames (with $\sigma = 0$), including, e.g., the following.

Lemma 6. *The following conditions on a curve $\gamma \subset E^3$ are equivalent:*

a) *γ lies on a sphere of radius $R = 1/c$, and has geodesic curvature κ_g. Here, $c = 0$ is allowed, for the planar case.*
b) *There exists a natural frame along γ having natural curvatures u_1, u_2 with $\sigma = 0$, such that $u_2 = c =$ constant, and $u_1 = \kappa_g$.*
c) *If u_1, u_2 are natural curvatures along γ with $\sigma = 0$, then the function $\psi(s) = u_1(s) + iu_2(s)$ maps into a line in the complex plane, and $\|\psi(s)\|^2 = \kappa_g^2 + c^2$, with c equal to the distance from the line to the origin.*

Proof. If $u_2 = c = constant \neq 0$, then $(\gamma + U_2/c)_s = T - cT/c = 0$, so γ lies on a sphere of radius $1/c$. The rest of the proof is also quite easy. □

So-called *frames of least rotation* (again $\sigma = 0$) have been considered also in the context of computer-aided design (see e.g., [W-J]), where the smoother or more regular behavior of natural frames is an advantage. Presently, natural Frenet systems will be seen to be intimately related to the structure of FM. Here it becomes important to include the *spectral parameter* σ—the reason for the term will be made clear in §5—and to allow *σ-frames* with $\sigma \neq 0$. For the moment, we simply note that σ-frames have distinct topological advantages: while 0-frames along a closed regular curve are generally not periodic, σ-frames realize the (p,q)-*cable construction* producing a new knot from an old knot, in the form $\gamma_{(p,q)} = \gamma + \epsilon U_1$, with $\int_\gamma \sigma(p,q) - \tau ds = 2\pi p/q$ (or one can produce non-cable knots, using larger ϵ). Of course, one can also "desingularize" a planar knot (which the standard Frenet frame cannot do).

Returning now to the FM hierarchy, we can obtain another version of Corollary 5 by applying Theorem 4 to the expressions $X_n = f_nT + g_nU_1 + h_nU_2$; here, g_n and h_n are polynomials in the u_i and their derivatives of order up to $(n-1)$, and f_n is one order lower. This is a good place to observe also that the normalized vectorfields $Y_n = a_nT + b_nN + c_nB = f_nT + g_nU_1 + h_nU_2$ have *homogeneous coefficients* with respect to both types of Frenet systems: a_n, b_n, c_n (respectively, f_n, g_n, h_n) all have *weight* n, each "factor" κ, τ, and $' = \frac{\partial}{\partial s}$ $(u_1, u_2, \sigma$, and $')$ contributing *one*. For example (using $\kappa^2 = u_1^2 + u_2^2$ for brevity):

$$Y_0 = -T,$$

$$Y_1 = \kappa B = -u_2U_1 + u_1U_2,$$

$$Y_2 = \frac{\kappa^2}{2}T + \kappa' N + \kappa\tau B = \frac{\kappa^2}{2}T + u_1'U_1 + u_2'U_2 + \sigma Y_1,$$

$$Y_3 = \kappa^2\tau T + (2\kappa'\tau + \kappa\tau')N + (\kappa\tau^2 - \kappa'' - \frac{\kappa^3}{2})B$$

$$= (u_1u_2' - u_2u_1')T + (u_2'' + u_2\frac{\kappa^2}{2})U_1 - (u_1'' + u_1\frac{\kappa^2}{2})U_2 + 2\sigma Y_2 - \sigma^2 Y_1,$$

$$Y_4 = a_4T + (-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \frac{3}{2}\kappa^2\kappa')N + (-\kappa\tau'' + \kappa\tau^3 - 3(\kappa'\tau)' - \frac{3}{2}\kappa^3\tau)B$$

$$= f_4T - (u_1''' + \frac{3}{2}\kappa^2u_1')U_1 - (u_2''' + \frac{3}{2}\kappa^2u_2')U_2 + 3\sigma Y_3 - 3\sigma^2Y_2 + \sigma^3Y_1$$

In the last term, $a_4 = -\kappa\kappa'' + \frac{1}{2}(\kappa')^2 + \frac{3}{2}\kappa^2\tau^2 - \frac{3}{8}\kappa^4$, and $f_4 = -\frac{1}{2}(\kappa^2)'' + \frac{3}{2}((u_1')^2 + (u_2')^2) - \frac{3}{8}\kappa^4$. In the above formulas one observes a *slipping phenomenon* associated with the spectral parameter σ. This will play a role in later sections.

Example 7. (*Spinning Lines*) $\Gamma_1 = \{\gamma : 0 = X_1 = A_1Y_0 + Y_1\}$ gives at once $0 = \kappa = u_1 = u_2 = A_1$ and $\gamma \in \Gamma_1$ is a straight line. While the classical Frenet system is not defined along γ, σ-frames satisfy $T_s = 0$ and $(U_1 + iU_2)' = -i\sigma(U_1 + iU_2)$. The trigonometric solution $U_1 + iU_2 = e^{-i\sigma s}(U_1(0) + iU_2(0))$ imparts a "spin" to the straight line $\gamma = sT + \gamma(0)$, which allows us to interpret γ as an *asymptotic helix*, as in the next example. Also (as pointed out by Tom Ivey), Bäcklund transformations of spinning lines give *Hasimoto filaments* (described below).

Example 8. (*Helices*) $\Gamma_2 = \{\gamma : 0 = X_2 = A_2Y_0 + A_1Y_1 + Y_2\}$, and $\gamma \in \Gamma_2$ is either a straight line, or satisfies $0 = (-A_2 + \kappa^2/2)T + \kappa_sN + (A_1 + \tau)\kappa B$. So $\gamma \notin \Gamma_1$ has constant curvature κ and torsion $\tau = -A_1$. Equations 12 give $r = \kappa\alpha^{-2}$, $z_s = \tau\alpha^{-1}$, $\theta_s = \alpha$, using $\alpha = \sqrt{\kappa^2 + \tau^2}$ and $\beta = -\tau\alpha^{-2}$. Thus, $\gamma = (x, y, z) = (\kappa\alpha^{-2}\cos\theta, \kappa\alpha^{-2}\sin\theta, \alpha^{-1}\tau s + z_0)$, with $\theta = \alpha s + \theta_0$. Note $X_1 = \tau T + \kappa B = \alpha\partial_z$ is a translation field, and $X_0 = -T$ is a screw field along the helix γ.

For $n = 2$, $\gamma(s; \lambda) = \int -Y\,ds$ turns out to be a homotopy of helices, whose nice behavior at $\lambda = \pm\infty$ completes the family of helices with spinning lines. Specifically, we have $X = X_0 + \lambda X_1 = -T + \lambda\alpha\partial_z$, and $p = 1 - 2\lambda\tau + \lambda^2\alpha^2$, from which we compute the tangent, $T(s; \lambda) = p^{-1/2}(T - \lambda\alpha\partial_z)$, normal, $N(s; \lambda) = N(s)$, curvature, $\kappa(s; \lambda) = p^{-1/2}\kappa$, and torsion, $\tau(s; \lambda) = p^{-1/2}(\lambda\alpha^2 - \tau)$, of the helix $\gamma(s; \lambda) = p^{-1/2}(\gamma(s) - \lambda\alpha s\partial_z)$. The framed curve defined by the *Frenet lift* has a limit at $\lambda = \pm\infty$: it is the spinning line $\gamma(s; \pm\infty) = \mp s\partial_z$, with $\sigma = \pm\alpha$.

Example 9. (*Elastic Rods*) $\Gamma_3 = \{\gamma : 0 = X_3 = A_3Y_0 + A_2Y_1 + A_1Y_2 + Y_3\}$. Reading off normal and binormal components ($\gamma \notin \Gamma_1$), one obtains the pair of equations: $2\kappa_s\tau + \kappa\tau_s + A_1\kappa_s = 0$, and $\kappa\tau^2 - \kappa_{ss} - \frac{\kappa^3}{2} + A_1\kappa\tau + A_2\kappa = 0$. The first integrals are: $\langle X_1, X_2\rangle = \kappa^2(\tau + \frac{A_1}{2}) + A_1A_2$ and $\langle X_2, X_2\rangle = (\kappa_s)^2 + \kappa^2\tau^2 + \kappa^2(\frac{1}{4}\kappa^2 + 2A_1\tau + A_1^2 - A_2) + A_2^2$. (The tangential component of $0 = X_3$ is just the lowest order first integral.) These equations can be solved for κ, τ, in terms of elliptic functions. Combined with Corollary 5, this provides one approach to integration of the equation $X_3 = 0$, to obtain an explicit parametrization $\gamma(s)$ of an elastic rod in terms of elliptic integrals.

Alternatively, the U_1 and U_2 components of the equation $0 = X_3$ give the following system for u_1, u_2: $0 = \partial^2 u_2 + u_2(u_1^2+u_2^2)/2 + (A_1+2\sigma)\partial^1 u_1 - (A_1\sigma+\sigma^2+A_2)u_2$, and $0 = \partial^2 u_1 + u_1(u_1^2+u_2^2)/2 - (A_1+2\sigma)\partial^1 u_2 - (A_1\sigma+\sigma^2+A_2)u_1$. This can be rewritten as a classical Hamiltonian system with two degrees of freedom, $q_i = u_i$, conjugate momenta $p_1 = \partial q_1 - (\frac{1}{2}A_1 + \sigma)q_2$, $p_2 = \partial q_2 + (\frac{1}{2}A_1 + \sigma)q_1$, Hamiltonian $H = \frac{1}{2}\langle X_2, X_2\rangle - (A_1+\sigma)\langle X_1, X_2\rangle = \frac{1}{2}(p_1^2+p_2^2) + \frac{1}{8}(q_1^2+q_2^2)^2 + (\frac{1}{2}A_1+\sigma)(q_2p_1 - q_1p_2) + (\frac{1}{4}A_1^2 - A_2)(q_1^2+q_2^2)/2 + const.$. Further, $K = \langle X_1, X_2\rangle = q_1p_2 - q_2p_1 + const.$ is a constant of motion for this system, which is therefore completely integrable.

The details are too lengthy to include here (see [L-S 4] and [I-S]). However, in the special case of the *Hasimoto filament*, the elliptic functions for curvature and torsion degenerate to $\kappa(s) = 2b \operatorname{sech} bs$, $\tau = \tau_0$, with b and τ_0 arbitrary constants. Further, one easily determines $A_1 = -2\tau$, $A_2 = \alpha = b^2 + \tau^2$, $X_1 = 2\tau T + \kappa B$ (a screw field), and $X_2 = (\frac{\kappa^2}{2} - \alpha)T + \kappa_s N - \tau\kappa B$ (a translation field). Finally, Equations 12 give $r = \kappa/\alpha$, $z_s = \frac{\kappa^2}{2\alpha} - 1 = (2b\alpha^{-1}\tanh bs - s)_s$, $\theta_s = -\tau$.

Example 10. (*Buckled Rings*) For $\gamma \in \Gamma_4$, we have the equation $X_3 = A_3Y_0 + A_2Y_1 + A_1Y_2 + Y_3 = const$. Here we consider only the planar curves in $\Gamma_4(\gamma \notin \Gamma_3)$, and note that *the odd constants A_{2k+1} vanish for planar soliton curves*, as a general proposition (a simple consequence of Corollary 3). Thus, X_3 is the binormal field $X_3 = A_2Y_1 + Y_3 = (A_2\kappa - \kappa_{ss} - \frac{\kappa^3}{2})B = PB$, for some constant $P \neq 0$. The ODE $\kappa_{ss} + \frac{\kappa^3}{2} - A_2\kappa = P$ is precisely the equation for the curvature of an elastic ring buckled under constant pressure P, according to a standard model (see [T-O]). (One may prefer to imagine the cross section of a symmetrically buckled cylindrical pipe under hydrostatic pressure.) In the present case, the first integral $\langle X_3, X_2\rangle$ is trivial (X_2 is planar), and $\langle X_2, X_2\rangle + 2\langle X_3, X_1\rangle = const.$ turns out to be equivalent to the obvious integral, $(\kappa_s)^2 + \frac{\kappa^4}{4} - A_2\kappa^2 - 2P\kappa = c$. Noting $\beta = 0$, Equation 12 gives the pair of equations: $\alpha^2r^2 = 2P\kappa + d$, and $\alpha^2r^2\theta_s = \frac{\kappa^2}{2} - A_2$ (where the integral has been used to simplify the first, and we have set $d = c + A_2^2$). It follows that $\kappa(s)$, $r(s)$, and $\theta(s)$ may be expressed in terms of elliptic functions and integrals; likewise for the closure condition, $\Delta\theta = 2\pi p/q$—a rationality condition for the change in the angle θ over a period of κ. More detailed computations for closed solutions and the related bifurcation problem (with pressure as bifurcation parameter) are given in [L-M-V].

The slipping phenomenon noted above (further illustrated in Example 9) hints at the following basic fact about the soliton class:

Proposition 11. *The class of all σ-natural curvature functions u_i for curves in Γ_n does not depend on σ. Thus, if $\gamma \in \Gamma_n$ has curvature κ and torsion τ, there exists also a curve $\gamma^\sigma \in \Gamma_n$ with curvature κ and torsion $\tau + \sigma$, for any σ. In particular,* *to any planar soliton curve $\gamma = \gamma^0$, we can associate the family of 'planar-like' soliton curves γ^τ with constant torsion τ and the same curvature function.*

It is convenient to defer the proof itself to §4.3, where it follows at once from Proposition 18. (Note, however, the second statement follows from the first, using the above formula for τ in terms of u_i and σ.) The associated *parametrized family construction* for the curves γ^σ will be discussed more explicitly in §5.1. The *planar-like solitons*—helices and Hasimoto filaments are the simplest examples—will play an important role in §4.4.

We remark also that the integrability statement in Example 9 is complementary to (not contained in) the integrability result of Corollary 5. On the other hand, the entire system $0 = X_n$ for γ may be cast as a Hamiltonian system on a cotangent bundle of the form $T^*(E(3) \times R^k)$, where $E(3)$ is the group of Euclidean motions. Using this formulation, the problems up to $0 = X_5$ were exhibited in [L-S 3] as completely integrable Hamiltonian systems in the Liouville sense.

The nice variational and Hamiltonian descriptions of soliton curves lend themselves to detailed computations for curves in $\Gamma = \bigcup \Gamma_n$. Such computations may be found in [L-S 2], [C-I 1], [C-I 2], and [I-S], where issues of closure and knottedness are discussed for the class Γ_3. Whereas knots in Γ_3 are precisely the torus knots, more exotic knots in higher Γ_n have been constructed recently by Calini and Ivey using Bäcklund transformations of Γ_3 knots. In this connection, an interesting open problem is to prove a density result for Γ as a subset of smooth curves (say, *closed* or *asymptotically linear*); in particular, all knot types should be represented in Γ.

4. Dynamics of curves

4.1. **PDE's for curve motion.** We begin by collecting some of the immediate consequences of Theorems 2, 4, Lemma 1, and Corollary 3 for curve dynamics:

Proposition 12. a) *For $n = 0, 1, 2, \ldots$, the equation $\gamma_t = Y_n[\gamma]$ may be regarded as an $(n+1)^{st}$ order polynomial partial differential equation for an evolving unit speed curve, $\gamma(s,t)$. The even equations $\gamma_t = Y_{2n}[\gamma]$ restrict to planar curves.*

b) *Suppose $\gamma(s)$ satisfies $X_{n+1}[\gamma] = 0$. Then γ is an initial curve for a 'translation solution' to $\gamma_t = X_n[\gamma]$. Similarly, suppose γ satisfies $X_{n+2}[\gamma] = 0$. Then γ yields a 'congruence solution' of the equation $\gamma_t = X_n[\gamma]$; i.e., γ evolves by a one-parameter group of rigid motions* (*generally 'screw motion'*).

In the case $n = 1$ of b), the conclusion is that helices translate, and elastic rods perform screw motions, under the evolution $\gamma_t = X_1 = \gamma_s \times \gamma_{ss} - A_1\gamma_s = \kappa B - A_1 T$ (where the constant A_1 depends on the curve). Since the term $A_1 T$ just induces sliding of the curve along itself, elastic rods are seen to correspond in a simple way to congruence solutions to FM. In particular, the screw motion of the Hasimoto filament was the first step towards the discovery of the soliton nature of FM ([Ha]).

For general curves, $\gamma_t = Y_1$, $\gamma_t = Y_2$, etc., describe interesting evolutions of non-stretching filaments. It's worth taking a moment to contrast such equations with the well-known *curve shortening flow* (CS), $\gamma_t = \frac{\partial^2}{\partial s^2}\gamma = \kappa N$ (see, e.g., [G-H]). Often described as the (negative) *gradient flow of arclength* (in a formal L^2 sense), CS is a natural and interesting example of a *geometric evolution equation.* But it should not be mistaken for a PDE describing $\gamma(s,t)$ directly; rather, CS is compact notation for the PDE $\gamma_t = \frac{1}{v}\frac{\partial}{\partial u}(\frac{1}{v}\frac{\partial}{\partial u}\gamma)$, describing a curve $\gamma(u,t)$ of variable speed $v = \|\partial\gamma/\partial u\|$. In this respect, CS should be regarded as typical among geometric curve evolution equations— $\gamma_t = Y_n$ is exceptionally nice.

Of course, any curve motion $\gamma(u,t)$ can be made (locally) non-stretching by reparametrization, leaving the shape of γ unchanged for each t. In fact, Lemma 1 shows how to define a *reparametrization operator*, $\mathcal{P}$, which modifies the tangential component of a general variation field $\gamma_t = W$ to make it LAP. Since $\mathcal{P}$ plays an important role in §4.2, we give formulas in terms of the various notations $W = aT + bN + cB = fT + gU_1 + hU_2$:

$$\begin{aligned}\mathcal{P}W &= (\partial^{-1}\langle\partial^2\gamma, W\rangle)\ T + W^{\perp} \\ &= (\partial^{-1}\kappa b)\ T + bN + cB = (\partial^{-1}(u_1 g + u_2 h))\ T + gU_1 + hU_2\end{aligned}$$

(the appropriate specification of antiderivative ∂^{-1} depending on the application). For example, one may consider the *normalized* curve shortening flow, $\gamma_t = \mathcal{P}(\kappa N) = \int \kappa^2 ds\ T + \kappa N$ (this approach was used in [A-L]); the resulting $\gamma(s,t)$ is perhaps better behaved analytically than $\gamma(u,t)$ (but again, $\gamma(s,t)$ is not described by a PDE).

We remark that the LAP property of Y_n is closely related to the *first FM conservation law*; namely, if γ is a closed curve, its evolution under $\gamma_t = Y_n$ will preserve the arclength functional, $\mathcal{L}[\gamma] = \int_\gamma ds$. As we now briefly indicate, Equation 10 is key to a whole *infinite hierarchy of conservation laws* for FM. First we recall that Marsden and Weinstein [M-W] introduced a *Poisson structure*, $\{\mathcal{F},\mathcal{G}\} = \int_\gamma \langle J\nabla\mathcal{F}, \nabla\mathcal{G}\rangle ds$, on the space Ω of regular curves in E^3, giving FM a Hamiltonian form. Here we are considering *geometric* (parametrization-independent) functionals on Ω given by variational integrals $\mathcal{F}(\gamma) = \int_\gamma F[\gamma](s)ds$ and $\mathcal{G}(\gamma) = \int_\gamma G[\gamma](s)ds$, with respective Euler operators $\nabla\mathcal{F}$ and $\nabla\mathcal{G}$. Since Euler operators of geometric functionals have no tangential components, we may just as well write $\{\mathcal{F},\mathcal{G}\} = \int_\gamma \langle \mathcal{J}\nabla\mathcal{F}, \nabla\mathcal{G}\rangle ds$, where $\mathcal{J} = \mathcal{P}J$. For instance, the length functional $\mathcal{L}$ has Euler operator $\nabla\mathcal{L} = -\gamma_{ss}$, and the *Hamiltonian flow* of $\mathcal{L}$ induced by $\{\ ,\ \}$ may be written $\gamma_t = -\mathcal{J}\nabla\mathcal{L} = \gamma_s \times \gamma_{ss}$ (FM). In fact, all the equations in the FM hierarchy (after $\gamma_t = Y_0$) are Hamiltonian with respect to this structure; as proved by Yasui and Sasaki [Y-S], the Hamiltonians are given simply by $\mathcal{F}_n = \frac{1}{n-2}\int_\gamma f_{n-1} ds$, for $n = 1, 3, 4, 5, \ldots$. That is, one has $Y_n = Y_{\mathcal{F}_n} = \mathcal{J}\nabla\mathcal{F}$. Modulo this result, we easily prove:

Proposition 13. *For $n = 1, 3, 4, 5, \ldots$, the integrals $\mathcal{F}_n = \frac{1}{n-2}\int_\gamma f_{n-1} ds$ are FM constants of motion in involution. In terms of curvature and torsion, the first few conserved quantities are: $\mathcal{L} = \int_\gamma ds$, $\mathcal{F}_2 = \int_\gamma -\tau ds$, $\mathcal{F}_3 = \frac{1}{2}\int_\gamma \kappa^2 ds$, $\mathcal{F}_4 = \frac{1}{2}\int_\gamma \kappa^2\tau ds$, $\mathcal{F}_5 = \frac{1}{2}\int_\gamma (\kappa')^2 + \kappa^2\tau^2 - \frac{1}{4}\kappa^4 ds$.*

Proof. By Equation 10, $\{\mathcal{F}_m, \mathcal{F}_n\} = \int_\gamma \langle J\nabla\mathcal{F}_m, \nabla\mathcal{F}_n\rangle ds = \int_\gamma \langle Y_m, JY_n\rangle ds = \int_\gamma \partial \frac{1}{2}\sum_{k=1}^{n-m}\langle Y_{m+k-1}, Y_{n-k}\rangle ds$, for $1 \le i \le j$. For suitable boundary/decay conditions, the Poisson brackets $\{\mathcal{F}_i, \mathcal{F}_{j+1}\}$ will therefore vanish. The curious special case $n = 2$ may be verified directly. □

4.2. **The recursion operator and variation formulas.** The filament hierarchy may also be written as $X_n = \mathcal{R}^n X_0$, in terms of the integro-differential *recursion* *operator*,

$$\mathcal{R}X = -\mathcal{J}\partial X = -\mathcal{P}J\partial X = (\partial^{-1}\langle\partial^2\gamma, W\rangle)\, T - J\partial X \tag{14}$$

(The antidifferentiation ∂^{-1} leads to the arbitrary constants of integration $A_1, \dots A_n$ in X_n.)

It is an interesting fact that $\mathcal{R}$ has geometric meaning, quite independent of the filament hierarchy:

Theorem 14. *Let $\gamma(s,t)$ be a variation of unit speed curves in E^3. Let $\{T, U_1, U_2\}$ be a natural frame along $\gamma(s,t)$. Consider the complex curvature $\psi = u_1 + iu_2 = Z(\gamma_{ss})$, where, Z is the normal coordinate map $Z(fT + gU_1 + hU_2) = g + ih$. Then the infinitesimal curve variation $W = \frac{\partial\gamma}{\partial t}$ induces curvature variation $\frac{\partial\psi}{\partial t}$ according to the formula*

$$(\frac{\partial}{\partial t} - i\mu)\psi = -Z\mathcal{R}^2 W \tag{15}$$

Here, μ is an arbitrary constant in a "gauge term" $i\mu\psi$ which may be associated with the non-uniqueness of the natural frame.

Proof. Writing $W = fT + gU_1 + hU_2$, with $f_s = u_1 g + u_2 h$ (W is LAP), and using the natural Frenet equations, we compute the useful formulas:

$$(\mathcal{R} - \sigma)W = \alpha T + (h_s + u_2 f)U_1 - (g_s + u_1 f)U_2 \tag{16}$$

$$(\mathcal{R} - \sigma)^2 W = \beta T - ((g_s + u_1 f)_s - u_2\alpha)U_1 - ((h_s + u_2 f)_s + u_1\alpha)U_2, \tag{17}$$

where $\alpha_s = u_1 h_s - u_2 g_s$, and $\beta_s = -u_1(g_{ss} + \kappa^2 g) - u_2(h_{ss} + \kappa^2 h) - \frac{1}{2}(\kappa^2)_s f$.

On the other hand, let $\omega = AT + BU_1 + CU_2$ be the *angular velocity* of the natural frame; i.e., $F_t = \omega \times F$ for $F = T$, U_1, or U_2. Using $\gamma_{st} = \gamma_{ts}$, one finds $B = -(h_s + u_2 f + \sigma g)$ and $C = g_s + u_1 f - \sigma g$. Further, the U_2-component of $(U_1)_{st} = (U_1)_{ts}$ yields $A_s = -(\alpha + \sigma f)_s$. We thus obtain a noteworthy intermediate result: $\omega = -\mathcal{R}W - \mu T$, for some constant μ.

Next, the T-component of $(U_1)_{st} = (U_1)_{ts}$ yields $C_s + u_2 A = (u_1)_t - \sigma B$, which can be expressed as $(u_1)_t = -\langle U_1, \mathcal{R}^2 W\rangle - \mu u_2$. Similarly, $(U_2)_{st} = (U_2)_{ts}$ gives $(u_2)_t = -\langle U_2, \mathcal{R}^2 W\rangle + \mu u_1$, and the result follows. □

This formula (in case $\sigma = 0$) appeared in [L-P 1], and was generalized in [L-P 3] to the context of Hermitian symmetric Lie algebras. Such results appear to argue in favor of natural curvatures (especially in higher dimensions, where they seem to be particularly advantageous). Nevertheless, κ and τ play important roles, below.

Returning to the FM hierarchy, a remarkable fact now emerges. The hierarchy Y_n not only determines distinguished geometric evolution equations for curves, but it simultaneously provides the corresponding evolution equations for natural curvatures. Namely, we have the following

Corollary 15. *If $\gamma_t = Y_n$, then $\psi_t = -ZY_{n+2}$. (Here we have suppressed the term $i\mu\psi$, which vanishes for appropriately chosen natural frames.) In particular, a curve evolving by FM, $\gamma_t = \gamma_s \times \gamma_{ss} = \kappa B$, has complex curvature $\psi(s,t)$ (with $\sigma = 0$) satisfying the nonlinear Schrödinger equation,*

$$\psi_t = i(\psi_{ss} + \frac{1}{2}|\psi|^2\psi) \tag{18}$$

Similarly, $\gamma_t = \frac{3}{2}\langle\gamma_{ss},\gamma_{ss}\rangle\gamma_s + \gamma_{sss} = \frac{\kappa^2}{2}T + \kappa_s N + \kappa\tau B$ *induces the* (*complex*) *modified Korteweg-de Vries equation,*

$$\psi_t = \psi_{sss} + \frac{3}{2}|\psi|^2\psi_s. \tag{19}$$

Proof. We use the table in §3.2. For FM, $\psi_t = -ZY_3 = -(\partial^2 u_2 + u_2\frac{\kappa^2}{2}) - i(-\partial^2 u_1 - u_1\frac{\kappa^2}{2}) = i(\psi_{ss} + \frac{1}{2}|\psi|^2\psi)$. The case $n = 4$ is simpler to read off (and we'll have more to say about even n, below.) □

Corollary 16. *The variations of curvature and torsion induced by an LAP curve variation* $\gamma_t = W$ *are given by:* $\kappa_t = -\langle\mathcal{R}^2W, N\rangle$*, and* $\tau_t = -[\frac{1}{\kappa}\langle\mathcal{R}^2W, B\rangle]_s$*. In particular,* W *is constant torsion-preserving if and only if* $\langle\mathcal{R}^2W, B\rangle = C\kappa$*, where* C *is constant along* γ*.*

Proof. The formulas follow easily from Equation 15 using $\psi = \kappa e^{i\theta}$, $U_1 + iU_2 = (N + iB)e^{i\theta}$, $\theta_s = \tau - \sigma$, $\psi_t = (\kappa_t + i\kappa\theta_t)e^{i\theta}$, etc. Note that the ambiguous gauge term $i\mu\psi$ drops out of the formulas, as does σ. □

4.3. **FM vectorfields preserving special classes of curves.** Here we discuss special sequences of vectorfields belonging to the FM hierarchy which preserve the classes of *planar, spherical*, or *constant torsion* curves. This topic well illustrates the approach of §2; while the formulas (16), (17) will be very helpful heuristically, we require here exact specification of constants of integration (e.g., via the normalization $\langle Y, Y\rangle = 1$). Corollary 3 easily implies the desired results, for the planar case, while spherical curves require additional inductive formulas. The constant torsion case is the most interesting, not only because of the connection to pseudospherical surfaces, but also because this case leads to further insight into the recursion process itself (as discussed in §5).

We begin by recalling that, along planar curves, the normalized FM hierarchy alternates between planar vectorfields, $Y_{2m} = a_{2m}T + b_{2m}N$ and binormal fields $Y_{2m+1} = c_{2m+1}B$, and that the even equations $\gamma_t = Y_{2m}[\gamma]$ therefore restrict to planar flows. In particular, for $m = 1$, we have the planar curve evolution and corresponding curvature evolution equations:

$$\gamma_t = \frac{\kappa^2}{2}T + \kappa_s N, \qquad \kappa_t = (\kappa_{ss} + \frac{1}{2}\kappa^3)_s \tag{20}$$

The second equation is the well-known (real) mKdV equation. The first equation, arguably the simplest geometric realization of a soliton equation, has been considered as a model of planar vortex patch dynamics [G-P]. We mention that a (unit speed) solution $\gamma(s,t) = (x(s,t), y(s,t))$ of this equation yields a solution $z = x+iy$ of the *Schwarzian KdV equation*, $z_t = S(z)z'$, while the Schwarzian derivative itself, $u = S(z) = (z''/z')' - \frac{1}{2}(z''/z')^2$, satisfies the (complex) *Korteweg-de Vries equation*, $u_t = u''' + 3uu'$. (We recall that the Schwarzian derivative is the basic differential invariant of Möbius transformations.)

The higher order equations of the planar subhierarchy, $\gamma_t = Y_{2m}[\gamma]$, satisfy the LAP condition, $a'_{2m} = \kappa b_{2m}$, and also $b_{2m+2} = -(b'_{2m} + \kappa a_{2m})'$ (an easy consequence of Corollary 3). Thus, the coefficients $B_j = b_{2j}$, $A_j = a_{2j}$ constitute a (normalized) solution to the *mKdV recursion scheme*:

$$A'_j = \kappa B_j, \quad B_j = -(B'_{j-1} + \kappa A_{j-1})', \qquad A_0 = -1,\ B_0 = 0 \tag{21}$$

Further, Corollary 16 implies $\gamma_t = Y_{2m}[\gamma]$ induces the curvature evolution equation $\kappa_t = -B_{m+1}[\kappa]$—the m^{th} *higher order equation of the mKdV hierarchy.* [These equations may also be written as $\kappa_t = K^m \kappa_s$, in terms of the recursion operator $K = -(\partial^2 + \kappa^2 + \kappa_s \partial^{-1}\kappa)$. The corresponding operator on the curve level has the simpler appearance, $\mathcal{R}^2 = -\mathcal{P}\partial^2$. The antidifferentiation operator introduces the usual ambiguity; however, starting with $A_1 = \frac{\kappa^2}{2}$, Corollary 3 yields local expressions for the (normalized) coefficients: $2A_j = \sum_{i=1}^{j-1}(A_i A_{j-i} + B_i B_{j-i}) + \sum_{i=1}^{j}(B'_{i-1} + \kappa A_{i-1})(B'_{j-i} + \kappa A_{j-i})$, $j = 2, 3, \ldots$. Finally, for comparison with the better known *Lenard recursion scheme* for the Korteweg-de Vries equation, we observe that the A_j satisfy $\partial A_{j+1} = DA_j$, where D is the third order operator $D = -\kappa\partial[\partial \frac{1}{\kappa}\partial + \kappa]$, and the m^{th} equation may be written $\kappa_t = \frac{1}{\kappa}\partial A_{m+1}$.]

Note the B_m are perfect derivatives—all equations in the mKdV hierarchy are in *conservation form* (unlike Equations (18), (19)). Consequently, all planar flows preserve the (algebraic) enclosed area $\mathcal{A}(\gamma)$ of a closed planar curve: since the Euler operator of enclosed area is the normal, $\nabla\mathcal{A} = N$, we have $\frac{\partial}{\partial t}\mathcal{A}(\gamma(s,t)) = \int \langle Y_{2m}, N\rangle \, ds = \int B_m ds = 0$, for closed curves. One could say the results just mentioned are *topologically obvious!* Namely, the *total curvature* is a topological invariant for closed curves, $\int_\gamma \kappa ds = 2\pi Ind[\gamma]$, where $Ind[\gamma]$ is the *rotation index* of γ. Thus, for each m, we can write $0 = \frac{\partial}{\partial t}\int \kappa ds = \int B_{m+1}[\kappa]ds$—the only reasonable explanation being that $B_{m+1}[\kappa]$ is a *perfect derivative* (null Lagrangian).

Next, we observe that there are two ways to generalize planar curves slightly—maintaining a single *functional* shape parameter $\kappa(s)$. Regarding planar curves as having $u_1(s) = \kappa(s)$, and $u_2 = \sigma = 0$, we can extend either to spherical curves by allowing $u_2 = constant \neq 0$, or to constant torsion curves by allowing $\sigma = constant \neq 0$. We begin with the former.

Proposition 17. *The even FM flows $\gamma_t = Y_{2n}$ restrict to spherical curves, and induce evolution of geodesic curvature by equations in the mKdV hierarchy. The algebraic area of a closed spherical curve is preserved under each of these flows.*

Proof. Expressing $Y_n = f_n T + g_n U_1 + h_n U_2$ in terms of a natural Frenet frame with $\sigma = 0$, the FM recursion scheme may be written:

$$(22) \qquad f'_n = u_1 g_n + u_2 h_n, \qquad g_n = h'_{n-1} + u_2 f_{n-1}, \qquad -h_n = g'_{n-1} + u_1 f_{n-1}$$

We specialize these equations to spherical curves with natural curvatures $u = u_1 = \kappa_g$ (*geodesic curvature*) and $v = u_2 = 1/R$ (R = *spherical radius*). Using also our closed form expression for $f_n = \langle T, Y_n\rangle$ (in Equation 8), and $f_1 = 0$, $g_1 = -v$, one establishes the following set of formulas by induction: $f_{2n+1} = -vg_{2n}$, $g_{2n+1} = vf_{2n}$, $h_{2n+2} = 0$. Thus, Y_{2n+2} is tangent to the sphere. Further, one easily checks: $h_{2n+1} = -(g'_{2n} + uf_{2n})$, $g_{2n+2} = -(g'_{2n} + uf_{2n})' - v^2 g_{2n}$. The latter shows inductively that the g_{2n} are perfect derivatives—in fact, linear combinations of the mKdV operators B_m (applied to u). The results on area and evolution of curvature now follow as in the planar case, the only difference being that the u-evolutions include linear combinations of lower order mKdV equations. Note that a topological argument is not available here; by the Gauss-Bonnet Theorem, total curvature and enclosed area are the same functional! □

We mention two minor variations on the equations just described. One may define an "intrinsic spherical recursion operator" $\mathcal{S} = -\mathcal{P}\nabla_T^2$—simply replacing ∂

in the planar recursion operator $\mathcal{R}^2$ with the covariant derivative in the sphere. Since the Frenet equations ($\nabla_T T = \kappa_g N$, $\nabla_T N = -\kappa_g T$) have not changed form, the resulting (normalized) hierarchy, $\gamma_t = S^v_{2m} = a_{2m}T + b_{2m}N$, involves the very same differential operators, a_{2m}, b_{2m}, as above. The evolution equations for κ_g will again belong to the mKdV hierarchy; however, there is a "slippage" relative to the planar hierarchy, due to the curvature variation formula $(\kappa_g)_t = \langle N, (\nabla_T^2 + G)W\rangle$ for curves in a surface of constant Gauss curvature G. Now, as it turns out, the vectorfields S^v_{2m} are just linear combinations of the Y_{2n}, restricted to the spherical curves. Finally, we note that by taking slightly different combinations of the Y_{2n}, one can arrange for κ_g to evolve by the normalized equations of the mKdV hierarchy, as in the planar case.

Turning now to constant torsion curves, one might attempt to use the *binormal indicatrix* construction—we recall this sets up a correspondence $\gamma \leftrightarrow B$ between unit speed curves of curvature κ and constant torsion $\tau = 1$, and unit speed curves with geodesic curvature $\kappa_g = \kappa$ in the unit sphere—to define unit speed and constant torsion-preserving flows, inducing the mKdV hierarchy for κ. It is perhaps not *a priori* clear that the implied flows are given directly by FM vectorfields along such curves (or that the flows are even PDE's on the curve level). In any event, we choose to 'start from scratch', for the following reason: though spherical curves may at first appear to be a simpler generalization of the planar case, we discover in the end a much more satisfactory explanation of the constant torsion case. Part of what's at stake is the correct geometric interpretation of the *spectral parameter* (see the discussion at the end of §5.2).

We begin with the observation, due to Lamb [L], that the following curve evolution (a special case of a vortex model considered by Fukumoto and Miyazaki) preserves constant torsion, with curvature evolving by mKdV:

$$\gamma_t = (\frac{\kappa^2}{2} - 3\tau^2)T + \kappa_s N - 2\tau\kappa B \tag{23}$$

In fact, letting Z_2 denote the vectorfield on the right-hand-side of this equation, one straightforwardly obtains $\mathcal{R}^2 Z_2 = a(s)T - (\kappa_{ss} + \frac{1}{2}\kappa^3)_s N + c\kappa B$, where c is constant. (This computation involves many cancellations, a few of which require $\tau = \mathit{constant}$) Then the claim follows at once from Corollary 16.

Underlying this example is the *slipping phenomenon* mentioned in §3.2: the coefficients of $Y_n = f_n T + g_n U_1 + h_n U_2$ depend on σ in a very simple way, involving lower order Y_k with binomial coefficients. In fact, the formulas (16), (17) suggest the definition of a hierarchy, $Y^\sigma_{n+1} = (\mathcal{R} - \sigma)^n Y_1$, $n = 0, 1, \ldots$, whose σ-frame coefficients, f_n, g_n, h_n, do not depend on σ. Now we regard the Frenet frame along a curve of constant torsion as a σ-frame with $\sigma = \tau$, $u_1 = \kappa$, $u_2 = 0$, $U_1 = N$, and $U_2 = B$; the coefficients of the shifted hierarchy $Y^\tau_n = f_n T + g_n N + h_n B$ must therefore be independent of torsion, and ought to resemble the planar case. It follows that vectorfields Z_{2m} defined so as to satisfy $\mathcal{R}^2 Z_{2m} = Y^\tau_{2m+2} + aT + c\kappa B$, for constants a and c, will preserve constant torsion, and induce evolution of curvature by equations of the mKdV hierarchy.

Actually, as noted above, the precise definition of Y^σ_n should not use $\mathcal{R}$, because of the ambiguities in constants of integration. Thus, we instead define the *shifted*

FM hierarchy by:

$$Y_n^\sigma = \sum_{k=0}^{n-1} \binom{n-1}{k} (-\sigma)^k Y_{n-k}, \qquad n = 1, 2, \ldots \tag{24}$$

Proposition 18. a) *The vectorfields $Y_0^\sigma = Y_0$, Y_n^σ, $n = 1, 2, \ldots$, satisfy the shifted FM recursion scheme, $\partial Y_n^\sigma = J(Y_{n+1}^\sigma + \sigma Y_n^\sigma)$. The series $Y^\sigma = \sum_{n=0}^\infty \lambda^n Y_n^\sigma$ satisfies $\lambda \partial Y^\sigma = (1+\sigma\lambda) J Y^\sigma$, and the normalization $\langle Y^\sigma, Y^\sigma \rangle = 1$ (which uniquely determines Y^σ).*

b) *Along a curve of constant torsion τ, the odd vectorfields of the shifted FM hierarchy are binormal and even fields are osculating: $Y_{2m}^\tau = a_{2m} T + b_{2m} N$, $Y_{2m+1}^\tau = c_{2m+1} B$. Here, $a_{2m}[\kappa]$, $b_{2m}[\kappa]$, and $c_{2m+1}[\kappa]$, are precisely the differential operators associated with planar curves, above.*

c) *For $n = 1, 2, \ldots$, the equation $\gamma_t = Z_{2n}^\tau = \sum_{k=0}^{2n} \binom{2n+1}{k} (-\tau)^k Y_{2n-k}$ preserves constant torsion τ and induces evolution of curvature κ by the equation of the mKdV hierarchy, $\kappa_t = -B_{n+1}[\kappa]$.*

Proof. The recursion equation for the Y_n^σ follows from Equation 1 and the recursion rule for binomial coefficients. The equation for Y^σ and $\partial \langle Y^\sigma, Y^\sigma \rangle = 0$ then follow nearly as before. The general solution to the shifted recursion scheme may then be written: $X_n^\sigma = f_n T + J(\sigma J - \partial) X_{n-1}^\sigma$, with $2f_n = -C_n + \sum_{k=1}^{n-1} \langle X_k^\sigma, X_{n-k}^\sigma \rangle$. The fact that the coefficients in the σ-frame expression for X_n^σ do not involve σ may be proved inductively. Using this fact, and setting $\sigma = 0$ in the definition of Y_n^σ establishes the normalization.

Now let γ have curvature κ and constant torsion τ. Using the above, one establishes the direct analogue of Corollary 3 by almost the same proof, and then b) follows easily. Finally, noting $\mathcal{R}^2 Z_{2n} = Y_{2n+2}^\tau + aT + c\kappa B$, for constants a, c, the rest of the claim follows from Corollary 16 as above. □

We remark that setting $\lambda = -\sigma^{-1}$ in the shifted series Y^σ gives $\partial Y^\sigma = 0$; such a condition of *covariant constancy* could have been used as definition of the Y_n^σ (writing $Y^\sigma = \sum_{n=0}^\infty (-\sigma)^{-n} Y_n^\sigma = \sum_{n=0}^\infty (-\sigma)^{-n} (f_n T + g_n U_1 + h_n U_2)$, where $\{T, U_1, U_2\}$ is a σ-frame, and the coefficients f_n, g_n, h_n are assumed not to depend on σ). For illustration, suppose γ has curvature κ and constant torsion τ. Term by term differentiation of the following series results in telescope cancellations:

$$\begin{aligned} Y^\tau &= -T - \tau^{-1}\kappa B + \tau^{-2}(\frac{\kappa^2}{2} T + \kappa' N) + \tau^{-3}(\kappa'' + \frac{\kappa^3}{2}) B \\ &\quad -\tau^{-4}((\kappa\kappa'' - \frac{1}{2}(\kappa')^2 + \frac{3}{8}\kappa^4) T + (\kappa''' + \frac{3}{2}\kappa^2\kappa') N) \ldots \end{aligned}$$

Of course, convergent examples may be constructed, using the soliton class.

The hierarchies preserving planar, spherical, and constant torsion curves were described in [L-P 2] (however, without the benefit of the detailed information on solutions to the FM and shifted FM recursion schemes). We mention also that soliton curves for the spherical evolutions are naturally viewed from an intrinsic point of view, as above; thus one considers *geodesics*, *elastica*, *buckled rings*, etc., in a Riemannian manifold M (as in [L-S 1], [L-S 3], [L-M-V]). Such special curves have some remarkable connections to objects of Euclidean geometry; e.g., using Hopf lifts of elastica in S^2, U. Pinkall [Pi] gave the first examples of *Willmore surfaces* in

R^3 (critical surfaces for the total squared mean curvature integral) not coming from stereographic projections of minimal surfaces in S^3; in a similar spirit, buckled rings in the hyperbolic plane H^2 (and the hyperbolic analogue of Equation 20) have been used ([G-L]) to construct explicit examples of the *Konopelchenko-Taimanov motions of immersed Riemann surfaces in* R^3 ([Ta 1], [Ta 2])- these surface evolutions preserve conformal type, the Willmore functional, and the infinite list constants of motion of the *modified Novikov-Veselov equation*.

4.4. **The swept-out surfaces.** Thus far, we have considered $\gamma(s,t)$ as an evolving unit speed curve. Of course, the (generally singular) parametrized surface *swept-out* by $\gamma(s,t)$ may also be interesting, to the extent that features of the curve geometry and evolution are closely related to the surface geometry. The following proposition identifies relevant cases of curve evolution from this standpoint.

Proposition 19. *Let Σ be the surface swept out by $\gamma(s,t)$, where γ satisfies the LAP curve evolution $\gamma_t = W[\gamma]$. Let Σ have Gaussian curvature $G = \kappa_1\kappa_2$ and mean curvature $H = \frac{1}{2}(\kappa_1 + \kappa_2)$, where κ_1, κ_2 are the principal curvatures of Σ. Let γ as a curve have curvature κ, torsion τ, and Frenet frame $\{T, N, B\}$. Then*

a) *If $W = aT + bN$, then $\gamma(s,t)$ foliates Σ by asymptotic curves. Further, $G = -\tau^2$, and $H = (b^2\tau)_s/2b^2\kappa$.*
b) *If $W = aT + cB$, then $\gamma(s,t)$ foliates Σ by geodesics, $G = -c_{ss}/c$, and $H = (c\kappa^2 + c\tau^2 - c_{ss})/2\kappa c$.*
c) *If $W = fT + gU$, then $\gamma(s,t)$ foliates Σ by principal curves, $G = (v\partial^{-1}vg_s)/g$, and $H = (vg + \partial^{-1}vg_s)/2g$; here, γ has natural curvatures u, v and frame U, V, and $\partial^{-1}vg_s$ is the appropriate antiderivative of vg_s.*

We omit the proof, which uses basic definitions and formulas of surface theory. However, we note that part a) is essentially the *Beltrami-Enneper Theorem* (see [Sp], vol. III, which also gives a general discussion of the three special classes of curves in a surface, appearing in a)–c)). We also note that variation formulas for Frenet frames (as that included in the proof of Theorem 14, for natural frames) may be used as the main computational technique for establishing several of the above formulas.

Now we observe, for instance, that FM yields surfaces (*Hasimoto surfaces*) foliated by geodesics, as in b), while c) includes the spherical curve evolutions of Proposition 17 with $G = v^2 = constant$. However, we will focus here on using part a) to give a dynamical description of pseudospherical surfaces.

We begin with a simple observation, based on the proposition and on our variation formula for torsion: to sweep out a surface of constant negative curvature $G = -\tau^2$, it suffices to find an *eigenvector* of $\mathcal{R}$ along a curve of constant torsion τ. In fact, one easily checks that the vectorfield $W = \cos\nu T - \sin\nu N$, $\nu_s = \kappa$, is such an eigenvector (up to an unimportant gauge term): $\mathcal{R}W = \tau W + \mu T$. Combining this with our variation formula for κ, we thus obtain a dynamical description of a pseudospherical surface in terms of curve evolution by the *trigonometric flow*, and corresponding evolution of $\nu = \partial^{-1}\kappa$ by the *sine-Gordon equation*:

$$\gamma_t = \cos\nu T - \sin\nu N, \qquad \frac{\partial^2\nu}{\partial s\partial t} = \tau^2 \sin\nu \tag{25}$$

In fact, $\gamma(s,t)$ describes the well-known foliation of a pseudospherical surface of constant curvature $G = -\tau^2$ by one family of asymptotic curves (and ν is the angle between the two asymptotic directions).

As the trigonometric vectorfield may be applied to any constant torsion curve, it does not necessarily point the way to explicit examples. However, the shifted FM hierarchy provides a more concrete realization of the above strategy.

Proposition 20. *Let $\gamma \in \Gamma_{2n+1}$ be a planar-like soliton curve of constant torsion τ. I.e., γ satisfies an equation of the form $0 = X^\tau_{2n+1}[\gamma] = \sum_{k=0}^n A_{2n-k} Y^\tau_{2n+1}[\gamma] = (\sum_{k=0}^n A_{2n-k} c_{2n+1}[\kappa])B$. Then $X^\tau_{2n} = \sum_{k=0}^n A_{2(n-k)} Y^\tau_{2n} = \sum_{k=0}^n A_{2(n-k)}(a_{2n}T + b_{2n}N)$ is an eigenvector of $\mathcal{R}$ along γ (modulo the usual aY_0 term), and the evolution $\gamma_t = X^\tau_{2n}[\gamma]$, with initial condition $\gamma(s,0) = \gamma(s)$, describes a foliation of a surface of constant curvature $G = -\tau^2$, by asymptotic curves of unit speed. In fact, X^τ_{2n} is a vectorfield of constant length which, up to scaling, may be identified with the trigonometric field along γ.*

Proof. We know from Proposition 11 and §4.3 that the curvature of a planar-like soliton curve satisfies one of the mKdV stationary equations, $\sum_{k=0}^n A_{2n-k} c_{2n+1}[\kappa] = 0$. By Proposition 18, this can be re-expressed in the form $0 = X^\tau_{2n+1}[\gamma]$ as above, where $X^\tau_0 = Y_0, X^\tau_1, \dots X^\tau_{2n+1}$ satisfy the shifted FM recursion scheme, $\partial X^\sigma_n = J(X^\sigma_{n+1} + \sigma X^\sigma_n)$. In particular, we have $\mathcal{R}X^\tau_{2n} = (\mathcal{R} - \tau)X^\tau_{2n} + \tau X^\tau_{2n} = X^\tau_{2n+1} + aY_0 + \tau X^\tau_{2n}$. Thus, along γ, $\mathcal{R}X^\tau_{2n} = \tau X^\tau_{2n} + aY_0$, as claimed. To complete the proof, one needs to observe that the stationary equation $0 = \sum_{k=0}^n A_{2n-k} c_{2n+1}[\kappa]$ remains satisfied as γ evolves. This is a consequence of the fact that all the FM flows commute, which in turn follows from the FM conservation laws given in Proposition 13; we omit the relevant arguments, which are part of the standard abstract theory of Hamiltonian systems. The last comment is obtained by writing $X^\tau_{2n} = f_{2n}T + g_{2n}N$. Then $f'_{2n} - \kappa g_{2n} = 0$ is the LAP condition, while $g'_{2n} + \kappa f_{2n} = -\langle B, X_{2n+1}\rangle = 0$ follows by recursion. But the resulting linear ODE, $(f_{2n} + ig_{2n})' = -i\kappa(f_{2n} + ig_{2n})$, is exactly that satisfied by $e^{-i\nu}$, with $\nu' = \kappa$. □

The trigonometric flow was discussed in [L-P 2] and also in [Mc-S]. The technique using planar-like FM solitons to generate pseudospherical surfaces was developed by R. Perline ([Pe 1]), who went on to construct closely related examples of *Weingarten systems* of triply orthogonal coordinates ([Pe 2]); the description of the latter makes use of both osculating and binormal vectorfields to evolve a constant torsion curve. Finally, it should be noted that the pseudospherical surface/sine-Gordon equation relationship is one of the oldest and most famous connections between geometry and soliton equations. We have merely described a particular aspect of the latter topic tying it to our discussion of FM; for a treatment of pseudospherical surfaces using modern methods of soliton theory, we refer the reader to [M-S].

5. The $SU(2)$ spectral problem; curves and NLS

5.1. **Lie equations on $SU(2)$ and representations for curves.** Up to this point, it has been convenient to formulate everything in E^3 and to use vector notation—most notably, the cross product. Of course, the rotation group $SO(3)$ and its Lie algebra $so(3)$ have been lurking in the background all along, but this point would have been little more than a distraction in the foregoing discussion! In this last section, however, we wish to describe some constructions related to curves which tie our subject more directly to the standard machinery of soliton theory.

Thus, we need to introduce notation for the rotation group—or rather, its double cover, $SU(2)$. An element of the latter will be written as $\Phi = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, where $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$; thus, $\Phi \in SU(2)$ is a 2×2 unitary matrix with determinant $+1$. For the Lie algebra $su(2)$, consisting of 2×2 skew-Hermitian matrices of trace zero, we use the $su(2)$ basis $e_0 = \frac{-i}{2}\sigma^3$, $e_1 = \frac{-i}{2}\sigma^1$, $e_2 = \frac{-i}{2}\sigma^2$, where σ^1, σ^2, σ^3 are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The commutator bracket $[A, B] = AB - BA$ on $su(2)$ may be written $[e_i, e_j] = \epsilon_{ijk} e_k$, where ϵ_{ijk} is skewsymmetric in i, j, k and $\epsilon_{ijk} = 1$, with the convention that repeated indices are summed over. Using the Cartan-Killing form, $K(B, C) = tr(ad_B ad_C)$, $su(2)$ is naturally identified with E^3; in fact, e_0, e_1, e_2 form an orthonomal basis with respect to the Euclidean inner product, $\langle\,,\rangle = -\frac{1}{2}K$. Further, the adjoint representation, $Ad_\Phi B = \Phi B \Phi^{-1}$, defines the well-known two-to-one homomorphism of $SU(2)$ onto $SO(3)$.

As usual, the tangent space to $G = SU(2)$ at the identity will be identified with $su(2)$, and the tangent space at $\Phi \in SU(2)$ may be represented as $G_\Phi = \{B\Phi : B \in su(2)\}$. Further, a matrix differential equation of the form $\Phi_s = Q\Phi$ with $Q = Q(s) \in su(2)$ may be regarded as an ODE on $SU(2)$, to which the usual existence/uniqueness theorems apply, with $SU(2)$ as the underlying manifold. Of course, the simplest case for such a *Lie equation* occurs when $Q(s) = Q_0 = constant$, in which case the general solution has the form $\Phi(s) = e^{sQ_0}\Phi_0$. In particular, the choices $Q = e_j$, with initial condition $\Phi(0) = Id$, result in the standard one-parameter subgroups:

$$e^{se_0} = \begin{pmatrix} e^{-i\frac{s}{2}} & 0 \\ 0 & e^{i\frac{s}{2}} \end{pmatrix}, \; e^{se_1} = \begin{pmatrix} \cos\frac{s}{2} & -i\sin\frac{s}{2} \\ -i\sin\frac{s}{2} & \cos\frac{s}{2} \end{pmatrix}, \; e^{se_2} = \begin{pmatrix} \cos\frac{s}{2} & -\sin\frac{s}{2} \\ \sin\frac{s}{2} & \cos\frac{s}{2} \end{pmatrix}$$

By virtue of the adjoint representation, $SU(2)$ Lie equations may be used to induce Frenet systems for curves in E^3. We will use a non-standard notation to discuss this construction. Namely, we consider a given $su(2)$-valued *curvature function*, $Q(s) = \sum_{j=0}^{2} q_j(s) e_j$, and corresponding $\Phi(s)$ solving the Lie system $\Phi_s = Q\Phi$ (with arbitrarily specified initial conditions). Now let $B = B(s)$ be any $su(2)$-valued function, and let $\{B\}$ denote the corresponding E^3-valued vector function, defined by $\{B\} = Ad_{\Phi^{-1}} B = \Phi^{-1} B \Phi$. Also, we will use the shorthand $\{B, C\} = \{[B, C]\}$. In this situation, one has the following computational fact (which is standard, and easy to verify):

Lemma 21. *absolute velocity* $=$ *relative velocity* $+$ *transferred velocity*

$$\{B\}_s = \{B_s\} + \{B, Q\}$$

In particular, by *Ad*-invariance of the Cartan-Killing form, we may define an orthonormal frame in E^3 according to $E_i = \{e_i\}$, and by antidifferentiation, we may regard $T = E_0$ as the tangent indicatrix of a curve γ in E^3. By the lemma, we then obtain (generalized) Frenet equations for γ:

$$E_i' = \{e_i, Q\} = \epsilon_{ijk} q_j E_k.$$

Further, computing $T = Ad_{\Phi^{-1}} e_0$ in terms of the usual quadratic expressions in α, β, one obtains the following "Weierstrass representation" for the resulting curve $\gamma = \int T ds = x^0 e_0 + x^1 e_1 + x^2 e_2$:

$$x^0 = \int \alpha\bar{\alpha} - \beta\bar{\beta} ds, \qquad x^1 + ix^2 = \int 2\alpha\bar{\beta} ds$$

The classical Frenet equations for $\{T = E_0, N = E_1, B = E_2\}$ are recovered in the special case $Q = -\tau e_0 - \kappa e_2$, while the choice $Q = -\sigma e_0 + u_2 e_1 - u_1 e_2$ yields the natural Frenet system, Equation 13, for $\{T = E_0, U_1 = E_1, U_2 = E_2\}$. By analogy with rigid body mechanics, one may also write the Frenet equations as $E_j' = \Omega \times E_j$, where the *Darboux vector* is given by $\Omega = -\{Q\}$; for the standard Frenet system, $\Omega = \tau T + \kappa B$, and for natural frames, $\Omega = \sigma T - u_2 U_1 + u_1 U_2$.

We note that if Φ satisfies the Lie equation $\Phi_s = \sum_{j=0}^{2} q_j(s) e_j \Phi$ as above, then the ratio $z = \beta/\bar{\alpha}$ solves the *Riccati equation* $iz' = \frac{1}{2}(q_1 - iq_2) + q_0 z - \frac{1}{2}(q_1 + iq_2) z^2$. For the standard Frenet frame, this becomes $z' = i\tau z + \frac{\kappa}{2}(1+z^2)$, and for the natural frame one gets $z' = \frac{1}{2}\bar{\psi} + i\sigma z + \frac{1}{2}\psi z^2$, where $\psi = u_1 + iu_2$. As is well known, such equations are not integrable by quadrature for general coefficients. On the other hand, it follows from our earlier discussion that for invariants κ, τ (or ψ) of a soliton curve, the above Riccati equations are indeed integrable by quadrature. In fact, if the Frenet frame F has been constructed (say, using Corollary 5), then one can lift F to a curve Φ in $SU(2)$ via the adjoint representation, and set $z = \beta/\bar{\alpha}$; from one solution, the general solution can be constructed by quadrature. Solutions z may also be described more geometrically as stereographic images of fixed Euclidean basis vectors, say, projecting from the *pole* T onto the *equatorial complex plane* determined by the remaining frame vectors.

We remark that the theory of Riccati equations reflects the richer setting of the Möbius group, and the representation of a curve γ in terms of a general solution $z = \frac{aP+Q}{aR+S}$ leads to consideration of curves in $C^3 \cong sl(2, C)$. This representation, developed by Lie and Darboux (see [Ei], [St]), expresses the coordinates of $\gamma = (x^0, x^1, x^2)$ as antiderivatives of ratios of quadratic expressions in P, Q, R, S, resulting in x^j which are generally complex.

Still on the theme of representations of unit speed curves, we consider now the *Sym-Pohlmeyer construction* (see [Sym]), which takes advantage of the spectral parameter σ in the Lie system for the natural Frenet equations.

Lemma 22. *Let* $q = u_2(s)e_1 - u_1(s)e_2$, $A = e_0$, *let* $\Phi(s;\sigma)$ *solve the Lie equation*

$$\Phi_s = (q - \sigma A)\Phi \tag{26}$$

for each value of σ *(with initial conditions possibly depending on* σ*), and define a family of curves in* $su(2)$ *by*

$$\gamma(s;\sigma_0) = -\left[\Phi^{-1}\frac{\partial\Phi}{\partial\sigma}\right]_{\sigma=\sigma_0} \tag{27}$$

Then for each fixed σ_0, $\gamma(s;\sigma_0)$ *is a unit speed curve with natural curvatures* u_1, u_2 *and spectral parameter* σ_0.

Proof. Applying the formula , $\frac{\partial}{\partial s}\Phi^{-1} = -\Phi^{-1}\Phi_s\Phi^{-1}$, we compute $\frac{\partial}{\partial s}\Phi^{-1}\Phi_\sigma = -\Phi^{-1}\Phi_s\Phi^{-1}\Phi_\sigma + \Phi^{-1}\Phi_{\sigma s} = -\Phi^{-1}(q-\sigma A)\Phi_\sigma + \Phi^{-1}\frac{\partial}{\partial\sigma}(q-\sigma A)\Phi = -\Phi^{-1}A\Phi$. It follows that s is indeed a unit speed parameter for $\gamma(s,\sigma_0)$, whose unit tangent vector may be written $T = \{A\}$. Further differentiation recovers the natural Frenet equations, as above. □

Note that this construction realizes the parametrized family γ^σ of Proposition 11—in particular, it may be used to represent a family of planar-like solitons with a single formula. It has the further interesting feature that no final antidifferentiation is required to produce the curve, after the frame equations have been solved. (Of course, one must first solve the σ-dependent Lie system, but for many purposes, *analytic dependence of solutions on the parameter* may be invoked.) We mention that the technique is actually rather general, as may be inferred from the proof; starting with appropriate Lie groups and making suitable specializations, one obtains useful representations of curves in higher dimensional Euclidean spaces (see [L-P 3]), as well as spherical, hyperbolic, and Lorentzian geometries. Returning to curves in R^3, one could also adapt the technique for standard Frenet systems (though introduction of the parameter appears more artificial), or one could use $SO(3)$ Lie equations. But the above version is of particular relevance here, because Equation 26 is precisely the *spectral equation for NLS*, to be discussed below.

We conclude this section by describing a *Bäcklund transformation* for constant torsion curves, followed by simple examples. Our treatment of this interesting topic is cursory; we include it to tie together a number of previous topics and examples. (For more extensive discussions and interesting applications, see [Ca], [C-I 1], [Iv].) The construction we describe here is really just the classical Bäcklund transformation for pseudospherical surfaces (i.e., for the sine-Gordon equation), restricted to a single asymptotic curve. We recall that transformation moves a fixed distance from the "old" curve (surface) to the "new" one, preserving unit speed parametrization.

Given the representation Equation 27, it is natural to try to make use of a *gauge transformation*: namely, if a curve γ corresponds to $\Phi_s = Q\Phi$, we can ask what new curve $\tilde\gamma$ and new curvature vector $\tilde Q$ correspond to $\tilde\Phi = G\Phi$, for a given G. Writing $G_s = gG$ and $G_\sigma = G\delta$, one finds that $\tilde\Phi_s = \tilde Q\tilde\Phi$ holds for $\tilde Q = g + Ad_G Q$, and one obtains $\tilde\gamma = -\left[\tilde\Phi^{-1}\frac{\partial\tilde\Phi}{\partial\sigma}\right] = \gamma - Ad_{\Phi^{-1}}\delta$ and $\tilde T = Ad_{\Phi^{-1}}(e_0 + \delta_s + [\delta, Q])$. Evidently, we are looking for a special G, depending on Q, such that δ and $(e_0 + \delta_s + [\delta, Q])$ have constant norm. The actual story is a bit subtle; however, the following result can also be verified by direct computation:

Proposition 23. *Let γ be a unit speed curve in R^3, with curvature κ, constant torsion τ, and Frenet frame T, N, B. For C a constant, let $w = \tan(\eta/2)$ satisfy the Riccati equation*

$$w_s = Cw + \frac{\kappa}{2}(1+w^2), \quad \textit{i.e.}, \qquad \eta_s = C\sin\eta - \kappa \tag{28}$$

Then the formulas

$$\tilde\gamma = \gamma + \frac{2C}{C^2+\tau^2}(\cos\eta T + \sin\eta N), \qquad \tilde\kappa = \kappa - 2C\sin\eta \tag{29}$$

describe a new unit speed curve $\tilde\gamma$ with curvature $\tilde\kappa$, and torsion τ.

Now we observe that Equation 28 would be exactly that satisfied by $z = \beta/\bar{\alpha}$ given earlier, if only γ had constant *imaginary torsion* $\tau = -iC$! (Which would mean γ actually lies in the Lorentz space R^2_1.) More to the point, if the starting curve γ is known, solving Equation 28 for w amounts to analytic continuation in τ of a known quantity z.

Example 24. (*Lines and loops, circles and rings*) Since we're doing geometry, one-parameter subgroups of $SU(2)$ are not all alike! For $Q = -\sigma e_0$, the Lie equation $\Phi_s = Q\Phi$ has solution $\Phi = e^{-\sigma s e_0}\Phi_0$, and $Ad_{\Phi^{-1}}$ yields (the frame of) a spinning line γ. The Riccati equation for $z = \alpha/\bar{\beta}$ reduces to $z' = i\sigma z$. The Bäcklund transform of γ is quick to compute, and gives Hasimoto loops $\tilde{\gamma} \in \Gamma_3$; the appearance of hyperbolic functions should be no surprise, since one just replaces $i\sigma$ by C in $z = z_0 e^{i\sigma s}$, to go from solutions of one Riccati equation to the other.

On the other hand, for $Q = -\kappa_0 e_2$, $\kappa_0 = constant$, the Lie equation has solution $\Phi = e^{-\kappa_0 s e_2}\Phi_0$, and $Ad_{\Phi^{-1}}$ yields a circle $\gamma \in \Gamma_2$. This time, the Bäcklund transform gives (not necessarily closed) "buckled rings" $\tilde{\gamma} \in \Gamma_4$, with curvature functions given by rational expressions in sines and cosines or exponential functions. To do the computations by analytic continuation, one needs to use the "spinning circle", which satisfies $z' = i\sigma z + \frac{\kappa_0}{2}(1 + z^2)$, and again replace $i\sigma$ by C.

Example 25. (*Helices and the Clifford torus*) For $Q = -\sigma e_0 + u_2 e_1 - u_1 e_2 = constant$, we obtain all helices, and can interpolate between circles and spinning lines. The computations for Bäcklund transforms of helices are not essentially different from the circular case, and give either *quasiperiodic* or *asymptotic* perturbations of the original helix (examples of which are pictured in [C-I 1]).

We now reconsider how the homotopy of Theorem 4c) achieves the interpolation just mentioned (essentially continuing Example 8), with a normalization fixing $\kappa^2 + \tau^2$. Starting from a given helix $\gamma(s)$, the standard Frenet frame $F(s;\lambda)$ of $\gamma(s;\lambda)$ may be regarded as an immersed cylinder in $SO(3)$, such that the projection $p : SO(3) \mapsto S^2$, $p(F) = T$, is onto and one-to-one, except that the boundary circles $F(s; \pm\infty)$ project to north and south poles $T(s; \pm\infty)$. Now $F(s;\lambda)$ lifts to $SU(2)$ and then extends as follows. Set $\theta = \alpha s$, $\cot\phi = \frac{\tau - \lambda\alpha^2}{\kappa}$, and let $\Phi(\theta, \phi) = e^{\phi e_1} e^{-\theta e_2}$. Then one can check that Φ and F are identified via the mapping $\Phi \mapsto Ad_{\Phi^{-1}}$. Further, the new variables θ, ϕ allow us to extend Φ *fourfold* by formula; the result is a conformal parametrization of the *Clifford torus*—a flat minimal submanifold of S^3. As a two-dimensional surface in $C^2 \cong R^4$, $\Phi = (\alpha, \beta) = (X_1, X_2, X_3, X_4)$ satisfies not only $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$, but also $Im[\alpha^2 + \beta^2] = 0$. A more standard representation of this surface is obtained by making the isometric coordinate transformation $Y_1 = \frac{1}{\sqrt{2}}(X_1 + X_2)$, $Y_2 = \frac{1}{\sqrt{2}}(X_3 + X_4)$, $Y_3 = \frac{1}{\sqrt{2}}(X_1 - X_2)$, $Y_4 = \frac{1}{\sqrt{2}}(X_3 - X_4)$, and the conformal change of angular variables $\theta = \eta + \zeta$, and $\varphi = \eta - \zeta$. Then $\Phi = (Y_1, Y_2, Y_3, Y_4) = \frac{1}{\sqrt{2}}(\cos\zeta, \sin\zeta, \cos\eta, \sin\eta)$ parametrizes the Clifford torus, with angular range $-\pi \leq \zeta, \eta < \pi$. It would be interesting to know if Y is similarly well-behaved at $\lambda = \pm\infty$ for a large class of (quasiperiodic) soliton curves—in particular, whether corresponding constructions result in immersed tori $\Phi : T^2 \mapsto S^3$, as above.

5.2. **The NLS hierarchy.** We begin by recalling the setting of the NLS hierarchy as a family of compatibility conditions for the following overdetermined linear system (see [Pa] for more background, and a survey of related topics in soliton theory):

$$\Phi_s = Q\Phi = (q - \sigma A)\Phi, \quad \Phi_t = P\Phi \tag{30}$$

Here, the *eigenfunction* $\Phi(s,t;\sigma)$ is $SU(2)$-valued while $Q(s,t;\sigma)$ and $P(s,t;\sigma)$ have values in the Lie algebra $su(2)$. Further, A is the fixed element $A = e_0$, and the *potential*, $q = q(s,t) = q_1e_1 + q_2e_2$, is meant to evolve *isospectrally*—this may be regarded as the essence of integrability—hence the lack of dependence on the *the spectral parameter* σ. (Note that both equations may be regarded as $SU(2)$-Lie equations, depending on parameters.) Cross-differentiating the pair of equations gives the *zero curvature condition* (ZCC), $Q_t - P_s + [Q,P] = 0$, i.e.,

$$q_t = P_s + [P,q] + \sigma[A,P] \tag{31}$$

The procedure for finding suitable P satisfying this compatibility condition begins with the *polynomial ansatz* $P = \sum_{j=0}^{m} \tilde{X}_j(s,t)(-\sigma)^{m-j}$. (One may prefer to write $\tilde{X}_j[q]$—we are actually seeking *ordinary differential operators*, acting on potentials q.) Substituting into the zero curvature condition and solving for the coefficients of $\sigma^1,\ \sigma^2, \dots,\ \sigma^m$, one straightforwardly obtains the *NLS recursion scheme*,

$$\tilde{J}\tilde{X}_n = \partial\tilde{X}_{n-1} + [\tilde{X}_{n-1}, q], \quad n = 1, \dots, m \tag{32}$$

Here, $\partial = \frac{\partial}{\partial s}$, and $\tilde{J}$ is the operator on $su(2)$ defined by $\tilde{J}B = ad_A B = [A,B]$, and the starting term for the recursion is $\tilde{X}_0 = -A$ (forced, up to scalar factor, by the σ^{m+1}-term in the expansion). Finally, the constant (σ^0) term describes isospectral evolution of the potential: $q_t = \partial\tilde{X}_m + [\tilde{X}_m, q]$.

To express the latter in the usual scalar form, we use the linear map $\tilde{Z} : su(2) \to C$ defined by $\tilde{Z}(ae_0 + be_1 + ce_2) = b + ic$. In particular, restriction of $\tilde{Z}$ to the two-dimensional subspace $\mathbf{m} = span(e_1, e_2)$ gives a convenient identification of $\mathbf{m}$ with the complex plane. Thus, e.g., we may associate to q the complex-valued function $\psi = u_1 + iu_2 = i\tilde{Z}q = -q_2 + iq_1$. Note that with the identification $\mathbf{m} \cong C$, $\tilde{J}|_{\mathbf{m}}$ corresponds to multiplication by i. The equations of the NLS hierarchy take the form $\psi_t = i\tilde{Z}(\partial\tilde{X}_m + [\tilde{X}_m, q]) = -\tilde{Z}\tilde{X}_{m+1}$.

Proposition 26. *The n^{th} equation in the NLS hierarchy is an $(n-1)^{st}$-order polynomial partial differential equation for $\psi(s,t)$. In fact, applying our usual normalization, it may be written*

$$\psi_t = -ZY_n = -(g_n + ih_n)[\psi], \tag{33}$$

where g_n and h_n are the operators computed in §3.2. This yields exactly NLS for $n = 3$, mKdV for $n = 4$, and for n even, reality of an initial function $\psi(s,t_0)$ is preserved in time. Finally, Equations 30 and 27 may be used to construct an evolving curve $\gamma(s,t)$ with complex curvature satisfying 33.

Proof. We may solve the NLS recursion scheme by imitating the argument for FM. Setting $\tilde{X} = \sum_{n=0}^{\infty} \lambda^n \tilde{X}_n$, (32) becomes $\tilde{J}\tilde{X} = \lambda\partial\tilde{X} + \lambda[\tilde{X}, q]$, hence, $\lambda\partial\langle\tilde{X},\tilde{X}\rangle = 2\langle\lambda\partial\tilde{X},\tilde{X}\rangle = 2\langle\tilde{J}\tilde{X} - \lambda[\tilde{X},q],\tilde{X}\rangle = 0$. One then solves for $\tilde{f}_n$ in terms of $\tilde{X}_1, \dots, \tilde{X}_{n-1}$ just as in §2, and obtains the inductive formula $\tilde{X}_n = \tilde{f}_n e_0 - \tilde{J}(\partial\tilde{X}_{n-1} + \lambda[\tilde{X}_{n-1}, q])$. The first claim follows.

Alternatively, writing $\tilde{X}_n = \tilde{f}_n e_0 + \tilde{g}_n e_1 + \tilde{h}_n e_2$, we note that (32) gives precisely Equation 22, with $\tilde{f}_n = f_n$, $\tilde{g}_n = g_n$, $\tilde{h}_n = h_n$. Thus, with the normalization $\langle\tilde{Y},\tilde{Y}\rangle = 1$, we can identify coefficients of $\tilde{Y}$ with 0-frame coefficients of Y. For the last claim, we solve Equation 30 for $\Phi(s,t;\sigma)$ and then apply Equation 27 for each time t, using $\sigma_0 = 0$. The result still depends on the choice of initial condition, say $\Phi(0,t;\sigma)$, and one may conveniently choose to eliminate the usual gauge term. □

For simplicity, we have used $\sigma = 0$ in the above proposition; however, a fuller interpretation of the FM-NLS recursion scheme equivalence is obtained by the following observation. Assume $\tilde{X}_0 = -A, \tilde{X}_1, \ldots, \tilde{X}_n \ldots$ satisfy (32). Define corresponding $X_n^\sigma = \{\tilde{X}_n\} = \Phi^{-1}\tilde{X}_n\Phi$ using a solution Φ to the Lie system (26), *with σ not necessarily zero.* Then

$$\partial X_{n-1}^\sigma = \{\partial \tilde{X}_{n-1} + [\tilde{X}_{n-1}, q - \sigma A]\} = \{\tilde{J}(\tilde{X}_n + \sigma\tilde{X}_{n-1}\} = J(X_n^\sigma + \sigma X_{n-1}^\sigma)$$

In other words, the X_n^σ solve the *shifted* FM recursion scheme, discussed in §4.3 (and this one-line computation might have sufficed as a proof of the proposition).

To pursue this one more step, we can define vectorfields Z_n^σ (as in §4.3) so that $\mathcal{R}^2 Z_n^\sigma = Y_n^\sigma$ (ignoring the gauge terms). Then the evolution equations $\gamma_t = Z_n^\sigma$ have the property that the evolution of σ-curvatures is independent of σ—a version of *isospectrality* on the curve level. In any event, one could choose to regard the FM recursion scheme *as a consequence* of this (or the usual) isospectrality ansatz.

Thus we have come full circle. We have not touched on interesting geometric aspects of many closely related topics—e.g., analogues of the FM hierarchy in hyperbolic, Lorentzian, and other geometric settings, connections to the Schwarzian KdV equation [C-L]. Certainly, much of our discussion could be merged nicely with these areas; however, to maintain the direct approach and narrow focus of this paper, we have disallowed topics which might argue for a broader synthesis (say, in the $SL(2,C)$ context). We have also not begun to introduce many of the powerful techniques of soliton theory into the picure—Lie algebraic, algebraic geometric, analytic—which are obviously relevant, but well beyond the scope of this paper. Some of these related topics may be found amoung the references, but we have not attempted to compile a comprehensive or representative bibliography.

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