ALMOST PSEUDO-VALUATION MAP AND
PSEUDO-ALMOST VALUATION MAP

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Abstract. Recently author (with A. Taouti) have introduced pseudo-
valuation maps and discussed pseudo-valuation domains through these
maps. In continuation we also introduced P-Krull domains with the help
of defined maps as well. In this note we generalize a pseudo-valuation
map \(\nu\) in the form of almost pseudo-valuation map and a pseudo-almost
valuation map \(\eta\). Furthermore, we construct and discuss an almost
pseudo-valuation domain and a pseudo-almost valuation domain through
the defined maps. Moreover, a few relationships between both integral
domains through the defined maps have been proved.

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domains

1. Introduction and Preliminaries

There are numerous studies on PVDs through various aspects. In \([8]\), the
group of divisibility of semi-valuation domains has discussed on the basis of a
semi-valuation map. Further, \([7]\) deals with the group of divisibility of quasi-
local domains, for example see \([7]\), Proposition 3.10]. An integral domain \(R\)
is said to be a pseudo-valuation domain (PVD) if every prime ideal of \(R\) is a
strongly prime \([5]\), Definition, p. 2]. A prime ideal \(P\) of \(R\) is called strongly
prime if \(xy \in P\), where \(x, y \in K\), then \(x \in P\) or \(y \in P\) (alternatively \(P\) is
strongly prime if and only if \(x^{-1}P \subseteq P\), whenever \(x \in K \setminus R\) \([3]\), Definition,
p. 2]. Every valuation domain is a PVD \([4]\, Proposition 1.1] but converse is
not true. A quasi-local domain \((R, M)\) is a PVD if and only if \(x^{-1}M \subseteq M\)
whenever \(x \in K \setminus R\) \([3]\, Theorem 1.4].

By \([1]\, p. 12], an integral domain \(D\) with the quotient field \(K\), is said to be a
valuation domain if it satisfies either of the (equivalent) conditions: (i) For any
two elements \(x, y \in D\), either \(x\) divides \(y\) or \(y\) divides \(x\). (ii) For any element
\(x \in K\), either \(x \in D\) or \(x^{-1} \in D\). When \(D\) is a valuation domain, \(G(D)\) is
merely the value group; and in this case, ideal theoretic properties of \(D\) are
easily derived from the corresponding properties of \(G(D)\), and conversely. \(D\)
is said to be an almost valuation domain (AVD) if for every \(0 \neq x \in K\), there
is a positive integer \(n\) such that either \(x^n\) or \(x^{-n} \in D\).

Following \([2]\), \(D\) is said to be a pseudo-almost valuation domain (PAVD) if
each prime ideal \(P\) of \(D\) is pseudo-strongly prime ideal (that is if, whenever

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$x, y \in K$ and $xyP \subseteq P$, then there is a positive integer $m \geq 1$ such that either $x^m \in R$ or $y^mP \subseteq P$). Equivalently, $D$ is a PAVD if and only if $D$ is quasi-local and for every nonzero element $x \in K$, there is an integer $n \geq 1$ such that either $x^n \in D$ or $ax^{-n} \in D$ for every nonunit $a \in D$.

Following [1], an integral domain $D$ is said to be an almost pseudo-valuation domain (APVD) if each prime ideal $P$ of $D$ is strongly primary ideal, in the sense that $xy \in P, x, y \in K$ implies that either $x^n \in P$ for some $n \geq 1$ or $y \in P$. Equivalently, $D$ is an APVD if and only if $D$ is quasi-local with maximal ideal $M$ such that for every nonzero element $x \in K$, either $x^n \in M$ for some positive integer $n \geq 1$ or $ax^{-1} \in M$ for every nonunit $a \in D$.

In general,

$VD \Rightarrow AVD \Rightarrow PAVD$

$\downarrow$

$PVD \Rightarrow APVD$

But none of the above implications is reversible.

As every PVD is necessarily quasi-local [3, Cor 1.3] and a quasi-local domain is PVD if and only if its maximal ideal is strongly prime [3, Theorem 1.4].

Author (with T. Shah) has introduced almost pseudo valuation monoids and pseudo almost valuation monoids in [9]. In [12], authors introduced a generalization of valuation maps by using some different conditions. Recently author, (with A. Taouti) introduced pseudo-valuation maps, and discussed pseudo-valuation domains through these maps [10, Theorem 1.4]. Also, author (with T. Shah and A. Taouti) introduced the class of domains (P-Krull domains) through these maps [11].

In this note we continue our study, and first generalize the pseudo-valuation map as an almost pseudo-valuation map $\nu$ and pseudo-almost valuation map $\eta$, then we constructed almost pseudo-valuation domain and pseudo-almost valuation domain through the defined maps. Finally, we discuss a few relationships between these domains.

2. Almost pseudo-valuation map and pseudo-almost valuation map

Here we consider $K^* = K\backslash\{0\}$ is a field and $G$ a partially ordered group. Now we begin with the following definition.

**Definition 1.** Let $\nu : K^* \to G$ be an onto map, which has the following properties. For $x, y \in K^*$;

(a) $\nu(xy) = \nu(x) + \nu(y)$
(b) $\nu(x) < \nu(y)$ implies $\nu(x + y) = \nu(x)$.
(c) $n\nu(x) = ng > 0$, for $n \in \mathbb{Z}^+$ or $g = \nu(x) < n\nu(y) = nh$ such that $nh > 0$, where $g, h \in G$ and $h > 0$.

In Definition 1, the map $\nu$ is an extended semi-valuation map. No doubt, (b) implies that it is a quasi-local domain as discussed in [1, p. 180]. Moreover, condition (c) plays an important role, hereafter we call $\nu$, the almost pseudo-valuation map. From Definition 1 we see that $\nu$ inherit a specific characteristic in $G$. We manipulate $G$ in the definition below.
Definition 2. Let \( (G, \leq) \) be a partially ordered group. The partial order \( \leq \) is an almost total order if for all \( g, h \in G \), there exists some fixed positive integer \( n \) such that either \( g \leq nh \) (and also \( ng \leq nh \)) or \( h \leq ng \) (and also \( nh \leq ng \)). We will denote such a group by \( G^\# \).

Definition 3. A partially ordered set \( X \) is said to be directed if every two elements have both an upper bound and a lower bound. A partially ordered group \( G \) whose partial order is directed is called directed group [3, p. 2].

Remark 1. The group \( G^\# \) is directed and not a torsion free.

Let \( R_v = \{ x \in K : v(x) \geq 0 \} \) be a subset of \( K^* \) which is related through the map \( v \) to \( G \). We derive the nature of \( R_v \) and we will find that \( R_v \) is a basically APVD.

Proposition 1. \( R_v = \{ x \in K : v(x) \geq 0 \} \) is an almost pseudo-valuation domain.

Proof. Clearly \( 1 \in R_v \) and, by Definition 4(a), \( R_v \) is closed under multiplication. If \( x, y \in R_v \) then \( v(x - y) \geq v(1) = 0 \) since \( v(x) \geq v(1) \) and \( v(y) \geq v(1) \). Thus \( R_v \) is a subring of \( K \) with identity. The map \( v \) is, no doubt, a group homomorphism and its kernel is \( U = \{ x \in K : v(x) = 0 \} \) which shows that \( U \) is a group of units of \( R_v \). So \( R_v \) is an integral domain. Definition 4(b) shows that \( R_v \) is a quasi local domain. Let \( M \) be a maximal ideal of \( R_v \). Furthermore, let \( x \in K \setminus R_v \) so by definition 4(c), \( x^n \in M \). Thus \( R_v \) is an almost pseudovaluation domain.

Below we give a crucial proposition for a better utilization of a group \( G^\# \).

Proposition 2. Let \( D \) be an integral domain with quotient field \( K \) and group of divisibility \( G \). The following are equivalent.

(i) \( D \) is an APVD (and hence quasi-local).

(ii) For each \( g \in G \), there exist \( n \in \mathbb{Z}^+ \) such that either \( ng > 0 \) or \( g < nh \) for all \( h \in G \).

Proof. \((1) \Rightarrow (2)\)

Let \( M \) be the only maximal ideal in \( D \). Let \( g \in G \) such that \( g = xU \), where \( x \in K^* \). So, the definition implies if \( x^n \in M \) for some positive integer \( n \geq 1 \), then we have \( ng = x^nU > 0 \) and if \( ax^{-1} \in M \) for any nonunit \( a \in D \) such that \( nh = aU > 0 \), then \( ax^{-1}U = h - g > 0 \). Thus \( g < nh \).

\((2) \Rightarrow (1)\)

We first show that \( D \) is quasi-local. If \( D \) has two distinct maximal ideals \( M \) and \( N \), then choose \( x \in M \setminus N \) and \( y \in N \setminus M \). Let \( ng' = x^n y^{-n} U \) and \( h = yU \). Clearly, \( ng' \neq 0 \) and \( g' \not< nh \), while \( nh > 0 \) because if \( g' < nh \) then this means that \( xy^{-1}U < yU \) implies that \( xU < y^2U \) and equivalently \( y^2 D \subset xD \subset M \) and hence \( y \in M \) contradiction to our supposition, therefore \( g' \not< nh \). This contradicts the hypothesis, so \( D \) must be local. Let \( x \in K \) such that \( xU = g \) and \( ng = x^nU \), if \( ng > 0 \) this implies that \( x^n \in M \) and if \( g < nh \) then \( xU < nh \). Let \( a \) be a nonunit element in \( D \) such that \( nh = aU \), obviously \( nh > 0 \) and hence \( xU < aU \Rightarrow ax^{-1}U > 0 \Rightarrow ax^{-1} \in M \). Hence, \( D \) is APVD.
Remark 2. By above Proposition 4(ii), it becomes clear that \( G \cong G^\# \) (\( G \) is isomorphic to \( G^\# \)). Thus \( G^\# \) is a group of divisibility of an almost pseudo-valuation domain.

We can discuss all the characteristics of an APVD through the map \( v \). Now, after defining an almost pseudo-valuation map we define pseudo-almost valuation map. In Definition 4 we consider \( K^* = K\setminus\{0\} \) is a field and \( G \), a partially ordered group.

Definition 4. Let \( \eta : K^* \rightarrow G \) be an onto map, which has the following properties. For \( x, y \in K^* \);

(a) \( \eta(xy) = \eta(x) + \eta(y) \).
(b) \( \eta(x) < \eta(y) \) implies \( \eta(x + y) = \eta(x) \).
(c) \( \eta(x^n) = ng > 0 \) or \( \eta(y) = h \) such that \( ng < h \) for all \( h \in G \), where \( h > 0 \).

We call \( \eta \), the pseudo-almost valuation map.

In Definition 4\( \eta \) inherits a specific property in \( G \), hereafter we denote such a \( G \) by \( G^\## \).

Let \( R_\eta = \{ x \in K : \eta(x) \geq 0 \} \) be a subset of \( K^* \) which is related through the map \( \eta \) to \( G \). We derive the nature of \( R_\eta \) and we will find that \( R_\eta \) is a basically APVD.

Proposition 3. \( R_\eta = \{ x \in K : \eta(x) \geq 0 \} \) is a pseudo almost valuation domain.

Proof. Clearly, \( 1 \in R_\eta \) and, by definition 4(a) \( R_\eta \) is closed under multiplication. If \( x, y \in R_\eta \) then \( \eta(x - y) \geq \eta(1) = 0 \) since \( \eta(x) \geq \eta(1) \) and \( \eta(y) \geq \eta(1) \). Thus \( R_\eta \) is a subring of \( K \) with identity. The map \( \eta \) is no doubt a group homomorphism and its kernel is \( U = \{ x \in K : \eta(x) = 0 \} \), which shows that \( U \) is a group of units of \( R_\eta \). So \( R_\eta \) is an integral domain. Definition 4(b) shows that \( R_\eta \) is a quasi local domain. Let \( x \in K \setminus R_\eta \), by Definition 4(c), \( x^n \in D \) for \( n \geq 1 \). Thus \( R_\eta \) is a pseudo almost valuation domain. \( \square \)

Further we check the validity of our defined map \( \eta \) and group of divisibility \( G^\## \) in Proposition 4.

Proposition 4. Let \( D \) be an integral domain with quotient field \( K \) and group of divisibility \( G^\## \), then the following are equivalent

(i) \( D \) is a PAVD.
(ii) For each \( g \in G^\## \), there exist \( n \in \mathbb{Z}^+ \) and is fixed, such that either \( ng > 0 \) or \( ng < h \) for all \( h \in G^\## \), where \( h > 0 \).

Proof. (1) \( \Rightarrow \) (2), let \( E(D) = \{ x \in K \mid x^n \notin D \text{ for every } n \geq 1 \} \) if \( x \in E(D) \), then clearly \( ng \geq 0 \). As in \( D \), every prime ideal is a pseudo strongly prime, so \( x^{-n}M \subseteq M \). Then for each \( m \in M \), \( x^{-n}m \in M \). Let \( xU = g \) and \( mU = h \geq 0 \), so \( (x^{-n}m)U = x^{-n}U + mU = -ng + h > 0 \) implies \( ng < h \) for each \( h > 0 \) otherwise \( g \geq nh \), which follows that \( x^n \in D \).

(2) \( \Rightarrow \) (1) Let \( M \) be the maximal ideal of \( D \), to show \( D \) is a PAVD we only need to show \( M \) is the pseudo strongly prime ideal. For this let \( x \in E(D) \)
such that \( xU = g \in G \), for each integer \( n \geq 1 \). So for each \( m \in M \), we choose \( mU = h > 0 \). Then by the hypothesis \( ng < h \) implies that \( x^nU < mU \). This implies \( mx^{-n}U > 0 \) and so \( mx^{-n} \in M \). Hence \( x^{-n}M \subset M \). So \( M \) is a pseudo strongly prime ideal.

**Remark 3.** From Definition \( \text{(2)} \) and by Proposition \( \text{(4)} \) it is clear that \( \text{APVD} \implies \text{PAVD} \). Also, we have \( G^\# \subset G^\## \).

**Conclusion 1.** This study brings a method by which one can discuss each of the characteristics of an almost pseudo-valuation domain and a pseudo almost valuation domain with the help of maps \( \nu \) and \( \eta \). Furthermore at the base of the maps \( \nu \) and \( \eta \) we can construct new integral domains which can be written as an intersection of almost pseudo-valuation overrings and pseudo-almost valuation overrings. Authors have already introduced the domains that can be written as intersection of pseudo-valuation overrings.

**References**


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