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MICROLOCAL TIME DECAYS FOR HYPERBOLIC EQUATIONS WITH LOWER ORDER TERMS[1]

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Abstract. In this paper we present microlocalised time-decay rates of solutions to hyperbolic equations with constant coefficients with arbitrary lower order terms. A particular attention is paid to regions with multiplicities.

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1. Introduction

In this note we give an overview of microlocal time-decay rates of oscillatory integrals appearing in the solution to the Cauchy problem for hyperbolic partial differential equations with constant coefficients and arbitrary lower order terms. We will be most interested in microlocal decay rates corresponding to different regions in the phase space. These decay rates can be then put together in the standard way as explained, for example, in [10]. Thus, we consider equations of the form

$$\begin{cases} L(D_t, D_x) \equiv D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, \\ D_t^l u(0,x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1, \end{cases} \tag{1}$$

with $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. Results on the time-decay rates of its solutions can be expressed in terms of its characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$, which are solutions to the characteristic polynomial equation $L(\tau, \xi) = 0$ with respect to τ. Symbol $P_j(\xi)$ of $P_j(D_x)$ (where as usual $D_x = -i\partial_x$) is assumed to be a homogeneous polynomial of order j, and the $c_{\alpha,r}$ are complex constants. We assume that equation (1) is strictly hyperbolic in order not to worry about its well-posedness.

Throughout, we assume the stability conditions

$$\operatorname{Im} \tau_k(\xi) \geq 0 \quad \text{for } k = 1, \dots, m\,, \tag{2}$$

for all $\xi \in \mathbb{R}^n$. Solution to the Cauchy problem (1) can be written in the form

$$u(t,x) = \sum_{j=0}^{m-1} E_j(t) f_j(x),$$

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where propagators $E_j(t)$ are defined by

$$E_j(t)f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\Big(\sum_{k=1}^{m} e^{i\tau_k(\xi)t} A_j^k(t,\xi)\Big)\widehat{f}(\xi)\,d\xi\,, \tag{3}$$

with suitable amplitudes $A_j^k(t,\xi)$. In the areas where roots are simple, phases and amplitudes are smooth, and we can analyse the sum (3) termwise, reducing the analysis to single integrals. In the case of multiple characteristics we can group terms in (3) in a special way to obtain suitable decay estimates. We note that properties of characteristics may be different in different regions. To isolate different types of behaviour in different areas we can microlocalise with cut-off functions $\chi(\xi)$, to study integrals of the form

$$\chi(D)E_j(t)f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\Big(\sum_{k=1}^{m} e^{i\tau_k(\xi)t} A_j^k(t,\xi)\Big)\chi(\xi)\widehat{f}(\xi)\,d\xi\,. \tag{4}$$

We will denote various constants throughout the paper by the same letter C. Balls with radius R centred at $\xi \in \mathbb{R}^n$ will be denoted by $B_R(\xi)$. We will use the notation $\langle\xi\rangle = \sqrt{1+|\xi|^2}, \langle D\rangle = \sqrt{1-\Delta}$ and $|D| = |-\Delta|^{1/2}$. The Sobolev space W_p^l is then defined as the space of measurable functions f for which $\langle D\rangle^l f \in L^p(\mathbb{R}^n_x)$.

We will also use the standard notation for the symbol class $S^\mu = S^\mu_{1,0}$, as a space of smooth functions $a = a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying symbolic estimates $|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{\mu-|\alpha|}$, for all $x,\xi \in \mathbb{R}^n$, and all multi-indices α,β.

If the function $a = a(\xi)$ is independent of x, we will sometimes also write $a \in S^\mu_{1,0}(U)$ for an open set $U \subset \mathbb{R}^n$, if $a = a(\xi) \in C^\infty(U)$ satisfies $|\partial_\xi^\alpha a(\xi)| \le C_\alpha(1+|\xi|)^{\mu-|\alpha|}$, for all $\xi \in U$, and all multi-indices α.

Estimates for wave type equations have been thoroughly analysed in [1, 2, 4, 6, 11, 14]. The case of dissipative wave equations was treated in [5] and for the analysis of the Klein–Gordon equation we refer to [3] and references therein. Equations with homogeneous symbols were analysed in [12, 13].

The present paper is based on the lecture of the author at the 12^{th} Serbian Mathematical Congress, which was based on papers [7, 8, 9, 10]. For the detailed proofs we refer to [10].

2. Away from the real axis

We begin by looking at the zone where roots are separated from the real axis. If the roots are smooth, we can analyse solution (3) termwise:

Theorem 2.1. *Let $\tau : U \to \mathbb{C}$ be a smooth function, $U \subset \mathbb{R}^n$ open. Let $a \in S^{-\mu}_{1,0}(U)$, i.e. assume that $a = a(\xi) \in C^\infty(U)$ satisfies*

$$|\partial_\xi^\alpha a(\xi)| \le C_\alpha(1+|\xi|)^{-\mu-|\alpha|},$$

for all $\xi \in U$ and all multi-indices α. Let $\chi \in S^0_{1,0}(\mathbb{R}^n)$ be such that $\chi = 0$ outside U. Assume further that:

(i) *there exists* $\delta > 0$ *such that* $\operatorname{Im}\tau(\xi) \geq \delta$ *for all* $\xi \in U$;

(ii) $|\tau(\xi)| \leq C(1 + |\xi|)$ *for all* $\xi \in U$.

Then for all $t \geq 0$ *we have*

$$\Big\| D_t^r D_x^\alpha \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau(\xi)t)} a(\xi)\chi(\xi)\widehat{f}(\xi)\, d\xi \Big) \Big\|_{L^q(\mathbb{R}^n_x)} \leq Ce^{-\delta t} \|f\|_{W_p^{N_p + |\alpha| + r - \mu}}, \tag{5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \leq 2$, $N_p \geq n\big(\frac{1}{p} - \frac{1}{q}\big)$, $r \geq 0$, α *a multi-index and* $f \in C_0^\infty(\mathbb{R}^n)$. *If* $p = 1$, *we take* $N_1 > n$.

Moreover, let us assume that equation $L(\tau, \xi) = 0$ *has only simple roots* $\tau_k(\xi)$ *which satisfy condition* (i) *above, in the open set* $U \subset \mathbb{R}^n$, *for all* $k = 1, \ldots, m$. *Then solution* u *to* (1) *satisfies*

$$||D_t^r D_x^\alpha \chi(D) u(t, \cdot)||_{L^q(\mathbb{R}^n_x)} \leq Ce^{-\delta t} \sum_{l=0}^{m-1} ||f_l||_{W_p^{N_p + |\alpha| + r - l}}, \tag{6}$$

where $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* N_p, r, α *are as above.*

We note that we may have different norms on the right.hand side of (6). For example, we also have the following estimate:

$$||D_t^r D_x^\alpha \chi(D) u(t, \cdot)||_{L^q(\mathbb{R}^n_x)} \leq Ce^{-\delta t} \sum_{l=0}^{m-1} ||f_l||_{W_2^{N'_q + |\alpha| + r - l}}, \tag{7}$$

where $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $N'_q \geq \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$, and $N'_\infty > \frac{n}{2}$ for $p = 1$.

To be able to derive time decay in the case of multiple roots, we will group terms in (3) in the following way. Assume that roots $\tau_1(\xi), \ldots, \tau_L(\xi)$ coincide on a set contained in some $\mathcal{M}$, that is

$$\mathcal{M} \supset \{\tau_1(\xi) = \cdots = \tau_L(\xi)\}.$$

For $\varepsilon > 0$, we define

$$\mathcal{M}^\varepsilon := \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) < \varepsilon\}.$$

Choose $\varepsilon > 0$ so that these roots $\tau_1(\xi), \ldots, \tau_L(\xi)$ do not intersect with any of the other roots $\tau_{L+1}(\xi), \ldots, \tau_m(\xi)$ in $\mathcal{M}^\varepsilon$. If different numbers of roots intersect in different sets, we can apply the following theorem to such sets one by one. We note that by the strict hyperbolicity of (1) the set $\mathcal{M}^\varepsilon$ is bounded. Here we will estimate the sum

$$\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t, \xi) \Big) \chi(\xi)\widehat{f}(\xi)\, d\xi. \tag{8}$$

Theorem 2.2. *Let the sum* (3) *be the solution to the Cauchy problem* (1)*. Assume that roots* $\tau_1(\xi),\ldots,\tau_L(\xi)$ *coincide in a set contained in* $\mathcal{M}$ *and do not intersect other roots in the set* $\mathcal{M}^\varepsilon$*. Let* $\chi\in C_0^\infty(\mathcal{M}^\varepsilon)$*. Assume that there exists* $\delta>0$ *such that* $\operatorname{Im}\tau_k(\xi)\geq\delta$ *for all* $\xi\in\mathcal{M}^\varepsilon$ *and* $k=1,\ldots,L$*.*

Then for all $t\geq 0$ *we have*

$$\left\| D_t^r D_x^\alpha\Big(\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi}\Big(\sum_{k=1}^L e^{i\tau_k(\xi)t}A_j^k(t,\xi)\Big)\chi(\xi)\widehat{f}(\xi)\,dx\Big)\right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{L-1}e^{-\delta t}\|f\|_{L^p},$$

where $\frac{1}{p}+\frac{1}{q}=1$*,* $1\leq p\leq 2$*.*

Thus, if characteristic roots are separated from the real axis on the support of some $\chi\in C_0^\infty(\mathbb{R}^n)$, we can separate the solution (3) into groups of multiple roots for which the L^p-L^q norms still decay exponentially as stated in Theorem 2.2. We also note that since $\mathcal{M}^\epsilon$ is bounded, assumption (ii) of Theorem 2.1 is automatically satisfied and, therefore, it is omitted in the formulation of Theorem 2.2.

2.1. Roots with non-degeneracies

The following case that we consider is the one when roots satisfy certain non-degeneracy conditions. These may be conditions on the Hessian, convexity conditions, or simply the information on the index of the corresponding level surfaces. In this section we will give the corresponding statements. We always assume the stability condition (2) but no longer assume that roots are separated from the real axis.

First we state the result for phases with the non-degenerate Hessian. The behaviour depends on critical points ξ^0 with $\nabla\tau(\xi^0)=0$ and the behaviour of the Hessian at such points. As usual, we say that the critical point ξ^0 is non-degenerate if the Hessian $\mathrm{Hess}\tau(\xi^0)$ is non-degenerate.

Theorem 2.3. *Let* $U\subset\mathbb{R}^n$ *be a bounded open set, and let* $\tau:U\to\mathbb{C}$ *be smooth and such that* $\operatorname{Im}\tau(\xi)\geq 0$ *for all* $\xi\in U$*. Assume that there are some constants* C_0 *and* M *such that*

$$|\det\mathrm{Hess}\tau(\xi)|\geq C_0(1+|\xi|)^{-M}$$

for all $\xi\in U$*. Let* $\chi\in S^0_{1,0}(\mathbb{R}^n)$ *be such that* $\chi=0$ *outside* U *and let* $a\in S^{-\mu}_{1,0}(U)$*.*

Assume that τ *has only one non-degenerate critical point in* U*, and that* U *is sufficiently small. Then there is a constant* $C>0$ *independent of the position of* U *such that for all* $t\geq 0$ *we have*

$$\left|\left|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a(\xi)\chi(\xi)\widehat{f}(\xi)d\xi\right|\right|_{L^q(\mathbb{R}^n_x)}\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}||f||_{W_p^{N_p}}, \tag{9}$$

with $1\leq p\leq 2$*,* $\frac{1}{p}+\frac{1}{q}=1$*,* $N_p=\frac{M}{2}(\frac{1}{p}-\frac{1}{q})-\mu$*.*

For example, the case of the Klein–Gordon equation corresponds to $M = n+2$ in this theorem. Since we want to have estimate (2.3) uniformly over all U of fixed volume but independent of its position, we use the norm $||f||_{W_p^{N_p}}$. If we want an estimate just for a single U, norm $||f||_{W_p^{N_p}}$ on the right-hand side of (9) can be replaced by $||f||_{L^p}$. For details of this we refer to [10]. The condition that critical points are isolated and therefore can be localised by different sets U may follow from certain properties of τ.

If we apply different versions of the stationary phase method under different conditions, we can reach different conclusions here. For example, we also have:

Theorem 2.4. *Let $U \subset \mathbb{R}^n$ be a bounded open and let $\tau : U \to \mathbb{C}$ be smooth and such that $\operatorname{Im} \tau(\xi) \geq 0$ for all $\xi \in U$. Let $\chi \in S^0_{1,0}(\mathbb{R}^n)$ be such that $\chi = 0$ outside U and let $a \in S^{-\mu}_{1,0}(U)$. Assume that τ has only one critical point ξ^0 in U, and that U is sufficiently small.*

Suppose that there are constants $C_0, M > 0$ independent of the size and position of U and of ξ^0, with the following conditions. Suppose that

$$\operatorname{rank} \operatorname{Hess}\tau(\xi^0) = k,$$

that this rank is attained on a $k \times k$ submatrix $A(\xi^0)$ and that

$$|\det A(\xi^0)| \geq C_0(1 + |\xi^0|)^{-M}.$$

Then for all $t \geq 0$ we have

$$\left|\left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau(\xi)t)} a(\xi)\chi(\xi)\widehat{f}(\xi)d\xi \right|\right|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{k}{2}(\frac{1}{p}-\frac{1}{q})}||f||_{W_p^{N_p}},$$

with $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $N_p = \frac{M}{2}(\frac{1}{p} - \frac{1}{q}) - \mu$.

The proof of this theorem is similar to the proof of Theorem 2.3 once we restrict to the set of k variables (possibly after a suitable change) on which the rank of the Hessian is attained on $A(\xi^0)$.

This result can be improved depending on further properties of $A(\xi^0)$. For example, if $\operatorname{rank} A(\xi^0) = n-1$ and this is attained on variables $\xi_1, \ldots, \xi_{n-1}$, the analysis reduces to the behaviour of the oscillatory integral with respect to ξ_n. If the l-th derivative of the phase with respect to ξ_n is non-zero, we get an additional decay by $t^{-1/l}$. This follows from the stationary phase method, or from an appropriate use of van der Corput lemma. We will not formulate further statements here since they are quite straightforward.

The next theorem is an estimate of oscillatory integrals with real-valued phases under convexity condition. The convexity condition is weaker than (but does not contain) the condition that the Hessian of τ is positive definite and the result can be compared with Theorem 2.3, dependent on suitable properties of roots.

Let us first give the necessary definitions. Given a smooth function $\tau : \mathbb{R}^n \to \mathbb{R}$ and $\lambda \in \mathbb{R}$, set

$$\Sigma_\lambda \equiv \Sigma_\lambda(\tau) := \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \ .$$

A smooth function $\tau : \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the convexity condition if surface Σ_λ is convex for each $\lambda \in \mathbb{R}$. Note that the empty set and the point set are considered to be convex. If the Gaussian curvature of Σ_λ never vanishes, Σ_λ is automatically convex (the converse is not true). This curvature condition corresponds to the case $k = n - 1$ in Theorem 2.4. Another important notion is that of the maximal order of contact of a hypersurface:

Definition 2.1. Let Σ be a hypersurface in $\mathbb{R}^n$. Let $\sigma \in \Sigma$, and denote the tangent plane at σ by T_σ. Now let P be a 2–dimensional plane containing the normal to Σ at σ and denote the order of the contact between the line $T_\sigma \cap P$ and the curve $\Sigma \cap P$ by $\gamma(\Sigma; \sigma, P)$. Then set

$$\gamma(\Sigma) := \sup_{\sigma\in\Sigma} \sup_{P} \gamma(\Sigma; \sigma, P) \, .$$

We note that $\gamma(\mathbb{S}^n) = 2$ since $\gamma(\mathbb{S}^n; \sigma, P) = 2$ for all $\sigma \in \mathbb{S}^n$ and all planes P containing σ and the origin. If $\varphi_l(\xi)$ is a characteristic root of an m^{th} order homogeneous strictly hyperbolic constant coefficient operator, then $\gamma(\Sigma_{\varphi_l}) \le m$ ([13]). Now we can formulate the corresponding theorem.

Theorem 2.5. *Suppose $\tau : \mathbb{R}^n \to \mathbb{R}$ satisfies the convexity condition and let $\chi \in C^\infty(\mathbb{R}^n)$; furthermore, on* $\operatorname{supp}\chi$, *we assume*:

- *for all multi-indices α there exists a constant $C_\alpha > 0$ such that*
$$|\partial_\xi^\alpha \tau(\xi)| \le C_\alpha (1 + |\xi|)^{1-|\alpha|};$$
- *there exist constants $M, C > 0$ such that for all $|\xi| \ge M$ we have $|\tau(\xi)| \ge C|\xi|$;*
- *there exists a constant $C > 0$ such that $|\partial_\omega \tau(\lambda\omega)| \ge C$ for all $\omega \in \mathbb{S}^{n-1}$, $\lambda > 0$; in particular, $|\nabla\tau(\xi)| \ge C$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$;*
- *there exists a constant $R_1 > 0$ such that, for all $\lambda > 0$,*
$$\frac{1}{\lambda}\Sigma_\lambda(\tau) \equiv \frac{1}{\lambda}\{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \subset B_{R_1}(0) \, .$$

Also, set $\gamma := \sup_{\lambda>0} \gamma(\Sigma_\lambda(\tau))$ and assume this is finite. Let $a_j = a_j(\xi) \in S^{-j}_{1,0}$ be a symbol of order $-j$ of type $(1,0)$ on $\mathbb{R}^n$. Then for all $t \ge 0$ we have the estimate

$$\left\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\chi(\xi)\widehat{f}(\xi)\, d\xi \right\|_{L^q(\mathbb{R}^n_x)} \le C(1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{W_p^{N_{p,j,t}}} \,, \tag{10}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1<p\le 2$, *and the Sobolev order satisfies* $N_{p,j,t}\ge n(\frac{1}{p}-\frac{1}{q})-j$ *for* $0\le t<1$, *and* $N_{p,j,t}\ge \left(n-\frac{n-1}{\gamma}\right)(\frac{1}{p}-\frac{1}{q})-j$ *for* $t\ge 1$.

In the case without convexity, we also introduce an analogue of the order of contact. Thus, if Σ is a hypersurface in $\mathbb{R}^n$, not necessarily convex, we define

$$\gamma_0(\Sigma):=\sup_{\sigma\in\Sigma}\inf_{P}\gamma(\Sigma;\sigma,P)\le\gamma(\Sigma),$$

where $\gamma(\Sigma;\sigma,P)$ is as in Definition 2.1. If $p(\xi)$ is a polynomial of order m, $\Sigma=\{\xi\in\mathbb{R}^n : p(\xi)=0\}$ is compact and $\nabla p(\xi)\ne 0$ on Σ, then $\gamma_0(\Sigma)\le\gamma(\Sigma)\le m$ ([13]).

Theorem 2.6. *Suppose* $\tau:\mathbb{R}^n\to\mathbb{R}$ *is a smooth function. Let* $\chi\in C^\infty(\mathbb{R}^n)$*; furthermore, on* $\operatorname{supp}\chi$*, we assume:*

- *for all multi-indices* α *there exist constants* $C_\alpha>0$ *such that*
$$|\partial_\xi^\alpha\tau(\xi)|\le C_\alpha(1+|\xi|)^{1-|\alpha|};$$
- *there exist constants* $M, C>0$ *such that for all* $|\xi|\ge M$ *we have* $|\tau(\xi)|\ge C|\xi|$;
- *there exists a constant* $C>0$ *such that* $|\partial_\omega\tau(\lambda\omega)|\ge C$ *for all* $\omega\in\mathbb{S}^{n-1}$ *and* $\lambda>0$;
- *there exists a constant* $R_1>0$ *such that, for all* $\lambda>0$,
$$\frac{1}{\lambda}\{\xi\in\mathbb{R}^n:\tau(\xi)=\lambda\}\subset B_{R_1}(0)\,.$$

Set $\gamma_0:=\sup_{\lambda>0}\gamma_0(\Sigma_\lambda(\tau))$ *and assume it is finite. Let* $a_j=a_j(\xi)\in S^{-j}_{1,0}$ *be a symbol of order* $-j$ *of type* $(1,0)$ *on* $\mathbb{R}^n$*. Then for all* $t\ge 0$ *we have the estimate*

$$\left\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\chi(\xi)\widehat{f}(\xi)\,d\xi\right\|_{L^q(\mathbb{R}^n_x)}\le C(1+t)^{-\frac{1}{\gamma_0}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{W_p^{N_{p,j,t}}},$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1<p\le 2$, *and the Sobolev order satisfies* $N_{p,j,t}\ge n(\frac{1}{p}-\frac{1}{q})-j$ *for* $0\le t<1$, *and* $N_{p,j,t}\ge\left(n-\frac{1}{\gamma_0}\right)(\frac{1}{p}-\frac{1}{q})-j$ *for* $t\ge 1$.

As a corollary and an example of these theorems, we get the following possibilities of decay for parts of solutions with roots on the axis. We can use a cut-off function χ to microlocalise around points with different qualitative behaviour (hence we also do not have to worry about Sobolev orders).

Colorallary 2.1. *Let* $\Omega\subset\mathbb{R}^n$ *be an open set and let* $\tau:\Omega\to\mathbb{R}$ *be a smooth real valued function. Let* $\chi\in C_0^\infty(\Omega)$*. Let us make the following choices of* $K(t)$*, depending on which of the following conditions are satisfied on* $\operatorname{supp}\chi$.

(1) *If* $\det \mathrm{Hess}\tau(\xi) \neq 0$ *for all* $\xi \in \Omega$*, we set* $K(t) = (1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$.

(2) *If* $\mathrm{rank}\mathrm{Hess}\tau(\xi) = n-1$ *for all* $\xi \in \Omega$*, we set* $K(t) = (1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$.

(3) *If* τ *satisfies the convexity condition with index* γ*, we set* $K(t) = (1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$.

(4) *If* τ *does not satisfy the convexity condition but has non-convex index* γ_0*, we set* $K(t) = (1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$.

Assume in each case that other assumptions of the corresponding Theorems 2.3–2.6 *are satisfied. Let* $1 \le p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$*. Then for all* $t \ge 0$ *we have*

$$\left|\left|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\chi(\xi)\widehat{f}(\xi)d\xi\right|\right|_{L^q(\mathbb{R}^n_x)} \le CK(t)||f||_{L^p(\mathbb{R}^n)}.$$

We note that no derivatives appear in the L^p–norm of f because the support of χ is bounded. In general, there are different ways to ensure the convexity condition for τ. We refer to [10] for a detailed discussion.

2.2. Roots meeting the real axis

In this section we will present the results for characteristic roots (or phase functions) in the upper complex plane near the real axis, that become real at some point or in some set.

For $\mathcal{M} \subset \mathbb{R}^n$, denote

$$\mathcal{M}^\varepsilon = \{\xi \in \mathbb{R}^n : \mathrm{dist}(\xi, \mathcal{M}) < \varepsilon\},$$

as before. The largest number $\nu \in \mathbb{N}$ such that $\mathrm{meas}(\mathcal{M}^\varepsilon) \le C\varepsilon^\nu$ for all sufficiently small $\varepsilon > 0$, will be denoted by $\mathrm{codim}\mathcal{M}$, and we will call it the codimension of $\mathcal{M}$.

We will say that the root τ_k meets the real axis at ξ^0 with order s_k if $\mathrm{Im}\ \tau_k(\xi^0) = 0$ and if there exists a constant $c_0 > 0$ such that

$$c_0|\xi-\xi^0|^{s_k} \le \mathrm{Im}\ \tau_k(\xi),$$

for all ξ sufficiently near ξ^0. More generally, if the root τ_k meets the axis on the set $Z_k = \{\xi \in \mathbb{R}^n : \mathrm{Im}\ \tau_k(\xi) = 0\}$, we will say that it meets the axis with order s if

$$c_0 \mathrm{dist}(\xi, Z_k)^s \le \mathrm{Im}\ \tau_k(\xi).$$

We will localise around each connected component of Z_k, e.g. around each point of Z_k, if it is a union of isolated points. As usual, when we talk about multiple roots intersecting in a set $\mathcal{M}$, we adopt the terminology introduced earlier. Since we are dealing with strictly hyperbolic equations, roots can meet each other only for bounded frequencies, so we may assume that set $\mathcal{M}$ is bounded.

Theorem 2.7. *Assume that the characteristic roots $\tau_1(\xi),\dots,\tau_L(\xi)$ intersect in the C^1 set $\mathcal{M}$ of codimension ℓ. Assume also that they meet the real axis in $\mathcal{M}$ with the finite orders $\le s$, i.e. that*

$$c_0 \mathrm{dist}(\xi,\mathcal{M})^s \le \mathrm{Im}\ \tau_k(\xi)\,,$$

for some $c_0>0$ and all $k=1,\dots,L$. Assume that (3) *is the solution of the Cauchy problem* (1) *and we look at its part* (8)*. Let $\chi\in C_0^\infty(\mathcal{M}^\varepsilon)$ for sufficiently small $\varepsilon>0$. Then for all $t\ge 0$ we have*

$$\Big\|D_t^r D_x^\alpha\Big(\int_{\mathcal{M}^\varepsilon} e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(t,\xi)\Big)\chi(\xi)\widehat{f}(\xi)\,d\xi\Big)\Big\|_{L^q(\mathbb{R}^n_x)} \le C(1+t)^{-\frac{\ell}{s}\left(\frac{1}{p}-\frac{1}{q}\right)+L-1}\|f\|_{L^p}\,, \tag{11}$$

where $\frac1p+\frac1q=1$, $1\le p\le 2$.

We assume $\varepsilon>0$ to be small enough to make sure that the type of behaviour assumed in the theorem is the only one that takes place in $\mathcal{M}^\varepsilon$. In the complement of $\mathcal{M}^\varepsilon$ we may use other theorems to analyse the decay rate. Moreover, we assume that set $\mathcal{M}$ is C^1. In fact, it is usually Lipschitz, but in order to avoid to go into depth about its structure and existence of almost everywhere differentiable coordinate systems, we make the technical C^1 assumption.

Let us now give a special case of this theorem where simple roots meet the axis at a point, so that we have $L=1$ and $\ell=n$. The following statement is also global in frequency, so we have the result in Sobolev spaces.

Theorem 2.8. *Consider the m^{th} order strictly hyperbolic Cauchy problem* (1) *for operator $L(D_t,D_x)$, with initial data $f_j\in W_p^{N_p+|\alpha|+r-j}$, for $j=0,\dots,m-1$, where $1\le p\le 2$ and $2\le q\le\infty$ are such that $\frac1p+\frac1q=1$, $r\ge 0$ and α is a multi-index. We assume that the Sobolev index N_p satisfies $N_p\ge n(\frac1p-\frac1q)$ for $1<p\le 2$ and $N_1>n$ for $p=1$.*

Assume that the characteristic roots $\tau_1(\xi),\dots,\tau_m(\xi)$ of $L(\tau,\xi)=0$ satisfy $\mathrm{Im}\ \tau_k\ge 0$ *for all k, and also the following conditions:*

- *for all $k=1,\dots,m$, we have*

$$\liminf_{|\xi|\to\infty}\ \mathrm{Im}\ \tau_k(\xi)>0\,;$$

- *for each $\xi^0\in\mathbb{R}^n$ there is at most one index k for which* $\mathrm{Im}\ \tau_k(\xi^0)=0$ *and there exists a constant $c>0$ such that*

$$|\xi-\xi^0|^s\le c\mathrm{Im}\ \tau_k(\xi),$$

  for ξ in some neighbourhood of ξ^0. Assume also that there are finitely many points ξ^0 with $\mathrm{Im}\ \tau_k(\xi^0)=0$*.*

Then the solution $u = u(t,x)$ to the Cauchy problem (1) *satisfies the following estimate for all $t \geq 0$:*

$$\|D_t^r D_x^\alpha u(t,\cdot)\|_{L^q} \leq C_{\alpha,r}(1+t)^{-\frac{n}{s}(\frac{1}{p}-\frac{1}{q})} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}}. \tag{12}$$

As a special case, such estimate together with (14) below (used with $s = s_1 = 2$), we improve the indices in Sobolev spaces over L^2 for the dissipative wave equation compared to [5].

If conditions of Theorem 2.8 hold only with $\xi^0 = 0$, namely if Im $\tau_k(\xi^0) = 0$ implies $\xi^0 = 0$, we will call the polynomial $L(\tau,\xi)$ strongly stable. Now we will give some improvements of (12) under additional assumptions on the roots:

Remark 2.1. The order of time decay in Theorem 2.8 may be improved in the following cases, if we make additional assumptions. If, in addition, we assume that Im $\tau_k(\xi^0) = 0$ in (H2) implies that $\xi^0 = 0$, then we actually get the estimate

$$\left\|D_t^r D_x^\alpha u(t,\cdot)\right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{|\alpha|}{2}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}},$$

where here and further in this remark N_p is as in Theorem 2.8.

Now, assume further that for all ξ^0 in (H2) we also have the estimate

$$|\tau_k(\xi)| \leq c_1|\xi-\xi^0|^{s_1}, \tag{13}$$

with some constant $c_1 > 0$, for all ξ sufficiently close to ξ^0.

If we have that Im $\tau_k(\xi^0) = 0$ in (H2) implies that we have (13) around such ξ^0, then we actually get

$$\left\|D_t^r D_x^\alpha u(t,\cdot)\right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{rs_1}{s}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}}.$$

And finally, assume that for all ξ^0 such that Im $\tau_k(\xi^0) = 0$ in (H2), we also have $\xi^0 = 0$ and (13) around such ξ^0. Then we actually get

$$\left\|D_t^r D_x^\alpha u(t,\cdot)\right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{|\alpha|}{s}-\frac{rs_1}{s}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}}. \tag{14}$$

Estimate (14) with $s = s_1 = 2$ gives the decay estimate for the dissipative wave equation.

Moreover, there are other possibilities of multiple roots intersecting each other while lying entirely on the real axis. For example, this is the case for the wave equation or for more general equations with homogeneous symbols, when several roots meet at the origin. In this case roots always lie on the real axis, but they become irregular at the point of multiplicity, which is the origin for homogeneous roots. In such cases we have to look at the structure of such multiple points by making cut-offs around them and studying their structure in more detail. In particular, there is an interaction between low frequencies and large times, which does not take place for homogeneous symbols. The detailed discussion of this topic and corresponding decay rates can be found in [10].

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