Novi Sad J. Math.
Vol. 38, No. 1, 2008, 127-135

DISTRIBUTION SEMIGROUPS ON FUNCTION SPACES WITH SINGULARITIES AT ZERO

Stevan Pilipović[1], Fikret Vajzović[2], Mirjana Vuković[3]

Abstract. We study various classes of distribution semigroups on the spaces of functions $\mathcal{F}_r$, $r \in \mathbf{R}$ distinguished by their behavior at the origin.

AMS Mathematics Subject Classification (2000): 47D06; 47D62

Key words and phrases: Integrated semigroups, distribution semigroups, smooth distribution semigroups

0. Introduction

Distribution semigroups of Lions [12] and Arendt's n-times integrated semigroups [1], have been studied by many authors, see e.g. [3], [4], [2], [8], [9] as well as the references therein. We refer especially to the paper [15], where we discussed various classes of distribution semigroups, following Kunstmann [11] (see also Wang [18]). As in [15], we drop Lions' denseness assumption and we investigate the condition that prescribes the behavior at the origin and strong distribution semigroups and distribution semigroups. Note that distribution semigroups are the same as quasi-distribution semigroups introduced in [18], [11], whereas strong distribution semigroups are characterized via the value 0 at the origin for their primitive, where the value is understood in the sense of Lojasiewicz. Moreover, a strong distribution semigroup is always a distribution semigroup and a distribution semigroup is always a weak distribution semigroup, but that the converse implications are false in general. For distributions of local order one, however, all these notions coincide ([15]).

The structural properties for strong distribution semigroups are given in [15], considering such semigroups on the test function space $\mathcal{F}_0$. In this way, it is shown that the class of strong distribution semigroups contains properly the class of smooth distribution semigroups introduced by Balabane and Emamirad [5], [6], [7] whose infinitesimal generators are always densely defined.

Further analysis of semigroups defined on test function spaces, consisting of functionds with appropriate integrability conditions at zero, is the subject of this paper. We introduce a scale of Fréchet spaces $\mathcal{F}_r, r \in \mathbf{R}$ and consider semigroups having extensions on such spaces. In the main assertion of Section 3

[1]Department of Mathematics and Informatics, University of Novi Sad, Trg D. Obradovića 4, 21 000 Novi Sad, Serbia, e-mail: pilipovic@im.ns.ac.yu
[2]Faculty of Natural sciences and mathematics, University of Sarajevo, Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina
[3]Faculty of Natural sciences and mathematics, University of Sarajevo, Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina, e-mail: mirjanav@yahoo.com

we characterize $r-$strong distribution semigroups of the form $\mathcal{G}(t,x)-(t^{k-r}F)^{(k)}$ as semigroups over $\mathcal{F}_r$ with an additional pointwise structural assumption. We note that M. Kostić have studied in [10] global version of such spaces with $r \geq 0$ and that our paper is related to the local version of semigroups of [10] for $r \geq 0$ and moreover, the most interesting case in our paper is $r < 0$.

1. Integrated, distribution and quasi-distribution semigroups

We use the usual notation: E is a Banach space with norm $\|\cdot\|$, $L(E) = L(E;E)$ is the space of bounded linear operators from E into E. For a linear operator A, its domain, range and null space are denoted by $D(A), R(A)$ and $N(A)$, respectively. We will always assume that A is a closed operator. Schwartz spaces of test functions on the real line $\mathbf{R}$ are denoted by $\mathcal{D} = C_0^\infty$ and $\mathcal{E} = C^\infty$ ([16]). Their strong duals are $\mathcal{D}'$ and $\mathcal{E}'$, respectively. We denote by $\mathcal{D}_0$ the subspace of $\mathcal{D}$ which consists of the elements with supports contained in $[0,\infty)$. Further on, $\mathcal{D}'(L(E)) = L(\mathcal{D}; L(E))$ and $\mathcal{E}'(L(E)) = L(\mathcal{E}; L(E))$ are spaces of continuous linear functions $\mathcal{D} \to L(E)$ and $\mathcal{E} \to L(E)$, respectively, equipped with the topology of uniform convergence on bounded subsets of $\mathcal{D}$ and $\mathcal{E}$, respectively; $\mathcal{D}'_+(L(E))$ and $\mathcal{E}'_+(L(E))$ are the subspaces of $\mathcal{D}'(L(E))$ consisting of elements supported in $[0,\infty)$ (for $E = \mathbf{C}$ we drop $(L(E))$ in notation). Note that a distribution $F \in \mathcal{D}'(L(E))$ is also a bilinear continuous mapping $f : \mathcal{D} \times E \to E$.

Let $\alpha \in C_0^\infty$, $\int_{\mathbf{R}} \alpha = 1$. We will use the following nets of smooth functions:

$$(1) \qquad \phi_\varepsilon(t) = \frac{1}{\varepsilon}\alpha(\frac{t}{\varepsilon}),\ \theta_\varepsilon(t) = \int\limits_{-\infty}^{t} \phi_\varepsilon(s)ds = \int\limits_{-\infty}^{t/\varepsilon} \alpha(s)ds\ ,\ t \in \mathbf{R},\ \varepsilon \in (0,1).$$

Note, $(\phi_\varepsilon)_\varepsilon$ is a delta net and $(\theta_\varepsilon)_\varepsilon$ is a net converging to the characteristic function of $[0,\infty)$ in the sense of $\mathcal{D}'(\mathbf{R})$.

J.L. Lions ([12]) introduced the notion of a distribution semigroup, which we shall call here a *distribution semigroup in the sense of Lions* or a DS-L for short: a $\mathcal{G} \in \mathcal{D}'_+(L(E))$ is a DS-L if it satisfies the properties (d.1), (d.2), (d.3), (d.4), where:

$$(d.1) \qquad \mathcal{G}(\phi * \psi, \cdot) = \mathcal{G}(\phi, \mathcal{G}(\psi, \cdot)),\ \phi, \psi \in \mathcal{D}_0,$$
where $\phi * \psi = \int_{\mathbf{R}} \phi(\cdot - t)\psi(t)dt$ is the usual convolution;

$$(d.2) \qquad \bigcap_{\phi \in \mathcal{D}_0} N(\mathcal{G}(\phi, \cdot)) = \{0\}\ ;$$

$$(d.3) \qquad \text{the linear hull } \mathcal{R} \text{ of } \bigcup_{\phi \in \mathcal{D}_0} R(\mathcal{G}(\phi, \cdot)) \text{ is dense in } E;$$

$$(d.4)\qquad \begin{array}{c}\text{for all } x \in \mathcal{R} \text{ there is a continuous function } u : [0,\infty) \to E \\ \text{satisfying } u(0) = x \text{ and } \mathcal{G}(\phi, x) = \int\limits_0^\infty \phi(t)u(t)dt, \phi \in \mathcal{D}.\end{array}$$

In [15] we were interested in dropping the assumption (d.3) and in replacing the assumption (d.4), which expresses a *regularity condition* at the origin.

In particular, if (d.1) and (d.2) hold for $\mathcal{G}$, then we can define the *generator* $A := \mathcal{G}(-\delta')$ of $\mathcal{G}$; it is a linear and closed operator in E.

We are also interested in the following case: Let $\psi \in \mathcal{D}$ and $\psi_+ := \psi 1_{[0,\infty)}$. Then $\psi_+ \in \mathcal{E}'_+$ and the operator $\mathcal{G}(\psi_+, \cdot)$ with domain $D(\mathcal{G}(\psi_+))$ is given by

$$x \in D(\mathcal{G}(\psi_+)), \mathcal{G}(\psi_+, x) = yx :\Longleftrightarrow \forall \phi \in \mathcal{D}_0 : \mathcal{G}(\phi, \mathcal{G}(\psi_+, x)) = \mathcal{G}(\phi * \psi_+, x).$$

We have considered in [15] the following conditions:

$$(d.5)\qquad \mathcal{G}(\psi, x) = \mathcal{G}(\psi_+, x), \text{ for all } \psi \in \mathcal{D}, \ x \in E;$$

$$(d.5)^s\qquad \begin{array}{c}\text{There is a dense subspace } E_0 \text{ of } E \text{ containing } \mathcal{R} \text{ such that} \\ \mathcal{G}(\phi\theta_\varepsilon, x) \to \mathcal{G}(\phi, x), \ \varepsilon \to 0^+, \text{ for all } \phi \in \mathcal{D}, \ x \in E_0 \\ \text{and for every } (\theta_\varepsilon)_\varepsilon \text{ of the form (1).}\end{array}$$

Every DS-L satisfies $(d.5)^s$ and (d.1), (d.2) and $(d.5)^s$ together imply (d.5) ([15]).

Definition 1. *([15]) Let $\mathcal{G} \in \mathcal{D}'_+(L(E))$. Then:*

a) $\mathcal{G}$ is called a weak distribution semigroup (or weak DS for short) if (d.1) and (d.2) hold.

b) $\mathcal{G}$ is called a strong distribution semigroup (or strong DS for short) if (d.1), (d.2) and $(d.5)^s$ hold.

c) $\mathcal{G}$ is called a distribution semigroup (or DS for short) if (d.1), (d.2) and (d.5) hold.

We refer to [18] and to [11] for quasi-distribution semigroups, QDS in short (also called pre-distribution semigroup).

Let $\mathcal{G} \in \mathcal{D}'_+(L(E))$. Then $\mathcal{G}$ is said to be of finite order $n \in \mathbf{N}$, resp., of local finite order n, if there exists a strongly continuous function $S \in \mathcal{C}([0,\infty), L(E))$, $S(0) = 0$, resp., $S \in \mathcal{C}([0,a), L(E))$, $a > 0, S(0) = 0$, (so we can put $S(t, \cdot) = 0$ for $t \leq 0$) such that

$$(2)\qquad \mathcal{G} = S^{(n)} \text{ in } \mathbf{R} \text{ (resp., } \mathcal{G} = S^{(n)} \text{ in } (-\infty, a)).$$

If $\mathcal{G}$ is of finite order, then we add this to the name of the corresponding distribution semigroup (for example, weak DS of finite order).

A densely defined operator A generates a DS-L (or exponentially bounded DS-L) if and only if A generates a local n-times integrated non-degenerated semigroup (or exponentially bounded one), see [1], p. 341. In this paper we consider the case when A is not necessarily densely defined in E and $(S(t))_{t\geq 0}$ might not be exponentially bounded.

In general, a weak DS is not a DS and a weak DS is not a strong DS. Wang [18] and Kunstmann [11] showed that a QDS is a weak DS. The next theorem establishes the relation between local integrated semigroups and QDS's.

Theorem 1. *([18], [11]) Let $n \in \mathbf{N}$ and let a family $(S(t))_{t\geq 0}$ be a local n-times integrated non-degenerate semigroup. Then its n-th distributional derivative is a QDS. Conversely, every QDS is n-th distributional derivative of a corresponding local n-times integrated non-degenerate semigroup on $[0, a)$, for some $n \in \mathbf{N}$ and $a > 0$.*

Concerning further relations between the various types of distribution semigroups introduced in Definition 1, we have the following proposition which contains assertions from [15].

Proposition 1. *([15]) Let $\mathcal{G} \in \mathcal{D}'_+(L(E))$.*

a) Assume (d.1), (d.2) and (d.5). Then $\mathcal{G}$ is a QDS. In particular (d.4) holds.

Moreover, with (d.1) and (d.2), (d.5)^s implies (d.5) and

$$\mathcal{G} \text{ is strong } DS \Rightarrow \mathcal{G} \text{ is } DS;$$

$$\mathcal{G} \text{ is } DS \Rightarrow \mathcal{G} \text{ is weak } DS.$$

b) Condition (d.5)^s implies that $\mathcal{G}^{(-1)}$ has the value 0 at the origin in the sense of Lojasiewicz on the set E_0.

c) Let $\mathcal{G}$ be of local order 1 with the corresponding S as in (2) (and $n = 1$). Then (d.5)^s holds for $\mathcal{G}$ with $x \in E$. In particular, (d.5)^s implies the equivalence of the following statements

(i) $\mathcal{G}$ is a weak DS. (ii) $\mathcal{G}$ is a strong DS. (iii) $(S(t))_{t\geq 0}$ is a 1-times local integrated non-degenerate semigroup.

d) $\mathcal{G}$ is a DS if and only if $\mathcal{G}$ is a QDS.

e) Let E_0 denote the set of all $x \in E$ such that

$$(\exists S_x \in C(\mathbf{R}; E),\ supp S_x \subset [0, \infty))(\exists a > 0)(\exists n \in \mathbf{N})$$

$$\mathcal{G}(\cdot, x) = S_x^{(n)} \text{ on } (-a, a) \quad \text{and } \|S_x(t)\| = o(t^{n-1}), x \in E_0, \quad \text{as } t \to 0.$$

If E_0 is dense in E, then $\mathcal{G}$ satisfies (d.5)^s.

In particular, $\mathcal{G} \in \mathcal{D}'_+(L(E))$ satisfies (d.5)^s if and only if $\mathcal{G}^{(-1)}$ has the value 0 at the origin on a dense set $E_0 \subset E$, i.e. $\mathcal{G} = (t^{n-1}F)^{(n)}$ on $(-a, a)$, where F is continuous and supported by $[0, \infty)$.

Thus a $\mathcal{G} \in \mathcal{D}'_+(L(E))$ is a strong DS if and only conditions (d.1),(d.2) hold and $\mathcal{G}^{(-1)}$ has the value 0 at the origin on a dense set $E_0 \subset E$.

Note ([15]) that a dense distribution semigroup cannot be of the form $\mathcal{G}(\cdot, x) = (t^k F)^{(k)}(\cdot, x),\ x \in D(A)$, where F is continuous on $(-a, a)$ $(a > 0)$ and supported in $[0, a)$ ([15]). Moreover, in [15] we have used the results of Propositions 2 and 3 and obtained a scale of strong DS with respect to their behavior at the origin.

2. Generalization of smooth DS's

Smooth DS's in the sense of [6]-[7] have been characterized in terms of integrated semigroups in [4]:

Let A be linear closed and densely defined. Then A generates a smooth DS $\mathcal{G}$, if and only if A generates a DS and there are $n \in \mathbf{N}$ and $C > 0$ such that $\mathcal{G} = S^{(n)}$ for an n-times integrated semigroup $S(t) = S_t, t \geq 0$, satisfying $\|S_t\| \leq Ct^n,\ t \geq 0$.

Hence, if A generates a smooth DS, then it generates a strong DS.

Recall ([5], [7]) that the underlying test function space for smooth distribution semigroups is the space $\mathcal{F}_0$: the completion of $\mathcal{D}((0,\infty))$ under the sequence of seminorms

$$q_j(\psi) = \|t^j \psi^{(j)}\|_{L^1((0,\infty))},\ j \in \mathbf{N}_0.$$

Clearly, $\psi_+ \in \mathcal{F}_0$ for every $\psi \in \mathcal{D}$. Hence every smooth semigroup can be extended on $\mathcal{D}$ to become a DS. Now we define a family of test function spaces.

Definition 2. *Let $r \in \mathbf{R}$. Then $\mathcal{F}_r$ is the completion of $\mathcal{D}((0,\infty))$ under the sequence of seminorms*

$$p_{r,j}(\psi) = \|t^j (\frac{\psi(t)}{t^r})^{(j)}\|_{L^1((0,\infty))},\ j \in \mathbf{N}_0.$$

Clearly, $p_{0,j} = q_j, j \in \mathbf{N}_0$ and if $\psi \in \mathcal{F}_r, r \geq 0$, has a bounded support, then $\psi \in \mathcal{F}_0$. The space $\mathcal{D}((0,\infty))$ is dense in all the spaces $\mathcal{F}_r, r \in \mathbf{R}$. Denote

$(d.1 - r - smooth)$ $$\mathcal{G}(\phi * \psi, \cdot) = \mathcal{G}(\phi, \mathcal{G}(\psi, \cdot)),$$

for all $\phi, \psi \in \mathcal{F}_r$;

Put $(d.2 - smooth) := (d.2)$.

Definition 3. *If $(d.1-r-smooth)$ and $(d.2-smooth)$ hold for $\mathcal{G} \in \mathcal{F}'_r(L(E))$, we call $\mathcal{G}$ a DS on $\mathcal{F}_r$, $r \in \mathbf{R}$.*

In the next section we will consider DS on $\mathcal{F}_r$.

The next result for non-densely defined infinitesimal generators is an extension of Theorem 4 in [6].

Proposition 2. *Let $\mathcal{G}$ be a DS on $\mathcal{F}_0$ of the form $\mathcal{G} = S^{(k)}$, where S is continuous and supported by $[0,\infty)$, and let A be the infinitesimal generator of $\mathcal{G}$. Then for any $x \in D(A^k)$, $\mathcal{G}(\cdot, x)$ is a continuous function on $\mathbf{R}$ supported by $[0,\infty)$ satisfying*

(3) $$||\mathcal{G}(t,x)|| \leq (||x|| + ||A^k x||)(1 + t^k),\ t \in \mathbf{R}.$$

3. DS on $\mathcal{F}_r, r \in \mathbf{R}$

We introduce one more condition:
$(d.6 - r - smooth)$ $(\exists D \subset X, \bar{D} = X)(\forall x \in D)$ $(\exists g_x \in C_b(\mathbf{R}; X), g_x(0) = 0)$

$$\mathcal{G}(\phi, x) = \int_0^\infty \phi(t) \frac{g_x(t)}{t^r} dt, \ x \in D, \ \phi \in \mathcal{F}_r. \tag{4}$$

Proposition 3. *Let A be linear, closed and densely defined on X. The following statements are equivalent.*
(i) A generates a DS on $\mathcal{F}_r$ satisfying (d.6-r-smooth).
(ii) A generates a strong DS $\mathcal{G}$ of the form

$$\mathcal{G}(t, x) = (t^{k-r} F(t, x))^{(k)},$$

where for every $x \in X$ $t \to F(t, x), t \in [0, \infty)$ is a continuous bounded function with respect to t and $F(0, x) = 0$, $x \in X$.

We remark that the regularity at the origin could not be larger than for smooth DS.

Proof. (i) $\Rightarrow$ (ii). We follow the proof of Theorem 4.4 in [4] with appropriate modifications. Note that for every $k \in \mathbf{N}$ the set of functions

$$\{(\phi(t)/t^r)^{(k)}; \phi \in \mathcal{D}((0, \infty))\} = \{\phi^{(k)}; \phi \in \mathcal{D}((0, \infty))\}$$

is dense in $L^1(\mathbf{R}_+, t^k dt)$. So we assume $\mathcal{G}$ is of order k and of the form (4). This implies

$$\mathcal{G}(\phi, x) = \int_0^\infty (-1)^k (\frac{\phi(t)}{t^r})^{(k)} H_x(t) t^k dt, \ \phi \in \mathcal{D}((0, \infty)),$$

where $H_x(t) = \frac{1}{t^k} \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} g_x(s) ds, \ t \geq 0$. Thus $H_x(0) = 0, \quad |H_x(t)| < \infty, \ x \in D$. In the same way as in [4] (proof of Theorem 4.4) we prove that for every $t > 0$, $x \mapsto H_x(t), \ x \in X$ is a bounded linear operator. Let $\phi \in \mathcal{F}_r, \ x \in X$. By Leibniz formula it follows

$$\mathcal{G}(\phi, x) = -\int_0^\infty \phi(t) \sum_{i=1}^k \binom{k}{i} (-1)^i r(r-1)...(r-i+1) \big(t^{k-r-i} H_x(t)\big)^{(k-i)} dt.$$

Integrating i-times the i-th member of the sum, we have

$$\mathcal{G}(\phi, x) = \langle (t^{k-r} H(t, x))^{(k)}, \phi(t) \rangle \ x \in X, \phi \in \mathcal{F}_r,$$

where $t \mapsto H(t, x)$ is continuous bounded and $H(0, x) = 0, x \in X$.

Let us show that $\mathcal{G}$ satisfies $(\text{d}.5)^s$. Let $\phi \in \mathcal{D}$ and θ_ε be defined by (1). Using the Leibniz formula for $(\phi\theta_\varepsilon)^{(k)}$, one can show that the integrals

$$\int_0^\infty \phi^{(k-j)}(t) \alpha_\varepsilon^{(j-1)}(t) t^{k-r} H(t, x) dt$$

converge to zero for $j = 1, \dots, k$, and for $k = 0$, to

$$\int_0^\infty \phi^{(k)}(t) t^{k-r} H(t,x) dt.$$

This proves (d.5)^s.

Similarly, $\mathcal{G}$ can be extended on $1_{[0,\infty]} e^{-\lambda t}$, which implies (as in [4]), for $k \geq r, k \in \mathbf{N}$, that

$$(\lambda I - A)^{-1} = \mathcal{G}(1_{[0,\infty]} e^{-\lambda t}) = \lambda^k \int_0^\infty e^{-\lambda t} t^{k-r} H(t,x) dt, \ x \in X,$$

and that $(t^{k-r} H(t,\cdot))_{t \geq 0}$ are k times integrated semigroups.

(ii)$\Rightarrow$ (i) Define

$$(-1)^k \mathcal{G}(\phi, x) = \int_0^\infty (\frac{\phi(t)}{t^r})^{(k)} t^k H(t,x) dt$$

$$- \sum_{j=1}^k \int_0^\infty \binom{k}{j} (\frac{\phi(t)}{t^r})^{(k-j)} (t^{k-r})^{(j)} H(t,x) dt, \ \phi \in \mathcal{F}_R.$$

One can prove that (d.1-smooth) holds for $\phi, \psi \in \mathcal{F}_r$. Let us prove (d.6-r-smooth). Note for $x \in D(A^k)$, $(t^{k-1} H(t,x))^{(k)}_{|t=0} = x$ and

$$t(t^{k-1} H(t,x))^{(k)} = t^k H(t, A^k x) + \frac{t^{k+1}}{k!} A^{k-1} x + \dots + tx.$$

for $x \in D = D(A^k)$ (a dense set of X). Let $x \in D = D(A^k)$ (a dense set of X). By partial integration we have

$$\mathcal{G}(\phi, x) = \int_0^\infty \frac{\phi(t)}{t^r} (t^{k-r} H(t,x))^{(k)} dt = \int_0^\infty (\frac{\phi(t)}{t^r}) g_x(t) dt.$$

Denoting the infinitesimal generator of $\mathcal{G}$ by B, in the same way as in [4] (last part of the proof of Theorem 4.4), one can prove that $B = A$.

Remark 1. Note that the homomorphism $a : \mathcal{F}_0 \to \mathcal{F}_r : \phi \mapsto a(\phi) = t^r \phi$, is an isomorphism on $\mathcal{D}((0,\infty))$. For $\psi \in \mathcal{D}((0,\infty))$ and $G \in \mathcal{F}_0$, we have

$$_{\mathcal{F}_0'}\langle G, a^{-1}\psi \rangle_{\mathcal{F}_0} = _{\mathcal{F}_0'} \langle G, \frac{\psi}{t^r} \rangle_{\mathcal{F}_0} = _{\mathcal{F}_r'} \langle \frac{G}{t^r}, \psi \rangle_{\mathcal{F}_r}.$$

This implies that every element $\tilde{G} \in \mathcal{F}_r'$, restricted on $\mathcal{D}((0,\infty))$, is of the form $\tilde{G} = G/t^r$ for some $G \in \mathcal{F}_0$. Applying this we can obtain another proof of $(i) \Rightarrow (ii)$ in Proposition 5.

If $\mathcal{G}$ is a strong DS, then it can be extended on $\mathcal{F}_r$ to be an element of $\mathcal{F}_r'(L(E))$ so that conditions (d.1-r-smooth) and (d.2-smooth) hold. This can be proved by the arguments of previous assertion.

In general, if $\mathcal{G} \in \mathcal{F}'_r$, then there exists $k \in \mathbf{N}, k > 1$, such that $\mathcal{G}(\cdot, x) = (t^{k-r}H(\cdot, x))^{(k+1)}$ for every $x \in E$, where $t \mapsto H(t, x)$ is continuous and supported by $[0, \infty)$. The proof is based on the Hahn-Banach theorem and partial integration.

Let $\mathcal{G}$ be a strong DS. We know,

$$\mathcal{G}(\cdot, x) = (t^{k-r}S(\cdot, x))^{(k)}, x \in E,$$

where S has the prescribed properties. Let $k \geq m \geq r, m \in \mathbf{N}$. Then

$$\mathcal{L}(\mathcal{G})(\lambda) = \lambda^m \mathcal{L}(\mathcal{G}^{-m})(\lambda),\ \Re\lambda > 0,$$

where $\mathcal{G}^{-m} = (t^{k-r}S(\cdot, x))^{(k-m)}$. It is clear that the restriction of $\mathcal{G}^{-m}$ on $\mathcal{D}(0, \infty)$ can be extended as an element of $\mathcal{F}_0$ denoted in the same way. So by Theorem III 8 in [5], for a given $q \in \mathbf{N}$ there exist constants $C > 0$ and $r > 0$ such that

$$||(\lambda I - A)^{-q}|| \leq Cq^k |\lambda|^{k-m+1} |\Re\lambda|^{-k+m-q}, \Re\lambda > 0.$$

Moreover, the same proof as for Theorem III 9 in [5] gives

Proposition 4. *Let A be a closed linear operator on E. Then the following is equivalent:*

(i) The spectrum of A lies in $\Im\lambda \leq 0$ and if its resolvent satisfies

$$||(\lambda I - A)^{-1}|| \leq C|\lambda|^{p-m+1} |\Re\lambda|^{-p+m-1}, \Re\lambda > 0$$

for some $C > 0$ and $p \geq m > 0$;

(ii) A is the infinitesimal generator of a strong DS.

References

[1] Arendt, W., Vector valued Laplace transforms and Cauchy problems. Israel J. Math. 59 (1987), 327-352.

[2] Arendt, W., Batty, C. J. K., Hieber, M., Neubrander, F., Vector-valued Laplace transforms and Cauchy problems. Berlin: Birkhäuser (2001).

[3] Arendt, W., El-Mennaoui, O., Keyantuo, V., Local integrated semigroups. J. Math. Anal. Appl. 186 (1994), 572-595.

[4] Arendt, W., Kellermann, K., Integrated solutions of Volterra integrodifferential equations and applications, Volterra integrodifferential equations in Banach spaces and applications. Proc. Conf., Trento/Italy 1987, Pitman Res. Notes Math. Ser. 190 (1989), 21-51.

[5] Balabane, M., Emami-Rad, H., Smooth distributions group and Schrödinger equation in $L^p(\mathbf{R}^n)$. J. Math. Anal. Appl. 70 (1979), 61-71.

[6] Balabane, M., Emami-Rad, H., Pseudo-differential parabolic systems in $L^p(\mathbf{R}^n)$, Contribution to Non-Linear P.D.E.. Research notes in Mathematics 89, New York: Pitman 1983, 16-30.

[7] Balabane, M., Emami-Rad, H., L^p- estimates for Schrödinger evolution equations. Trans. AMS 292 (1985), 357-373.

[8] Hieber, M., L^p spectra of pseudodifferential operators generating integrated semigroups. Trans. Amer. Math. Soc. 347 (1995), 4023-4035.

[9] Kellermann, K., Hieber, M., Integrated semigroups. J. Func. Anal. 84 (1989), 160-180.

[10] Kostić, M., On a Class of Quasi-Distribution Semigroups. Novi Sad J. Math. Vol. 36 No. 2 (2006), 137-152.

[11] Kunstmann, P. C., *Distribution Semigroups and Abstract Cauchy Problems.* Trans. Amer. Math. Soc. 351 (1999), 837-856.

[12] Lions, J. L.,Semi-groupes distributions. Portugal. Math. 19 (1960), 141-164.

[13] Melnikova, I. V., Alshansky, M. A., Well-posedness of the degenerate Cauchy problem in a Banach space . Dokl. Acad. Nauk 336(1) (1994), 17-20.

[14] Melnikova, I. V., Filinkov, A., Abstract Cauchy problems: Three Approaches. Boca Raton, London, New York, Washington: Chapman Hall 2001.

[15] Kunstmann, P. C., Mijatović, M., Pilipović, S., Classes of distribution semigroups. Submitted.

[16] Schwartz, L., Théorie des distributions, 2 vols. Paris: Hermann 1950-1951.

[17] Stein, E. M.,Harmonic analysis: Real variable methods. Princeton: University Press 1993.

[18] Wang, S., Quasi-Distribution Semigroups and Integrated Semigroups. J. Func. Anal. 146 (1997), 352-381.

Received by the editors April 9, 2007