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ON SPARR AND FERNANDEZ'S INTERPOLATION METHODS OF BANACH SPACES

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Abstract. We investigate the interpolation spaces defined by Sparr and Fernandez's methods for 4-tuples Banach $\overline{A} = (A_0, A_1, A_2, A_3)$, where A_i is of class $\mathcal{C}(\theta_i, X, Y)$. If θ_i is suitably chosen, then the J- and K- methods coincide and are equal to the space $(X, Y)_{\eta,p}$. Some concrete applications of this fact are also presented.

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0. Introduction

The theory of interpolation usually deals with interpolation of two topological vector spaces. In the case of Banach spaces, we mention in particular the K- and J-methods (see [1] or [2]). K- and J-methods are extended to n-tuples ($n \geq 3$) Banach by G. Sparr [7] and to 2^n-tuples ($n \geq 2$) Banach by D.L Fernandez [6]. In contrast to the case of Banach couples, the so-called Equivalence Theorem fails, in general, for the case of Banach n-tuples ($n \geq 3$). In this paper we apply Sparr's method and Fernandez's method to 4-tuples Banach $\overline{A} = (A_0, A_1, A_2, A_3)$, where A_i is of class $\mathcal{C}(\theta_i, X, Y)$. In this case, if θ_i is suitably chosen, the J- and K-methods coincide and are equal to the space $(X, Y)_{\eta,p}$. The paper is organized in the following way. Some definitions and other necessary preliminaries are collected in Section 1. The main theorems are in Section 2. Some applications are given in Section 3.

1. Preliminaries and notation

Our notation and terminology is standard and we refer to [1], [6] and [7]. For the reader's convenience, we give some definitions and results that will be used later.

We will briefly describe some well-known real interpolation methods between two and four spaces.

Let $\overline{A} := (A_0, A_1)$ be a compatible Banach pair, i.e. two Banach spaces both linearly and continuously embedded in a Hausdorff topological vector space U. The K functional is defined for all $f \in \Sigma(\overline{A}) := A_0 + A_1$ and $t > 0$ by

$$K(t, f, \overline{A}) := \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f_0 \in A_0, f_1 \in A_1 \text{ and } f = f_0 + f_1\}.$$

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The J functional is defined for all $u \in \Delta(\overline{A}) := A_0 \cap A_1$ and $t > 0$ by

$$J(t, u, \overline{A}) := \max(\|u\|_{A_0}, t\|u\|_{A_1}).$$

One defines the K-interpolation space $\overline{A}_{\theta,p,K}$, for $0 < \theta < 1$ and $1 \leq p < \infty$, as the set of $f \in \Sigma(\overline{A})$ for which

$$\|f\|_{\theta,p,K} := \left(\int_0^\infty (t^{-\theta} K(t, f, \overline{A}))^p \frac{dt}{t} \right)^{1/p} < \infty.$$

The function $f \mapsto \|f\|_{\theta,p,K}$ is used as a norm on this space. One also defines the J-interpolation space $\overline{A}_{\theta,p,J}$, for $0 < \theta < 1$ and $1 \leq p < \infty$, as the set of $f \in \Sigma(\overline{A})$ which can be represented by

$$f = \int_0^\infty u(t) \frac{dt}{t} \quad \text{(convergence in } \Sigma(\overline{A})) \tag{1}$$

where $u(t)$ is measurable with values in $\Delta(\overline{A})$ and

$$\|f\|_{\theta,p,J} := \inf \left(\int_0^\infty (t^{-\theta} J(t, u(t), \overline{A}))^p \frac{dt}{t} \right)^{1/p} < \infty.$$

The infimum is taken over all representations of f on the form (1). The function $f \mapsto \|f\|_{\theta,p;J}$ is used as a norm of this space. It is well-known that (see [1]) $\overline{A}_{\theta,p,J} = \overline{A}_{\theta,p,K}$ with equivalence of norms. In view of this we will omit the subcripts K and J, just writing $\overline{A}_{\theta,p}$.

An intermediate space X with respect to $\overline{A}$, i.e. X is a Banach space for which $\Delta(\overline{A}) \hookrightarrow X \hookrightarrow \Sigma(\overline{A})$, is said to be of class $\mathcal{C}_K(\theta, \overline{A})$ if

$$K(t, f, \overline{A}) \leq ct^\theta \|f\|_X, \quad f \in X$$

and of class $\mathcal{C}_J(\theta, \overline{A})$ if

$$\|u\|_X \leq ct^{-\theta} J(t, u, \overline{A}), \quad u \in \Delta(\overline{A}).$$

Here $0 \leq \theta \leq 1$. If X is of class $\mathcal{C}_k(\theta, \overline{A})$ and of class $\mathcal{C}_J(\theta, \overline{A})$ then it is said to be of class $\mathcal{C}(\theta, \overline{A})$. For $0 < \theta < 1$ one can prove (see[1]) that an intermediate space X is of class $\mathcal{C}(\theta, \overline{A})$ if and only if $\overline{A}_{\theta,1} \hookrightarrow X \hookrightarrow \overline{A}_{\theta,\infty}$. This means, in particular, that $\overline{A}_{\theta,p}$ is of class $\mathcal{C}(\theta, \overline{A})$.

We will now briefly discuss two extensions of the real method described above to the case when interpolating between four Banach spaces. The methods described below are Sparr's method [7] and Fernandez's method [6].

Let $\overline{A} := (A_0, A_1, A_2, A_3)$ be a compatible Banach 4-tuple, i.e. A_0, A_1, A_2, A_3 are four Banach spaces which are linearly and continuously embedded in a Hausdorff topological vector space U.

If (t_1, t_2, t_3) are three positive numbers and $f \in \Sigma(\overline{A})$ we define the K-functional by

$$K(t_1, t_2, t_3, f, \overline{A}) := \inf\Big\{ \|f_0\|_{A_0} + t_1\|f_1\|_{A_1} + t_2\|f_2\|_{A_2} + $$

$$+ t_3\|f_3\|_{A_3} : f = \sum_{i=0}^{3} f_i, f_i \in A_i \Big\}$$

and the J-functional, for $u \in \Delta(\overline{A})$ by

$$J(t_1, t_2, t_3, u, \overline{A}) := \max(\|u\|_{A_0}, t_1\|u\|_{A_1}, t_2\|u\|_{A_2}, t_3\|u\|_{A_3}).$$

For the positive numbers $(\theta_1, \theta_2, \theta_3)$ such that $\theta_1+\theta_2+\theta_3 < 1$ and $1 \le p < \infty$, one defines the K- interpolation space $\overline{A}_K^S = \overline{A}_{(\theta_1,\theta_2,\theta_3),p,K}$ (Sparr K-space), as the set of $f \in \Sigma(\overline{A})$ for which

$$\|f\|_{(\theta_1,\theta_2,\theta_3),p,K} := \left(\int_0^\infty\int_0^\infty\int_0^\infty \left(t_1^{-\theta_1} t_2^{-\theta_2} t_3^{-\theta_3} K(t_1, t_2, t_3, f, \overline{A}) \right)^p \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} \right)^{1/p} < \infty.$$

The function $f \mapsto \|f\|_{(\theta_1,\theta_2,\theta_3),p,K}$ is used as a norm on this space. One also defines the J-interpolation space $\overline{A}_J^S = \overline{A}_{(\theta_1,\theta_2,\theta_3),p,J}$ (Sparr J-space), for $\theta_1 > 0$, $\theta_2 > 0$, $\theta_3 > 0$, $\theta_1 + \theta_2 + \theta_3 < 1$ and $1 \le p < \infty$, as the set of $f \in \Sigma(\overline{A})$ which can be represented by

$$f = \int_0^\infty\int_0^\infty\int_0^\infty u(t_1, t_2, t_3) \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} \quad \text{(convergence in } \Sigma(\overline{A})) \tag{2}$$

where $u(t_1, t_2, t_3)$ is measurable with values in $\Delta(\overline{A})$ and
$\|f\|_{(\theta_1,\theta_2,\theta_3),p,J} :=$

$$= \inf \left(\int_0^\infty\int_0^\infty\int_0^\infty \left(t_1^{-\theta_1} t_2^{-\theta_2} t_3^{-\theta_3} J(t_1, t_2, t_3, u(t_1, t_2, t_3), \overline{A}) \right)^p \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} \right)^{1/p} < \infty.$$

The infimum is taken over all representations of f on the form (2). The function $f \mapsto \|f\|_{(\theta_1,\theta_2,\theta_3),p,J}$ is used as a norm on this space.

In order to interpolate the 4-tuple $\overline{A}$ by Fernandez's method, we need two independent positive parameters (t_1, t_2) and two numbers (α, β) such that $0 < \alpha$, $\beta < 1$; then spaces $\overline{A}_K^F = \overline{A}_{(\alpha,\beta),p,K}$ (Fernandez K-space) and $\overline{A}_J^F = \overline{A}_{(\alpha,\beta),p,J}$

(Fernandez J-space) are defined analogously as above, but this time using the functionals

$$K(t_1,t_2,f,\overline{A}) := \inf\Big\{ \|f_0\|_{A_0} + t_1\|f_1\|_{A_1} + t_2\|f_2\|_{A_2} + \\ + t_1t_2\|f_3\|_{A_3};\ f = \sum_{i=0}^{3} f_i \Big\} \tag{3}$$

$$J(t_1,t_2,u,\overline{A}) := \max\left(\|u\|_{A_0}, t_1\|u\|_{A_1}, t_2\|u\|_{A_2}, t_1t_2\|u\|_{A_3}\right). \tag{4}$$

2. The main results

It is a well-known fact that the so-called Equivalence Theorem does not hold for Sparr's method or for Fernandez's method (see [7], [8]). We only have the embedding $\overline{A}_J^S \hookrightarrow \overline{A}_K^S$, or $\overline{A}_J^F \hookrightarrow \overline{A}_K^F$. If we put $\theta_1 = \alpha(1-\beta)$, $\theta_2 = \beta(1-\alpha)$ and $\theta_3 = \alpha\beta$ then the following continuous inclusions holds:

$$\overline{A}_J^F \hookrightarrow \overline{A}_J^S \hookrightarrow \overline{A}_K^S \hookrightarrow \overline{A}_K^F \quad \text{(see [3])}. \tag{5}$$

In this section we study 4-tuples $\overline{A} := (A_0, A_1, A_2, A_3)$, where A_i is of class $\mathcal{C}(\theta_i, X, Y)$, $0 \le \theta_i \le 1$, for a compatible pair (X, Y).

Theorem 1. *Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a Banach 4-tuple, such that A_i is of class $\mathcal{C}(\theta_i, X, Y)$, with $0 < \theta_0 < \theta_1 < \theta_2 < \theta_3 < 1$. Assume that $(\alpha,\beta,\gamma) \in (0,1)^3$, $\alpha+\beta+\gamma < 1$ and $1 \le p < \infty$. Then, we have*

$$(X,Y)_{\eta,p} = \overline{A}_J^S = \overline{A}_{(\alpha,\beta,\gamma),p,J} = \overline{A}_K^S = \overline{A}_{(\alpha,\beta,\gamma),p,K} \tag{6}$$

where $\eta = (1-\alpha-\beta-\gamma)\theta_0 + \alpha\theta_1 + \beta\theta_2 + \gamma\theta_3$.

Proof. We start by checking that

$$(X,Y)_{\eta,p,J} \hookrightarrow \overline{A}_{(\alpha,\beta,\gamma),p,J} \hookrightarrow \overline{A}_{(\alpha,\beta,\gamma),p,K} \hookrightarrow (X,Y)_{\eta,p,K}. \tag{7}$$

Let $f \in (X,Y)_{\eta,p,J}$. This means that $f = \int_0^\infty u(t)\frac{dt}{t}, u(t) \in X \cap Y$ and $\int_0^\infty [t^{-\eta} J(t,u(t),X,Y)]^p \frac{dt}{t} < \infty$. Since $A_i \in \mathcal{C}(\theta_i, X, Y)$ we have

$$t^{\theta_i}\|u(t)\|_{A_i} \le c_i J(t,u(t),X,Y), \quad i = 0,1,2,3$$

and thus

$$t^{\theta_0} J(t^{\theta_1-\theta_0}, t^{\theta_2-\theta_0}, t^{\theta_3-\theta_0}, u(t), \overline{A}) \le cJ(t,u(t),X,Y). \tag{8}$$

We define

$$v(t_1,t_2,t_3)=\frac{1}{\theta_1-\theta_0}u\left(t_1^{\frac{1}{\theta_1-\theta_0}}\right)\chi_{\left[\frac{t_1^{\frac{\theta_2-\theta_0}{\theta_1-\theta_0}}}{e},t_1^{\frac{\theta_2-\theta_0}{\theta_1-\theta_0}}\right]}(t_2)\chi_{\left[\frac{t_1^{\frac{\theta_3-\theta_0}{\theta_1-\theta_0}}}{e},t_1^{\frac{\theta_3-\theta_0}{\theta_1-\theta_0}}\right]}(t_3).$$

This v satisfies

$$\int_0^\infty\int_0^\infty\int_0^\infty v(t_1,t_2,t_3)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}=\int_0^\infty\int_0^\infty\int_0^\infty u(t)\chi_{\left[\frac{t^{\theta_2-\theta_0}}{e},t^{\theta_2-\theta_0}\right]}(t_2)\chi_{\left[\frac{t^{\theta_3-\theta_0}}{e},t^{\theta_3-\theta_0}\right]}(t_3)\cdot$$

$$\cdot\frac{dt}{t}\frac{dt_2}{t_2}\frac{dt_3}{t_3}=\int_0^\infty u(t)\frac{dt}{t}=f.$$

According to (8), we have

$$\|f\|^p_{\overline{A}(\alpha,\beta,\gamma),p,J}\le\int_0^\infty\int_0^\infty\int_0^\infty[t_1^{-\alpha}t_2^{-\beta}t_3^{-\gamma}J(t_1,t_2,t_3,v(t_1,t_2,t_3),\overline{A})]^p\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}=$$

$$=c\int_0^\infty\int_0^\infty\int_0^\infty\Big[t^{-\alpha(\theta_1-\theta_0)}z^{-\beta}t^{-\beta(\theta_2-\theta_0)}y^{-\gamma}t^{-\gamma(\theta_3-\theta_0)}\chi_{\left[\frac{1}{e},1\right]}(t_2)\cdot$$

$$\cdot\chi_{\left[\frac{1}{e},1\right]}(t_3)\max(\|u(t)\|_{A_0},t^{\theta_1-\theta_2}\|u(t)\|_{A_1},zt^{\theta_2-\theta_0}\|u(t)\|_{A_2},yt^{\theta_3-\theta_0}\|u(t)\|_{A_3})\Big]^p\cdot$$

$$\cdot\frac{dt}{t}\frac{dz}{z}\frac{dy}{y}\le c_1\int_0^\infty[t^{-\eta}t^{\theta_0}J(t^{\theta_1-\theta_0},t^{\theta_2-\theta_0},t^{\theta_3-\theta_0},u(t),\overline{A})]^p\frac{dt}{t}\le$$

$$\le c_2\int_0^\infty[t^{-\eta}J(t,u(t),X,Y)]^p\frac{dt}{t}.$$

Thus we obtain

$$\|f\|_{\overline{A}_{(\alpha,\beta,\gamma),p,J}}\le c\|f\|_{(X,Y)_{\eta,p,J}}$$

and the first embedding (7) follows.

The embedding $\overline{A}_{(\alpha,\beta,\gamma),p,J}\hookrightarrow\overline{A}_{(\alpha,\beta,\gamma),p,K}$ is classical (see [7]). Next we establish the last embedding (7).

Let $f\in\overline{A}_{(\alpha,\beta,\gamma),p,K}$ and $f=f_0+f_1+f_2+f_3\in\Sigma(\overline{A})\hookrightarrow X+Y$. Since $A_i\in\mathcal{C}_K(\theta_i,X,Y)$ we have

$$K(t,f_i,X,Y)\le c_it^{\theta_i}\|f_i\|_{A_i},\quad i=0,1,2,3$$

and thus

$$K(t,f;X,Y)\le ct^{\theta_0}K(t^{\theta_1-\theta_0},t^{\theta_2-\theta_0},t^{\theta_3-\theta_0},f,\overline{A}),\quad f\in\Sigma(\overline{A}). \tag{9}$$

According to (9), we have

$$\int_0^\infty\int_0^\infty\int_0^\infty[t_1^{-\alpha}t_2^{-\beta}t_3^{-\gamma}K(t_1,t_2,t_3,f,\overline{A})]^p\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}=$$

$$=c'\int_0^\infty\int_0^\infty\int_0^\infty\Big[t^{-\alpha(\theta_1-\theta_0)}t^{-\beta(\theta_2-\theta_0)}t^{-\gamma(\theta_3-\theta_0)}z^{-\beta}y^{-\gamma}K(t^{\theta_1-\theta_0},zt^{\theta_2-\theta_0},yt^{\theta_3-\theta_0},f,\overline{A})\Big]^p\cdot$$

$$\cdot\frac{dt}{t}\frac{dz}{z}\frac{dy}{y}\ge c''\int_0^\infty[t^{-\eta}t^{\theta_0}K(t^{\theta_1-\theta_0},t^{\theta_2-\theta_0},t^{\theta_3-\theta_0},f,\overline{A})]^p\frac{dt}{t}\ge$$

$$\ge c'''\int_0^\infty[t^{-\eta}K(t,f,X,Y)]^p\frac{dt}{t}.$$

Thus obtain

$$\|f\|_{(X,Y)_{\eta,p,K}}\le c\|f\|_{\overline{A}_{(\alpha,\beta,\gamma),p,K}}$$

and the last embedding (7) follows.

According to (7) and Equivalence Theorem for Banach couples we obtain (6).

Theorem 2. *Let $\overline{A}=(A_0,A_1,A_2)$ be a Banach 4-tuple, such that A_i is of class $\mathcal{C}(\theta_i,X,Y)$, with $\theta_0+\theta_3=\theta_1+\theta_2$ and $\theta_0\ne\theta_1$. Assume that $0<\alpha$, $\beta<1$ and $1\le p<\infty$. Then, we have*

$$(X,Y)_{\eta,p}=\overline{A}_J^F=\overline{A}_{(\alpha,\beta),p,J}=\overline{A}_K^F=\overline{A}_{(\alpha,\beta),p,K} \tag{10}$$

where $\eta=(1-\alpha-\beta)\theta_0+\alpha\theta_1+\beta\theta_2$.

Proof. In order to interpolate the 4-tuples $\overline{A}$ by Fernandez's method we need to use the functionals K and J from (3) and (4) respectively. We start by checking that

$$(X,Y)_{\eta,p,J}\hookrightarrow\overline{A}_{(\alpha,\beta),p,J}\hookrightarrow\overline{A}_{(\alpha,\beta),p,K}\hookrightarrow(X,Y)_{\eta,p,K}. \tag{11}$$

Let $f\in(X,Y)_{\eta,p,J}$. This means that $f=\int_0^\infty u(t)\frac{dt}{t}$, $u(t)\in X\cap Y$ and $\int_0^\infty[t^{-\eta}J(t,u(t),X,Y)]^p\frac{dt}{t}<\infty$. Since $A_i\in\mathcal{C}(\theta_i,X,Y)$, with $\theta_0+\theta_3=\theta_1+\theta_2$ we have

$$t^{\theta_0}J(t^{\theta_1-\theta_0},t^{\theta_2-\theta_0},u(t),\overline{A})\le cJ(t,u(t),X,Y). \tag{12}$$

We define

$$v(t_1,t_2)=\frac{1}{\theta_1-\theta_0}u\left(t_1^{\frac{1}{\theta_1-\theta_0}}\right)\chi_{\left[\frac{t_1^{\frac{\theta_2-\theta_0}{\theta_1-\theta_0}}}{e},\,t_1^{\frac{\theta_2-\theta_0}{\theta_1-\theta_0}}\right]}(t_2).$$

This v satisfies

$$\int_0^\infty\!\!\int_0^\infty v(t_1,t_2)\frac{dt_1}{t_1}\frac{dt_2}{t_2}=\int_0^\infty\!\!\int_0^\infty u(t)\chi_{\left[\frac{t^{\theta_2-\theta_0}}{e},\,t^{\theta_2-\theta_0}\right]}(t_2)\frac{dt}{t}\frac{dt_2}{t_2}=\int_0^\infty u(t)\frac{dt}{t}=f$$

and according to (12), we have

$$\|f\|^p_{\overline{A}_{(\alpha,\beta),p,J}}\le\int_0^\infty\!\!\int_0^\infty[t_1^{-\alpha}t_2^{-\beta}J(t_1,t_2,v(t_1,t_2),\overline{A})]^p\frac{dt_1}{t_1}\frac{dt_2}{t_2}=$$

$$=c\int_0^\infty\!\!\int_0^\infty\left[t^{-\alpha(\theta_1-\theta_0)}z^{-\beta}t^{-\beta(\theta_2-\theta_0)}J(t^{\theta_1-\theta_0},zt^{\theta_2-\theta_0},u(t),\overline{A})\chi_{[\frac{1}{e},1]}(z)\right]^p\frac{dt}{t}\frac{dz}{z}=$$

$$=c'\int_0^\infty[t^{-\eta}t^{\theta_0}J(t^{\theta_1-\theta_0},t^{\theta_2-\theta_0},u(t),\overline{A})]^p\frac{dt}{t}\le c'\int_0^\infty[t^{-\eta}J(t,u(t),X,Y)]^p\frac{dt}{t}.$$

Thus we obtain

$$\|f\|_{\overline{A}_{(\alpha,\beta),p,J}}\le c\|f\|_{(X,Y)_{\eta,p,J}}$$

and the first embedding (11) follows.

The embedding $\overline{A}_{(\alpha,\beta),p,J}\hookrightarrow\overline{A}_{(\alpha,\beta),p,K}$ is classical (see [6]).

We prove the last embedding (11), under the restriction $\theta_0+\theta_3=\theta_1+\theta_2$, like in theorem 1.

According to (11) and Equivalence Theorem for Banach couples we obtain (10).

Corollary 3. *Let $\overline{A}=(A_0,A_1,A_2,A_3)$ be a Banach 4-tuple, such that A_i is of class $\mathcal{C}(\theta_i,X,Y)$, with $\theta_0+\theta_3=\theta_1+\theta_2$, $\theta_0\ne\theta_1$. Assume that $0<u$, $v<1$ and put*

$$\alpha=u(1-v),\quad \beta=v(1-u)\ \text{and}\ \gamma=uv$$

then the Sparr spaces $\overline{A}^S_{(\alpha,\beta,\gamma),p}=\overline{A}_{(\alpha,\beta,\gamma),p,J}=\overline{A}_{(\alpha,\beta,\gamma),p,K}$ and the Fernandez spaces $\overline{A}^F_{(u,v),p}=\overline{A}_{(u,v),p,J}=\overline{A}_{(u,v),p,K}$ are equal (with equivalent norms). Moreover

$$\overline{A}^F_{(u,v),p}=\overline{A}^S_{(\alpha,\beta,\gamma),p}=(X,Y)_{\eta,p}$$

where $\eta=(1-u-v)\theta_0+u\theta_1+v\theta_2$.

Proof. We apply (5), Theorems 1 and 2.

3. Applications

In this sections we present three applications of Theorem 1 and 2. First we consider the Lebesgue spaces L^p.

Theorem 4. *Let* $1 \le p_0 < p_1 < p_2 < p_3 \le \infty$ *and* $1 \le p \le \infty$. *Then*

$$(L^{p_0}, L^{p_1}, L^{p_2}, L^{p_3})^S_{(\alpha,\beta,\gamma),p} = L^{q,p} \tag{13}$$

where $\dfrac{1}{q} = \dfrac{1-\alpha-\beta-\gamma}{p_0} + \dfrac{\alpha}{p_1} + \dfrac{\beta}{p_2} + \dfrac{\gamma}{p_3}$.

If $\dfrac{1}{p_0} + \dfrac{1}{p_3} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$ *then*

$$(L^{p_0}, L^{p_1}, L^{p_2}, L^{p_3})^F_{(u,v),p} = L^{q,p} \tag{14}$$

where $\dfrac{1}{q} = \dfrac{1-u-v}{p_0} + \dfrac{u}{p_1} + \dfrac{v}{p_2}$.

Proof. Since L^{p_i} is of class $\mathcal{C}\left(1 - \dfrac{1}{p_i}, L^1, L^\infty\right)$ we obtain (13) from Theorem 1 and (14) from Theorem 2.

The second application is from semi-groups of operators. Let X be a Banach space and let $\{T(t) : 0 \le t < \infty\}$ be an equi-bounded, strongly continuous semi-group of operators on X (see for details [1] or [2]). We denote by Λ the infinitesimal operator of the semi-group $\{T(t)\}$ and by $D(\Lambda)$ the domain of Λ.

Theorem 5. *Let* r *be an arbitrary, but fixed integer* > 0. *For integer numbers* (k_0, k_1, k_2, k_3) *such that* $0 < k_0 < k_1 < k_2 < k_3 < r$ *we have*

$$(D(\Lambda^{k_0}), D(\Lambda^{k_1}), D(\Lambda^{k_2}), D(\Lambda^{k_3}))^S_{(\alpha,\beta,\gamma),p} = (X, D(\Lambda^r))_{\eta,p} \tag{15}$$

where $\eta = (1-\alpha-\beta-\gamma)\dfrac{k_0}{r} + \alpha\dfrac{k_1}{r} + \beta\dfrac{k_2}{r} + \gamma\dfrac{k_3}{r}$.

Proof. Since $D(\Lambda^k)$, $k \in \overline{0,r}$ belongs to the class $\mathcal{C}\left(\dfrac{k}{r}, X, D(\Lambda^r)\right)$ (see [2]) we obtain (15) from Theorem 1.

We end this section by the behaviour of compact operators under Sparr's interpolation method for Banach 4-tuples $\overline{A} = (A_0, A_1, A_2, A_3)$, with A_i of class $\mathcal{C}(\theta_i, X, Y)$.

Theorem 6. *Let* (X, Y) *and* (Z, W) *be compatible pairs of Banach spaces. Suppose that* $\overline{A} = (A_0, A_1, A_2, A_3)$ *and* $\overline{B} = (B_0, B_1, B_2, B_3)$ *are Banach 4-tuples such that* A_i *is of class* $\mathcal{C}(\theta_i, X, Y)$, *with* $0 < \theta_0 < \theta_1 < \theta_2 < \theta_3 < 1$ *and*

B_i is a class $\mathcal{C}(\psi_i, Z, W)$, with $0 < \psi_0 < \psi_1 < \psi_2 < \psi_3 < 1$. If $T : X \to Z$ is compact, then $T : \overline{A}^S_{(\alpha,\beta,\gamma),p} \to \overline{B}^S_{(\alpha,\beta,\gamma),p}$ compactly for all values of (α, β, γ) in $(0,1)^3$, with $\alpha + \beta + \gamma < 1$,

$$(1-\alpha-\beta-\gamma)\theta_0 + \alpha\theta_1 + \beta\theta_2 + \gamma\theta_3 = (1-\alpha-\beta-\gamma)\psi_0 + \alpha\psi_1 + \beta\psi_2 + \gamma\psi_3$$

and $p \in [1, \infty)$.

Proof. It is well-known that, if $T : X \to Z$ is compact, then $T : (X,Y)_{\theta,p} \to (Z,W)_{\theta,p}$ compactly for all values of θ in $(0,1)$ and $p \in [1,\infty)$ (see [4]). If we use this and Theorem 1 we obtain that $T : \overline{A}^S_{(\alpha,\beta,\gamma),p} \to \overline{B}^S_{(\alpha,\beta,\gamma),p}$ compactly.

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