NOVI SAD J. MATH.
VOL. 29, NO. 3, 1999, 369–375
XII Yugoslav Geometric Seminar
Novi Sad, October 8–11, 1998

ON THE SUBMANIFOLDS OF THE RIEMANNIAN MANIFOLDS

K. Trenčevski
Institute of Mathematics, St. Cyril and Methodius University
P.O.Box.162, Skopje, Macedonia
Email address: kostatre@iunona.pmf.ukim.edu.mk

Abstract

In this paper are introduced principal directions and principal normal vectors of arbitrary n-dimensional submanifold of $n+k$-dimensional Riemannian space. It is also introduced canonical normal vectors and curvatures for the submanifolds of the Riemannian spaces.

AMS Mathematics Subject Classification (1991): 53A05, 53B25.
Key words and phrases: principal directions, principal curvatures, canonical normal vectors, normal curvature, osculator space.

1 Introduction

In the paper [4] are introduced principal directions and so called principal normal vectors of an n-dimensional manifold imbedded in R^{n+k}, and also are introduced canonical k normal vectors fields and the corresponding curvatures, which do not depend on the parameterization of the manifold. The principal directions on the tangent manifold which are introduced in [4] are not intrinsic, i.e. they depend not only on the intrinsic metric induced of the neighboring flat space, but they also depend on the imbedding. Thus in the paper [5] are introduced another two definitions for principal directions which are intrinsic. They are defined only for dimension bigger than 2. In the paper [5] are also studied some special cases when these three definitions for principal directions yield to the same directions.

This paper is a generalization of the papers [4] and [5], such that we consider an n-dimensional submanifold (M', g) of an $n+k$-dimensional Riemannian space (M, g). Principal directions and principal normal vectors of arbitrary n-dimensional submanifold of $n+k$-dimensional Riemannian manifold are introduced. It is also introduced canonical normal vectors and curvatures for the submanifolds of the Riemannian spaces and they do not depend on the parameterization of the submanifold.

2 Principal Directions

Let (M, g) be an $n+k$-dimensional Riemannian manifold and let (M', g) be an n-dimensional submanifold imbedded in (M, g). We choose an arbitrary point P of M' and all considerations are local in a neighborhood U of P, because the examination of the principal directions has a local character. We will denote by $\nabla_{\mathbf{Y}}$ the covariant derivative with respect to the metric g in direction of the tangent vector $\mathbf{Y}$. Each norm will be induced by the metric g. Usually, by $\mathbf{N}$ we will denote a unit vector (field) which is orthogonal to the basic manifold.

Let φ be the linear mapping defined by

$$\varphi(\mathbf{X}) = -\nabla_{\mathbf{X}}\mathbf{N}. \tag{2.1}$$

Lemma 2.1. *For each tangent vector fields* $\mathbf{X}$ *and* $\mathbf{Y}$, *it holds*

$$g(\varphi(\mathbf{X}), \mathbf{Y}) = g(\mathbf{X}, \varphi(\mathbf{Y})). \tag{2.2}$$

Proof.

$$g(\varphi(\mathbf{X}), \mathbf{Y}) - g(\mathbf{X}, \varphi(\mathbf{Y})) = g(-\nabla_{\mathbf{X}}\mathbf{N}, \mathbf{Y}) - g(\mathbf{X}, -\nabla_{\mathbf{Y}}\mathbf{N}) =$$
$$= -g(\mathbf{N}, \nabla_{\mathbf{X}}\mathbf{Y}) + g(\nabla_{\mathbf{Y}}\mathbf{X}, \mathbf{N}) = g(\nabla_{\mathbf{Y}}\mathbf{X} - \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{N}) =$$
$$= g([\mathbf{Y}, \mathbf{X}], \mathbf{N}) = 0. \quad \|$$

According to (2.2), φ is a symmetric operator, and hence the principal curvatures are real numbers and the principal directions are orthogonal.

Let $\mathbf{N}_1, \cdots, \mathbf{N}_k$ be arbitrary k orthonormal vector fields on U, and let $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ be arbitrary n orthonormal tangent vector fields on U. Now we define tensor field T_{ij} as follows

$$T_{ij} = \sum_{\alpha=1}^{k}\sum_{r=1}^{n} g(\mathbf{Y}_r, \nabla_{\mathbf{Y}_i}\mathbf{N}_\alpha) g(\mathbf{Y}_r, \nabla_{\mathbf{Y}_j}\mathbf{N}_\alpha). \tag{2.3}$$

It is easy to verify that T_{ij} does not depend on the choice of the orthonormal system $\{\mathbf{N}_\alpha\}$. Further, let $\mathbf{Y}'_i = \sum_{s=1}^{n} c_{is}\mathbf{Y}_s$ be another orthonormal basis. Then

$$g(\mathbf{Y}'_r, \nabla_{\mathbf{Y}'_i}\mathbf{N}_\alpha) = g(\sum_{s=1}^{n} c_{rs}\mathbf{Y}_s, \sum_{t=1}^{n} c_{it}\nabla_{\mathbf{Y}_t}\mathbf{N}_\alpha) = \sum_{s=1}^{n}\sum_{t=1}^{n} c_{rs} g(\mathbf{Y}_s, \nabla_{\mathbf{Y}_t}\mathbf{N}_\alpha) c^T_{ti}.$$

But the matrix $C = (c_{ij})$ is an orthogonal matrix, and we obtain the following transformation law

$$T' = C \cdot T \cdot C^{-1}.$$

Thus the eigenvalues of the matrix T_{ij} are invariant of the basis $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$. Moreover T is a symmetric matrix, and thus the eigenvalues are real numbers and the corresponding eigenvectors are orthonormal. If $(t_1, \cdots, t_n)$ is an arbitrary eigenvector of T_{ij}, then $t_1\mathbf{Y}_1 + \cdots + t_n\mathbf{Y}_n$ is eigenvector of the corresponding (1.1) tensor field T^i_j and hence it does not depend on the basis $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$. The eigenvalues $\lambda_1, \cdots, \lambda_n$ of T also do not depend on the choice of the basis $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ and moreover they are non-negative because the matrix T is a sum of matrices of the form $A_{ij} = a_i a_j$ (see [4]).

The eigenvectors of T^i_j determine *the principal directions.* The eigenvalues as non-negative scalars are squares of the "principal curvatures" which we would like to be defined. But it is more convenient to define principal normal vectors.

Suppose that $\mathbf{Z}_1, \cdots, \mathbf{Z}_n$ are orthonormal tangent vectors which determine the principal directions. Then we define the corresponding *principal normal vectors* $\mathbf{U}_i (1 \leq i \leq n)$ as

$$\mathbf{U}_i = \sum_{\alpha=1}^{k} -g(\mathbf{Z}_i, \nabla_{\mathbf{Z}_i}\mathbf{N}_\alpha)\mathbf{N}_\alpha \tag{2.4}$$

and also define vectors

$$\mathbf{U}_{ij} = \sum_{\alpha=1}^{k} -g(\mathbf{Z}_i, \nabla_{\mathbf{Z}_j}\mathbf{N}_\alpha)\mathbf{N}_\alpha \tag{2.5}$$

such that $\mathbf{U}_i = \mathbf{U}_{ii}$. These vectors are canonical, i.e. they do not depend on the choice of the orthonormal system $\{\mathbf{N}_\alpha\}$ and do not depend on the signs of the vectors $\mathbf{Z}_i$. It follows from lemma 2.1 that $\mathbf{U}_{ij} = \mathbf{U}_{ji}$. Moreover, the matrix

$$T_{ij} = \sum_{\alpha=1}^{k}\sum_{r=1}^{n} g(\mathbf{Z}_r, \nabla_{\mathbf{Z}_i}\mathbf{N}_\alpha) g(\mathbf{Z}_r, \nabla_{\mathbf{Z}_j}\mathbf{N}_\alpha).$$

is diagonal, i.e.

$$\sum_{\alpha=1}^{k}\sum_{r=1}^{n} g(\mathbf{Z}_r, \nabla_{\mathbf{Z}_i}\mathbf{N}_\alpha) g(\mathbf{Z}_r, \nabla_{\mathbf{Z}_j}\mathbf{N}_\alpha) = 0 \quad \text{for} \quad i \neq j.$$

The diagonal elements are the eigenvalues, i.e.

$$\lambda_i = \sum_{\alpha=1}^{k}\sum_{r=1}^{n} g(\mathbf{Z}_r, \nabla_{\mathbf{Z}_i}\mathbf{N}_\alpha) g(\mathbf{Z}_r, \nabla_{\mathbf{Z}_i}\mathbf{N}_\alpha) = \sum_{r=1}^{n} \|\mathbf{U}_{ir}\|^2 \qquad (1 \leq i \leq n).$$

Hence we have proven the following proposition.

Proposition 2.2. *The sums* $\sum_{j=1}^{n} \|\mathbf{U}_{ij}\|^2$ *for* $i \in \{1, \cdots, n\}$ *are the eigenvalues of* T.

If $k = 1$, then the principal curvatures can be defined by the scalars

$$\lambda_i = -g(\mathbf{Z}_i, \nabla_{\mathbf{Z}_i}\mathbf{N})$$

and they are determined up to the sign, because they depend on the sign of the unit vector $\mathbf{N}$.

In the special case, when $k = 1$ and (M, g) is the Euclidean space, then the principal directions defined here coincide with the standard principal directions from the classical literature. Then $\mathbf{U}_{ij} = 0$ for $i \neq j$ and $\|\mathbf{U}_i\|^2$ $(1 \leq i \leq n)$ are squares of the principal curvatures.

Note that the principal directions defined now depend on the imbedding of the submanifold (M', g). In the paper [5] are introduced also two definitions for principal directions which are intrinsic, i.e. the principal directions depend only on the metric. One of them uses the sectional curvature and the other uses the Ricci tensor. Further are considered some special cases when some of these definitions give the same principal directions. Analogously to the theorem 2.2 [5], the following theorem for n-dimensional submanifold of $n+1$-dimensional Riemannian manifold (M, g) can be proven.

Theorem 2.3. *If the squres of the principal curvatures of an n-dimensional submanifold of $n+1$-dimensional manifold (M, g) are different numbers, then the corresponding principal directions and the principal curvatures up to the sign depend only on the metric, i.e. they are intrinsic property of the submanifold.*

3 Canonical Normal Vectors and Normal Curvatures for Submanifolds of Riemannian Spaces

Let us define the following $k \times k$ matrix

$$P^{(1)}_{\alpha\beta} = \sum_{i=1}^{n}\sum_{j=1}^{n} g(\mathbf{Y}_i, \nabla_{\mathbf{Y}_j}\mathbf{N}_\alpha) g(\mathbf{Y}_i, \nabla_{\mathbf{Y}_j}\mathbf{N}_\beta). \tag{3.1}$$

If $\mathbf{Y}'_i = \sum_{s=1}^{n} c_{is}\mathbf{Y}_s$ where c is an orthogonal matrix, i.e. $\{\mathbf{Y}'_i\}$ is another system of orthonormal tangent vectors, then according to the properties of ∇ it is easy to see that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} g(\mathbf{Y}'_i, \nabla_{\mathbf{Y}'_j}\mathbf{N}_\alpha) g(\mathbf{Y}'_i, \nabla_{\mathbf{Y}'_j}\mathbf{N}_\beta) = \sum_{i=1}^{n}\sum_{j=1}^{n} g(\mathbf{Y}_i, \nabla_{\mathbf{Y}_j}\mathbf{N}_\alpha) g(\mathbf{Y}_i, \nabla_{\mathbf{Y}_j}\mathbf{N}_\beta)$$

which means that $P^{(1)}_{\alpha\beta}$ do not depend on the choice of the basis $\{\mathbf{Y}_i\}$, and thus the matrix $P^{(1)}$ is well defined. It depends on the choice of the orthonormal basis $\{\mathbf{N}_\alpha\}$. Since the matrix $P^{(1)}$ is symmetric and non-negatively defined, because it is a sum of n^2 matrices of form $A_{\alpha\beta} = a_\alpha a_\beta$, its eigenvalues are non-negative real numbers ([4]) and the eigenvectors are orthogonal. Let $k_1 = rank(P^{(1)})$, and let us denote the corresponding positive k_1 eigenvalues by $\lambda^{(1)}_1, \cdots, \lambda^{(1)}_{k_1}$. For any such eigenvalue, let $(t_1, \cdots t_k)$ be the corresponding eigenvector and we consider the normal vector $t_1\mathbf{N}_1 + \cdots + t_k\mathbf{N}_k$. We convenient to call these normal vectors $\mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}$ as eigenvectors. It verifies that these k_1 eigenvalues and the corresponding eigenvectors do not depend on the choice of the basis $\{\mathbf{N}_\alpha\}$. The positive scalars $\lambda^{(1)}_1, \cdots, \lambda^{(1)}_{k_1}$ we define to be the squares of *the first normal curvatures* and the corresponding eigenvectors $\mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}$ we define to be *the first normal vectors*.

The geometrical interpretation is the following. If $P^{(1)}$ is zero matrix at each point, then it is easy to verify that locally (M', g) is autoparallel submanifold of (M, g). So suppose that $k_1 > 0$. Then the vectors $\mathbf{Y}_1, \cdots, \mathbf{Y}_n, \mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}$ generate the osculated space at the considered point.

Note that the normal vectors $\mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}$ are uniquely determined up to permutation if and only if the eigenvalues $\lambda^{(1)}_1, \cdots, \lambda^{(1)}_{k_1}$ are different numbers. If some of the eigenvalues are equal, then instead of normal vectors we have normal subspace.

Since the matrix $P^{(1)}$ is a sum of n^2 matrices of the form $A_{\alpha\beta} = a_\alpha a_\beta$, it follows that

$$k_1 \le \min(n^2, k). \tag{3.2}$$

Specially, if $n = 1$ then $k_1 = 1$ or $k_1 = 0$, the first normal vector is

$$\mathbf{N}^{(1)}_1 = -[g(\mathbf{Y}, \nabla_\mathbf{Y}\mathbf{N}_1)\mathbf{N}_1 + \cdots + g(\mathbf{Y}, \nabla_\mathbf{Y}\mathbf{N}_k)\mathbf{N}_k] \tag{3.3}$$

and $g(\mathbf{Y}, \nabla_\mathbf{Y}\mathbf{N}^{(1)}_1) \cdot g(\mathbf{Y}, \nabla_\mathbf{Y}\mathbf{N}^{(1)}_1)$ is the square of the first curvature.

Now we consider the second osculating space. Without loss of generality we suppose that $\mathbf{N}_1 = \mathbf{N}^{(1)}_1, \cdots, \mathbf{N}_{k_1} = \mathbf{N}^{(1)}_{k_1}$, where $\mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}$ were defined previously. Thus $g(\mathbf{N}_i, \mathbf{N}^{(1)}_j) = 0$ for $i > k_1$ and $j \le k_1$. Now we define the second $(k - k_1) \times (k - k_1)$ matrix

$$P^{(2)}_{\alpha\beta} = \sum_{i=1}^{k_1}\sum_{j=1}^{n} g(\mathbf{N}^{(1)}_i, \nabla_{\mathbf{Y}_j}\mathbf{N}_\alpha) g(\mathbf{N}^{(1)}_i, \nabla_{\mathbf{Y}_j}\mathbf{N}_\beta). \tag{3.4}$$

Analogously to (3.2) we obtain

$$k_2 = \operatorname{rank}(P^{(2)}) \leq \min(nk_1, k - k_1), \tag{3.5}$$

and hence

$$k_2 \leq min(n^3, nk, k - k_1). \tag{3.6}$$

It can be verified that if $k_2 \equiv 0$, i.e. $P^{(2)}_{\alpha\beta} \equiv 0$, then the manifold can locally be imbedded in $n + k_1$-dimensional autoparallel submanifold of (M, g). So, suppose that $k_2 > 0$. Since the matrix $P^{(2)}$ is symmetric and non-negative, there exist k_2 positive eigenvalues $\lambda^{(2)}_1, \cdots, \lambda^{(2)}_{k_2}$ and the other are zeros. Let $(t_1, \cdots, t_{k-k_1})$ be an eigenvector corresponding to a positive eigenvalue. Then we consider the following vector $t_1\mathbf{N}_{k_1+1} + t_2\mathbf{N}_{k_1+2} + \cdots + t_{k-k_1}\mathbf{N}_k$ as an eigenvector. According to this identification, all the eigenvectors of $P^{(2)}$ do not depend on the choice of the basis $\{N_\alpha\}$, and also the eigenvalues do not depend on the basis $\{N_\alpha\}$. These k_2 eigenvectors $\mathbf{N}^{(2)}_1, \cdots, \mathbf{N}^{(2)}_{k_2}$ of $P^{(2)}_{\alpha\beta}$ have the following geometrical meaning. The vectors $\mathbf{Y}_1, \cdots, \mathbf{Y}_n, \mathbf{N}^{(1)}_1, \cdots, \mathbf{N}^{(1)}_{k_1}, \mathbf{N}^{(2)}_1, \cdots, \mathbf{N}^{(2)}_{k_2}$ generate the osculating space of second order at the considered point. The eigenvalues $\lambda^{(2)}_1, \cdots, \lambda^{(2)}_{k_2}$ we define to be the squares of *the second normal curvatures* and the corresponding eigenvectors $\mathbf{N}^{(2)}_1, \cdots, \mathbf{N}^{(2)}_{k_2}$ we define to be *the second normal vectors.*

Note that the vectors $\mathbf{N}^{(2)}_1, \cdots, \mathbf{N}^{(2)}_{k_2}$ are uniquely determined up to permutation if and only if the eigenvalues $\lambda^{(2)}_1, \cdots, \lambda^{(2)}_{k_2}$ are different numbers.

Specially, if $n = 1$ then $k_2 = 1$ or $k_2 = 0$, the second normal vector is

$$\mathbf{N}^{(2)}_1 = -[g(\mathbf{N}^{(1)}_1, \nabla_{\mathbf{Y}}\mathbf{N}_{k_1+1})\mathbf{N}_{k_1+1} + \cdots + g(\mathbf{N}^{(1)}_1, \nabla_{\mathbf{Y}}\mathbf{N}_k)\mathbf{N}_k] \tag{3.7}$$

and $g(\mathbf{N}^{(1)}_1, \nabla_{\mathbf{Y}}\mathbf{N}^{(2)}_1)g(\mathbf{N}^{(1)}_1, \nabla_{\mathbf{Y}}\mathbf{N}^{(2)}_1)$ is the square of the second curvature.

In order to consider the third step, without loss of generality we suppose that $\mathbf{N}_{k_1+1} = \mathbf{N}^{(2)}_1, \cdots, \mathbf{N}_{k_1+k_2} = \mathbf{N}^{(2)}_{k_2}$. Hence $g(\mathbf{N}_i, \mathbf{N}^{(2)}_j) = 0$ for $i > k_1 + k_2$ and $j \in \{1, \cdots, k_2\}$. Analogously to (3.1) and (3.4) the third $(k - k_1 - k_2) \times (k - k_1 - k_2)$ matrix can be formed and the procedure can be continued. We give only the recurrent formulas

$$P^{(s+1)}_{\alpha\beta} = \sum_{i=1}^{k_s}\sum_{j=1}^{n} g(\mathbf{N}^{(s)}_i, \nabla_{\mathbf{Y}_j}\mathbf{N}_\alpha)g(\mathbf{N}^{(s)}_i, \nabla_{\mathbf{Y}_j}\mathbf{N}_\beta). \tag{3.8}$$

where $\alpha, \beta \in \{k_1 + \cdots + k_s + 1, \cdots, k\}$, and

$$k_{s+1} = rank(P^{(s)}) \leq \min(nk_s, k - k_1 - \cdots - k_s), \tag{3.9}$$

analogously to (3.2) and (3.5). Hence we obtain the following inequality

$$k_s \le n^{s+1}, \tag{3.10}$$

and it is convenient to define $k_0 = n$, comparing (3.1) and (3.8).

Thus finally we obtain canonical orthogonal vector fields

$$\mathbf{N}_1^{(1)}, \cdots, \mathbf{N}_{k_1}^{(1)}, \mathbf{N}_1^{(2)}, \cdots, \mathbf{N}_{k_2}^{(2)}, \cdots, \mathbf{N}_1^{(r)}, \cdots, \mathbf{N}_{k_r}^{(r)}$$

and canonical squares

$$\lambda_1^{(1)}, \cdots, \lambda_{k_1}^{(1)}, \lambda_1^{(2)}, \cdots, \lambda_{k_2}^{(2)}, \cdots, \lambda_1^{(r)}, \cdots, \lambda_{k_r}^{(r)}$$

of the normal curvatures, such that $k_1 + \cdots + k_r$=k. The normal curvatures are determined up to the sign. Indeed they are unique determined if we suppose that they are non-negative. The orthogonal vectors $\mathbf{N}_1^{(1)}, \cdots, \mathbf{N}_{k_r}^{(r)}$ can be chosen as unit vectors.

Note that all these formulas hold also in the special case when (M, g) is the Euclidean space. In [4] it is given an alternative geometrical representation of this special case, by using the Lie brackets.

References

[1] P. Dombrowski, Differentiable maps into Riemannian manifolds of constant stable osculating rank. I., J. Reine Angew. Math., **274-275** (1975), 310–341.

[2] P. Dombrowski, Differentiable maps into Riemannian manifolds of constant stable osculating rank. II. Frenet-theory, J. Reine Angew. Math., **289** (1977), 144–173.

[3] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. II, Interscience Publishers, New York, 1969.

[4] K. Trenčevski, New approach for the submanifolds of the Euclidean space, Balkan Journal of Geometry and Its Applications, **2** (1997), 117–127.

[5] K. Trenčevski, Principal directions for submanifolds imbedded in Euclidean spaces of arbitrary codimension, submitted for publication.