NOVI SAD J. MATH.
VOL. 29, NO. 3, 1999, 349–356
XII Yugoslav Geometric Seminar
Novi Sad, October 8–11, 1998



# POLAR CYLINDERS OF SURFACES OF REVOLUTION: CONTOUR LINE DETERMINATION

**Radovan Štulić** **Zoran Jandrić** **Zoran Milojević**

Faculty of Technical Sciences, University of Novi Sad
Trg Dositeja Obradovića 6, 21121 Novi Sad, Yugoslavia

**Abstract**

In this paper a general analysis on contour line determination of surfaces of revolution, as well as an algorithm for its computation and computer aided representation has been performed.

A contour line of a surface of revolution (being a plane curve, when surface of revolution represented in certain projection) is being obtained as a projection of a space curve lying on the surface, which divides its visible from its invisible part.

It has been shown that the space curve can be obtained as an intersection of the surface and its view-dependent polar cylinder which is generated by a particular family of spheres. The surface, being the envelope for the family of spheres, with each sphere has a parallel i.e. a circle of revolution in common. Thus, the space curve can be determined in the analytical form so that it can be treated by means of differential geometry. Once the curve and its projection that is the contour line of the surface are determined, a base for creation of a simple visibility criterion is got avoiding in that way any later hidden line removal.

## 1 Introduction

Among the problems of realistic surface representation, the contour line determination takes an important place. Once the contour line determined the visible part of the surface from its invisible part is directly separated. No more visibility determination of points of the surface is required so that the use of standard methods of computer graphics can be avoided. Knowing the contour line which is a view dependent property of the surface, the data storage of the surface is made easier and less demanding. Further more the boundary representation which is normally used in descriptive geometry, which is in computer graphics regarded as inconvenient (because of its complexity) for curved surface, can now been used for surface of revolution representation.

The problem of contour line determination is already known and developed in numerous investigations. It is usually based upon the tangent planes determination of the particular position using methods of differential geometry. On one hand it can be regarded as a general method but on the other hand it might sometimes appear difficult to apply.

Whenever a descriptive geometric approach can be applied to certain problem of surface representation, the solution of the problem turned out to be easier and more convenient for computer aided use [2, 7].

In this paper, based upon the results given in [2, 7], a pure analytical procedure of contour line determination of surfaces of revolution has been developed, as well as the algorithm for contour line computing and drawing.

## 2 Geometric Structure of Surfaces of Revolution

The representation of an object may appear as a need for representation of either already exiting or "new" one. In case of the latter, one may almost always use mathematical description of these objects. When the matter is on the surfaces of revolution, usual way of its modeling is by revolving two dimensional entity about an axis in space [5].

In this paper we used the following notation: vectors or vector functions are represented by bold letters f.e. i; **f(x)** and first derivatives with respect to certain parameter t by: $\dot{\tilde{}} = \frac{d}{dt}\tilde{}$.

Therefore surface of revolution **R** can be generated by rotating a plane curve **f**, called meridian, about its coplanar axis $o$. Let the meridian be given

$$\mathbf{f}(t) = x(t) \cdot \mathbf{i} + y(t) \cdot \mathbf{j} + z(t) \cdot \mathbf{k}; \qquad t \in [t_0, t_1] \tag{1}$$

where the condition of coplanarity is valid:

$$A \cdot x(t) + B \cdot y(t) + C \cdot z(t) + D = 0 \tag{2}$$

and the axis of revolution by its vector $\mathbf{o} = (o_x, o_y, o_z)$, and one point on it $O(O_x, O_y, O_z)$.

Since the contour line of a surface is only a view dependent line, in this paper, for the simplicity of the obtained formulae, we shall presume that the axis of revolution is coincident with one coordinate axis (for example $z$ axis) and its coplanar meridian lies in one coordinate plane ($Oyz$ plane). Therefore we have the meridian

$$\begin{aligned} &\mathbf{f}(t) = y(t) \cdot \mathbf{j} + z(t) \cdot \mathbf{k}, \\ &x(t) \equiv 0; \\ &t \in [t_0, t_1] \end{aligned} \tag{3}$$

and the axis $o$ of its vector $(0, 0, 1)$ through the coordinate origin.

The surface of revolution generated by revolving the meridian (3) about the axis has its vector equation:

$$\begin{aligned} &\mathbf{R}(t, \theta) = y(t) \cos\theta \cdot \mathbf{i} + y(t) \sin\theta \cdot \mathbf{j} + \mathbf{z}(t) \cdot \mathbf{k}; \\ &t \in [t_0, t_1] \\ &\theta \in [0, 2\pi] \end{aligned} \tag{4}$$

This means that certain point $F(0, y(t_F), z(t_F))$ lying on the meridian (3) rotates about the axis $o$ into the point $F_R(y(t_F) \cos\theta_{F_R},\ y(t_F) \sin\theta_{F_R},\ z(t_F))$.

The equations of meridians and parallels of the surface (5) are obtained when taking $t = const.$ and $\theta = const.$, respectively.

Meridians and parallels form an orthogonal net of curves on the surface of revolution. The tangent plane to the surface is plane determined by tangents to the meridian and parallel.

# 3 Contour Line Determination

It is known that the contour line of any curved surface is a space curve lying on the surface, which is in fact an envelope of points of tangency of tangent planes to the surface which pass through the centre of projection (both in perspective and parallel projection).

## 3.1 Family of Auxiliary Touching Spheres

Let us introduce a family of spheres given by the equation

$$f(X, Y, Z, t) = X^2 + Y^2 + (Z - \zeta(t))^2 - R^2(t) = 0 \tag{5}$$

where $t \in [t_0, t_1]$ is a parameter and terms $z(t)$ and $R(t)$ are given by the following expressions:

$$\begin{aligned} \zeta(t) &= z(t) + \frac{\dot{y}(t)}{\dot{z}(t)} \\ R(t) &= y(t)\sqrt{1 + \left(\frac{\dot{y}(t)}{\dot{z}(t)}\right)^2} \end{aligned} \tag{6}$$

which means that each sphere from the family (5) has one parallel with the surface (4) in common, that is, each sphere is a touching sphere of the surface of revolution.

### 3.2 Polar Plane of Sphere

It is known [6] that, for certain ideal point, the polar plane of the sphere intersects the sphere at its great circle (through the centre of the sphere). It means that for parallel projecting rays whose vector is $\mathbf{p} = (p_x, p_y, p_z)$ polar plane of each sphere has the equation

$$p_x\tilde{x} + p_y\tilde{y} + (p_z\tilde{z} - \zeta(t)) = 0. \tag{7}$$

### 3.3 Polar Cylinders

Polarity of the family of auxiliary touching spheres can be generalized onto the polarity of its surface of revolution. The equivalent to the polar plane of the sphere is a polar cylinder of the surface of revolution since each sphere from the family touches the surface at one parallel. Namely if the polar plane (7) is intersected by the plane of their common parallel an equation of the line is obtained:

$$p_x\tilde{X} + p_y\tilde{Y} + p_z y(t)\frac{\dot{y}(t)}{\dot{z}(t)} = 0; \qquad \tilde{Z} = z(t) \tag{8}$$

which can be written in the canonical form as

$$\frac{\tilde{X}}{p_y} = \frac{\tilde{Y} - \frac{p_z}{p_y}y(t)\frac{\dot{y}(t)}{\dot{z}(t)}}{-p_x} = \frac{\tilde{Z} - z(t)}{0} = \nu \tag{9}$$

or in the parametric form:

$$\begin{aligned} \tilde{X}(t,\nu) &= p_y\nu \\ \tilde{Y}(t,\nu) &= \frac{p_z}{p_y}y(t)\frac{\dot{y}(t)}{\dot{z}(t)} - p_x\nu \\ \tilde{Z}(t,\nu) &= z(t) \end{aligned} \tag{10}$$

which, more obviously, represents a straight line generated surface.

### 3.4 Contour Line

The intersecting curve between the surface of revolution (4) and its polar cylinder (10) is a contour line whose 3D coordinates are given as follows:

$$\begin{aligned}
\tilde{X}_{C_1}(t,\nu) &= p_y \nu_{C_1} \\
\tilde{Y}_{C_1}(t,\nu) &= \frac{p_z}{p_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} - p_x \nu_{C_1} \\
\tilde{Z}_{C_1}(t,\nu) &= z(t) \\
\tilde{X}_{C_2}(t,\nu) &= p_y \nu_{C_2} \\
\tilde{Y}_{C_2}(t,\nu) &= \frac{p_z}{p_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} - p_x \nu_{C_2} \\
\tilde{Z}_{C_2}(t,\nu) &= z(t) \\
&\text{for } t \in [t_0, t_1]
\end{aligned} \tag{11}$$

where

$$\nu_{C_1/C_2} = \frac{p_x \cdot p_z}{p_y \left(p_x^2 + p_y^2\right)} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} \pm \frac{y(t)}{p_x^2 + p_y^2} \sqrt{p_x^2 + p_y^2 - \left(p_z \frac{\dot{y}(t)}{\dot{z}(t)}\right)^2} \tag{12}$$

which being projected by parallel projecting rays gives generally an oblique projection of the surface of revolution which is in fact the transformation of 3D into 2D coordinates onto the frontal projection plane i.e. Oyz plane. The transformation formulae are taken from [7]:

$$\begin{aligned}
y_{2D} &= y_{3D} - \lambda \cdot x_{3D} \cos\varphi \\
z_{2D} &= z_{3D} - \lambda \cdot x_{3D} \sin\varphi
\end{aligned} \tag{13}$$

where $\lambda$ and $\varphi$ are "shortening" coefficient and angle $\angle(-xOy)$, respectively, usual parameters of an oblique projection in descriptive geometry. The relation between the vector of projection rays and $\lambda$ and $\varphi$ are:

$$\begin{aligned}
p_x &= 1 \\
p_y &= \lambda \cos\varphi \\
p_z &= \lambda \sin\varphi
\end{aligned} \tag{14}$$

### 3.5 Visibility Test

As we have already said, the contour line divides a surface of revolution on its visible and invisible part. Therefore, points on each parallel of the surface are either on the visible part or invisible, where the boundary points are contour points which lie on the contour line-space curve of the surface. In order to determine which points of the parallel are on the visible side of the surface we have performed simple visibility test according to which an arbitrary point $N$ of the parallel together with each of boundary-contour points $C_1$ and $C_2$ forms two vectors $\mathbf{NC}_1$ and $\mathbf{NC}_2$, whose sum is a vector directed out of the surface. Thus a scalar product between this sum and rays of projection gives:

$$(\mathbf{NC}_1 + \mathbf{NC}_2) \cdot \mathbf{p} > 0 \Longrightarrow \text{point } N \text{ lies on invisible side of the surface,}$$

$$(\mathbf{NC}_1 + \mathbf{NC}_2) \cdot \mathbf{p} < 0 \Longrightarrow \text{point } N \text{ lies on visible side of the surface.}$$

## 4 Examples

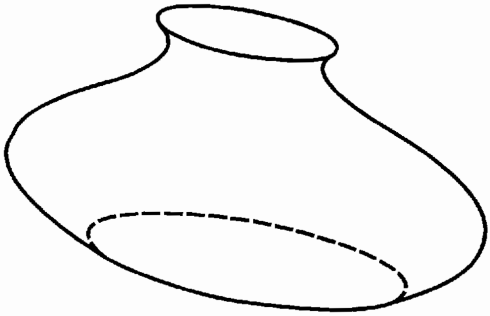
According to the previous we have created the following algorithm:

```
Program CONTOUR LINE;
begin
  INPUT:      meridian, axis, rays of projection;
  CALCULATE: contour line; transformation of 3D to 2D coordinates;
  DRAW:       contour of the surface, edges of upper and lower
              basis after having performed the visibility test
end.
```

Figures 1–3 illustrate three surfaces of revolution represented in oblique projection. The meridian which was revolved about $z$ axis is given beside each picture.

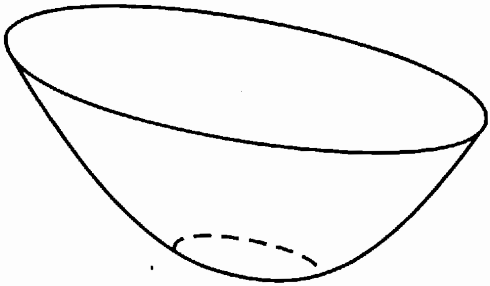
## 5 Conclusion

In the paper we have developed an analytical procedure for direct contour line determination of surfaces of revolution. The procedure is based upon descriptive geometric method of auxiliary touching spheres by generalizing the polarity of spheres onto the polarity of surfaces of revolution, that is, each surface of revolution generates one polar cylinder whose intersection is a space curve, lying on the surface, i.e. contour line. The restriction we have taken into account is that the surface does not hide itself, which means that



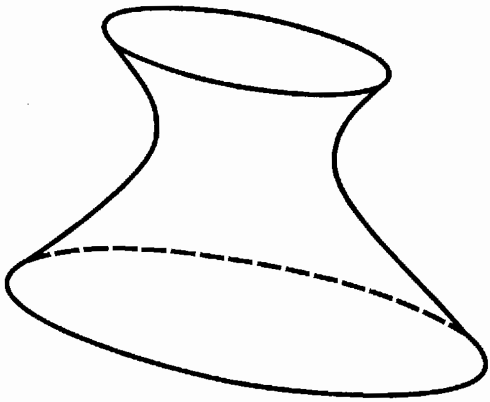
$$x(t) \equiv 0;$$
$$y(t) = 2a + a\sin t;$$
$$z(t) = t;$$
$$t \in [t_0, t_1] \tag{15}$$

Figure 1:



$$x(t) \equiv 0;$$
$$y(t) = t;$$
$$z(t) = t^2;$$
$$t \in [t_0, t_1] \tag{16}$$

Figure 2:



$$x(t) \equiv 0;$$
$$y(t) = \sqrt{1+t^2};$$
$$z(t) = t;$$
$$t \in [t_0, t_1] \tag{17}$$

Figure 3:

each parallel the surface of revolution has real contour points. More general case of self-hiding surfaces is to be analyzed in further investigations.

# References

[1] Foley, J., van Dam, A., Feiner, S. and Hughes, J.: Computer Graphics-Principles and Practise, Addison Wesley Publishing company Inc., 1996.

[2] Mihajlović Levi, V. and Stulić, R.: Surfaces of Revolution in CAD: Contour Line Determination, Proceedings of the 5th ASEE International Conference on Engineering Computer Graphics and Geometry, Melbourne, Australia, 1992. vol. 1, pp. 358-362.

[3] Pejović, T.: Mathematical Analysis V (in Serbian language), Naučna knjiga Beograd, 1978.

[4] Prvanović, M.: Projective Geometry (in Serbian language), Naučna knjiga Beograd, 1986.

[5] Rogers, F. D. and Adams, A. J.: Mathematical Elements for Computer Graphics, McGraw Hill International Editions, 1990.

[6] Sbutega, V.: Synthetic Geometry III (in Serbian language), Postgraduate Course, University of Beograd, 1986.

[7] Stulić, R.: Descriptive Geometric Methods in Computer Graphics: Contour Lines and Isophots of Surfaces of Revolution (in Serbian language), Master Thesis, Beograd, 1994.

[8] Wheatherburn, C. E.: Differential Geometry of Three Dimensions, Cambridge University Press, 1955.