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PARTICULAR F-STRUCTURE ON VECTOR BUNDLE AND COMPATIBLE D-CONNECTIONS

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Abstract

The structures determined by a tensor field of $(1,1)$ type of constant rank, with the property $f^3+f=0$, was studied by many authors. R. Miron and Gh. Atanasiu determined the set of all connections compatible with f-structures, the integrability of f-structures and studied the case of (f,g)-structures in [1].

In the present paper we shall consider f-structures on the total space E of the vector bundle $\xi=(E,\pi M)$ and we shall find the non-linear connections N on E so that the tensor field f has a particular form. In this manner f-structures of type I and type II are defined and in these cases the compatible d-connections from general case [2] are determined.

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Let $\xi=(E,\pi,M)$ be a vector bundle over n-dimensional manifold M, $\chi(E)$ the $F(E)$-modul of the vector fields, $\tau^p_q(E)$ the $F(E)$-modul of the tensor field of (p,q) type and $\tau_F\binom{p\ r}{q\ s}(E)$ the algebra of the Finsler tensor fields (d-tensor fields) of $\binom{p\ r}{q\ s}$ type.

Let us consider f-structures on E, as tensor fields $f\in\tau^1_1(E)$ of constant rank r, with the property $f^3+f=0$.

In the following we shall suppose that there exists an f-structure on E.

We shall consider a non-linear connection N on E; thus $\forall u\in E$, $TE_u=N_u\oplus E^v_u$, where $\dim N_u=n$ and $\dim E^v_u=m$, so it results that total space TE is decomposed $TE=HE\oplus VE$.

Proposition 1 *If f is f-structure on E then there is a unique decomposition of f in the following d-tensor fields:*

$$f={}^1f+{}^2f+{}^3f+{}^4f,\ \text{where}\ {}^1f\in\tau_F\begin{pmatrix}1\ 0\\1\ 0\end{pmatrix}(E),\ {}^2f\in\tau_F\begin{pmatrix}1\ 0\\0\ 1\end{pmatrix}(E),\quad (1)$$

${}^3f \in \tau_F\binom{0\ 1}{1\ 0}(E)$ *and* ${}^4f \in \tau_F\binom{0\ 1}{0\ 1}(E)$, *i.e.*

$$\begin{aligned} &{}^1f(\omega, X) = f(\omega^H, X^H), {}^3f(\omega, X) = f(\omega^H, X^V)\\ &{}^2f(\omega, X) = f(\omega^V, X^H), {}^4f(\omega, X) = f(\omega^V, X^V) \\ &\forall X \in \chi(E) \text{ and } \forall\omega \in X^*(E). \end{aligned} \tag{2}$$

$$\begin{cases} f(X^H) = {}^1f(X) + {}^2f(X) \\ f(X^V) = {}^3f(X) + {}^4f(X) \end{cases}, \forall X = X^H + X^V. \tag{3}$$

Let ∇ be a linear connection on E.

Definition 1 *A linear connection ∇ on E is a d-connection compatible with structure f on E if:*

a) ∇ preserves parallelism of the horizontal H and vertical E^V distribution.

b) $\nabla_X f - f\nabla_X = 0$, $\forall X \in \chi(E)$.

Theorem 1 *If ∇° is a fixed linear d-connection on E, then the following connection*

$$\begin{aligned} \nabla_X = {} & \nabla^\circ_X - \frac{1}{2}[{}^1f \circ \nabla^\circ_X \circ {}^1f + {}^2f \circ \nabla^\circ_X \circ {}^3f - 2({}^1f \circ {}^1f + \\ & {}^2f \circ {}^3f) \circ \nabla^\circ_X - 2\nabla^\circ_X \circ ({}^1f \circ {}^1f + {}^2f \circ {}^3f) - \\ & 3({}^1f \circ {}^1f + {}^2f \circ {}^3f) \circ \nabla^\circ_X \circ ({}^1f \circ {}^1f + {}^2f \circ {}^3f) - \\ & 3({}^1f \circ {}^2f + {}^2f \circ {}^4f) \circ \nabla^\circ_X \circ ({}^3f \circ {}^1f + {}^4f \circ {}^3f)] - \\ & \frac{1}{2}[{}^3f \circ \nabla^\circ_X \circ {}^2f + {}^4f \circ \nabla^\circ_X \circ {}^4f - 2({}^3f \circ {}^2f + {}^4f \circ {}^4f) \circ \nabla^\circ_X - \\ & 2\nabla^\circ_X \circ ({}^3f \circ {}^2f + {}^4f \circ {}^4f) - \\ & 3({}^3f \circ {}^1f + {}^4f \circ {}^3f) \circ \nabla^\circ_X \circ ({}^1f \circ {}^2f + {}^2f \circ {}^4f) - \\ & 3({}^3f \circ {}^2f + {}^4f \circ {}^4f) \circ \nabla^\circ_X \circ ({}^3f \circ {}^2f + {}^4f \circ {}^4f)]. \end{aligned} \tag{4}$$

is a d-connection compatible with the structure f.

Let $\pi^{-1}(U)$ be a local chart on E and (x^i, y^a), $i = 1, 2, \ldots, m$ the coordinates of point $u = (x^i, y^a) \in \pi^{-1}(U)$.

If N° is a fixed non-linear connection on E, then in the adapted frame $\left(\frac{\delta^\circ}{\delta x^i}, \frac{\partial}{\partial y^a}\right)$, the tensor field f is written in the form

$$f = f^i_j \frac{\delta^\circ}{\delta x^i} \otimes dx^j + f^i_b \frac{\delta^\circ}{\delta x^i} \otimes \delta^\circ y^b + f^a_j \frac{\partial}{\partial y^a} \otimes dx^j + f^a_b \frac{\partial}{\partial y^a} \otimes \delta^\circ y^b. \tag{5}$$

If $B \in \tau_F\binom{0\,1}{1\,0}(E)$ is a d-tensor field and $N_j^a = N_j^{\circ a} - B_j^a$ is another non-linear connection on E then the tensor field F has the components $(F_j^i, F_b^i, F_j^a, F_b^a)$,

$$\begin{cases} F_j^i = f_j^i - f_c^i B_j^c \\ F_b^i = f_b^i \\ F_j^a = f_j^a + f_j^s B_s^a - f_c^a B_j^c - f_c^s B_s^a B_j^c \\ F_b^a = f_b^a + f_b^s B_s^a \end{cases} \quad . \tag{6}$$

When on the total space E there exists a non-linear connection so that the tensor f has the form $f = (F_j^i, 0, 0, F_b^a)$, f will be called f-structure of type I, respectively f-structure of type II, when f has the form $f = (0, F_b^i, F_j^a, 0)$.

From (6) it is clear that the f-structure of type I exists only in the case $f_b^i = 0$ and a non-linear connection N for which $F_j^a = 0$ exists.

Proposition 2 *The f-structure is of type I if and only if there exists a tensor field $B \in \tau_F\binom{0\,1}{1\,0}(E)$ which satisfies the equation*

$$f_c^a B_j^c - f_j^k B_k^a = f_j^a. \tag{7}$$

Since, rank $f = r < n + m$, the system (7) has not always solution. In the affirmative case, the d-tensor field f has the form:

$$f = {}^1f + {}^4f, \text{ where } {}^1f \in \tau_F\begin{pmatrix}1\,0\\1\,0\end{pmatrix}(E),\ {}^4f \in \tau_F\begin{pmatrix}0\,1\\0\,1\end{pmatrix}(E) \tag{8}$$

and the relation ${}^3f + f = 0$ is equivalent with:

$$\begin{cases} {}^1f \circ {}^1f \circ {}^1f + {}^1f = 0 \\ {}^4f \circ {}^4f \circ {}^4f + {}^4f = 0 \end{cases} \quad . \tag{9}$$

Theorem 2 *If a fixed linear d-connection (distinguished connection) exists on E, then the distinguished f-connection are of type I, and one of them is given by*

$$\nabla_x = \nabla_x^\circ - \frac{1}{2}[{}^1f \circ \nabla_x^\circ \circ {}^1f + {}^4f \circ \nabla_x^\circ \circ {}^4f - \tag{10}$$
$$2({}^1f \circ {}^1f + {}^4f \circ {}^4f) \circ \nabla_x^\circ - 2\nabla_x^\circ \circ ({}^1f \circ {}^1f + {}^4f \circ {}^4f) -$$
$$3\,{}^1f \circ {}^1f \circ \nabla_x^\circ \circ {}^1f \circ {}^1f - 3\,{}^4f \circ {}^4f \circ \nabla_x^\circ \circ {}^4f \circ {}^4f].$$

In local coordinates, if we denote by $DT^\circ = (L^{\circ i}_{jk}, L^{\circ a}_{bk}, C^{\circ i}_{jc}, C^{\circ a}_{bc})$ the local components of linear d-connection ∇°, then the distinguished f-connection of type I is characterized by

$$\begin{cases} L^i_{jk} = L^{\circ i}_{jk} + {}^1f^{r}_j\,{}^1f^{i\circ}_{r|k} - {}^1f^{i}_r\,{}^1f^{r\circ}_{j|k} + \frac{3}{2}\,{}^1f^{i}_s\,{}^1f^{s}_e\,{}^1f^{r}_j\,{}^1f^{e\circ}_{r|k} \\ L^a_{bk} = L^{\circ a}_{bk} + {}^4f^{d}_b\,{}^4f^{a\circ}_{d|k} - {}^4f^{a}_d\,{}^4f^{a\circ}_{b|k} + \frac{3}{2}\,{}^4f^{a}_b\,{}^4f^{b}_d\,{}^4f^{c}_b\,{}^4f^{d\circ}_{c|k} \\ C^i_{jc} = C^{\circ i}_{jc} + {}^1f^{r}_j\,{}^1f^{i\circ}_{r|c} - {}^1f^{i}_r\,{}^1f^{r}_{j|c} + \frac{3}{2}\,{}^1f^{i}_s\,{}^1f^{s}_e\,{}^1f^{r}_j\,{}^1f^{e\circ}_{r|c} \\ C^a_{bc} = C^{\circ a}_{bc} + {}^4f^{d}_b\,{}^4f^{a\circ}_{d|c} - {}^4f^{a}_d\,{}^4f^{a\circ}_{b|c} + \frac{3}{2}\,{}^4f^{a}_b\,{}^4f^{b}_d\,{}^4f^{a}_b\,{}^4f^{d\circ}_{a|c}. \end{cases} \tag{11}$$

Theorem 3 *The set of all distinguished f-connections of type I is given by*

$$\bar{\nabla}_x = \nabla_x + \Omega(W_X), \tag{12}$$

where ∇_x is the connection (10), $W_X \in \tau^1_1(E)$ is an arbitrary tensor field so that $[\Omega(W_X)Y^H]^V = 0$, $[\Omega(W_X)Y^V]^H = 0$ and Ω is Obata operator [4].

The structures of type II, $f = {}^2f + {}^3f$, under some conditions, permit to study the general case of the f-structures.

For the study of these structures we will consider the following case:

a) $n = dimHE > dimVE = m$

If the rank $f > m$, then the system $f^k_b B^a_k + f^a_b = 0$, $k = 1, 2, \dots, n$; $a, b = 1, 2, \dots, m$ admits infinite solutions. Moreover, if there exist B^a_k for which $f^i_c B^c_j - f^i_j = 0$, then there exists a non-linear connection $N^c_j = N^{\circ c}_j - B^c_j$ and the tensor field f can be written in the form $(0, F^i_b, F^a_j, 0)$.

b) $n = m$

If we consider rank $f \geq m = n$, rank $(f^i_b) = \max$ and the equations $f^i_c B^c_j - f^i_j = 0$, $f^k_b B^a_k + f^a_b = 0$ are satisfied, then for the non-linear connection $N^c_j = N^{\circ c}_j - B^{\circ c}_j$, the tensor field f is of the form $(0, F^i_b, F^a_j, 0)$.

c) $n < m$

Let us suppose $F^i_j = 0$, rank $f \geq n$, then the equation $f^i_c B^c_j - f^i_j = 0$ admits infinite solutions. Let $(B^{\circ c}_j)$ be a solution for which $f^k_b B^a_k + f^a_b = 0$, then there exists a non-linear connection $N^c_j = N^{\circ c}_j - B^{\circ c}_j$ so that $f = (0, F^i_b, F^a_j, 0)$. In this case, Y. Ichijio had designed a natural example [2].

Let us denote by ${}^2f^i_b = F^i_b$ and $f^a_j = {}^3F^a_j$, in the adapted frame, in all the above cases.

Theorem 4 *If there exists a non-linear connection N for which $f = {}^2f + {}^3f$, then there exist distinguished f-connections of type II, one of them is given* *by:*

$$\begin{aligned} \nabla_X \;=\;& \nabla^\circ_X - \frac{1}{2}[f^\circ \circ \nabla^\circ_X \circ {}^3f - 2\,{}^2f \circ {}^3f \circ \nabla^\circ_X - \\ & 2\nabla^\circ_X \circ {}^2f \circ {}^3f - 3\,{}^2f \circ {}^3f \circ \nabla^\circ_X \circ {}^2f \circ {}^3f] - \\ & -\frac{1}{2}[{}^3f \circ \nabla^\circ_X \circ {}^2f - 2\,{}^3f \circ {}^2f \circ \nabla^\circ_X - \\ & 2\nabla^\circ_X \circ {}^3f \circ {}^2f - 3\,{}^3f \circ {}^2f \circ \nabla^\circ_X \circ {}^3f \circ {}^2f] \end{aligned} \tag{13}$$

where ∇°_X is a fixed linear d-connection on E,

$$\bar{\nabla} = \nabla_X + \Omega(W_X), \tag{14}$$

where ∇_X is the connection (13) $W_X \in \tau^1_1(E)$ an arbitrary tensor field and $[\Omega(W_X)Y^H]^V = 0$, $[\Omega(W_X)Y^V]^H = 0$.

If we denote by $DT^\circ = (L^{\circ i}_{jk}, L^{\circ a}_{bk}, C^{\circ i}_{jc}, C^{\circ a}_{bc})$ the local coordinates of the linear d-connection ∇° then the distinguished f-connection of type II, (13), is characterised by:

$$\begin{cases} L^i_{jk} = L^{\circ i}_{jk} - {}^2f^{i3}_c f^{c\circ}_{j|k} + {}^2f^{i\circ}_{c|k}\,{}^3f^c_j + \frac{3}{2}\,{}^2f^{i3}_d f^{d2}_r f^{r\circ}_{c|k}\,{}^3f^c_j \\ L^a_{bk} = L^{\circ a}_{bk} - {}^3f^{a2}_r f^{r\circ}_{b|k} + {}^3f^{a\circ}_{r|k}\,{}^2f^r_b + \frac{3}{2}\,{}^3f^{a2}_r f^{r3}_d f^{d\circ}_{s|k}\,{}^2f^s_b \\ C^i_{jc} = C^{\circ i}_{jc} - {}^2f^{i3}_d f^{d\circ}_{j|c} + {}^2f^{i\circ}_{d|c}\,{}^3f^d_j + \frac{3}{2}\,{}^2f^{i3}_d f^{d2}_r f^{r\circ 3}_{b|c} f^b_j \\ C^a_{bc} = C^{\circ a}_{bc} - {}^3f^{a2}_r f^{r\circ}_{b|c} + {}^3f^{a\circ 2}_{r|c} f^r_b + \frac{3}{2}\,{}^3f^{a2}_r f^{r3}_d f^{d\circ 2}_{s|c} f^s_b . \end{cases} \tag{15}$$

Remark. If we consider natural f-structure $F = \begin{pmatrix} 0 & -C^i_b \\ B^a_j & 0 \end{pmatrix}$ with $B^a_s C^s_b = \delta^a_b$, and if $f^i_b = -C^i_b$, $f^b_j = B^b_j$, $f^i_c f^c_j + \delta^i_j = v^i_j$ in (12), then the linear distinguished f-connection (15), represents the natural distinguished f-connection [6].

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