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ON ALMOST GEODESIC MAPPINGS π_2 BETWEEN SEMISYMMETRIC RIEMANNIAN SPACES

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Abstract

In this paper we investigated special almost geodesic mappings of semisymmetric Riemannian spaces onto semisymmetric Riemannian spaces. It is proved that by this conditions semisymmetric spaces are symmetric in the sense of Cartan.

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1 Introduction

The beginnings of the investigation semisymmetric spaces are connected with the names P.A. Shirokov, E. Cartan and A. Lichnerovicz. N.S. Sinyukov

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(see [2], [3], [4], [6]) called (pseudo-) Riemanniann space V_n *semisymmetric* if the following condition is fulfilled

$$R^h_{ijk,[lm]} = 0 \tag{1}$$

where R^h_{ijk} is the Riemannian tensor, comma denotes a covariant derivative of V_n and the square brackets $[lm]$ are the alternation of indices l and m.

On the basis of the Ricci identity conditions (1) have the following form

$$R^h_{\alpha jk}R^\alpha_{ilm} + R^h_{i\alpha k}R^\alpha_{jlm} + R^h_{ij\alpha}R^\alpha_{klm} - R^\alpha_{ijk}R^h_{\alpha lm} = 0. \tag{2}$$

Geometric aspects of this spaces are studied in the papers [2], [3], [9]. N.S. Sinyukov, J. Mikeš, P. Venzi, E.N. Sinyukova and others investigated geodesic and holomorphically-projective mappings and transformations of semisymmetric spaces (see [3]–[7]).

The mentioned mappings generalize almost geodesic mappings introduced by N.S.Sinyukov (see [6], [7]). Almost geodesic mappings π_2 of symmetric spaces are studied by V.S. Sobchuk [8] and A. Adamov [1].

The diffeomorphism f from the (pseudo-) Riemannian space V_n $(n \geq 2)$ onto the (pseudo-) Riemannian space $\bar{V}_n$ is called *almost geodesic mapping* π_2 if in the common coordinate system x with respect to the mapping f, the conditions

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} + \sigma_{(i}F^h_{j)} \tag{3}$$

$$F^h_{(i,j)} + \sigma_{(i}F^\alpha_{j)}F^h_\alpha = \nu_{(i}\delta^h_{j)} + \mu_{(i}F^h_{j)} \tag{4}$$

hold, where Γ^h_{ij} $(\overline{\Gamma}^h_{ij})$ are the Christoffel symbols of V_n $(\bar{V}_n)$, ψ_i, σ_i, ν_i, μ_i are covectors, F^h_i is an affinor, round bracket is the symmetrization of indices.

In [1] and [8] are studied special almost geodesic mappings π_2. We define a wider class of mappings π_2. These mappings (*special almost geodesic mappings* π_2) are characterized by the conditions (3), while the structure F^h_i satisfies the conditions:

$$F^h_\alpha F^\alpha_i = e\delta^h_i, \qquad F_{[ij]} = 0, \qquad F^h_{i,j} = 0 \tag{5}$$

where $e = \pm 1$, $F_{ij} = F^\alpha_i g_{\alpha j}$, and the vectors ψ_i and σ_i are gradient vectors.

2 Special almost geodesic mappings π_2 between semisymmetric spaces

From the equations (3) and the conditions (5) follows that the Riemannian tensors of the spaces V_n and $\bar{V}_n$ under special almost geodesic mappings π_2 satisfy the equations:

$$\overline{R}^h_{ijk} = R^h_{ijk} + \psi_{i[j}\delta^h_{k]} + \sigma_{i[j}F^h_{k]} \tag{6}$$

where ψ_{ij} and σ_{ij} are symmetric tensors.

We shall study special almost geodesic mappings π_2 of semisymmetric space V_n onto semisymmetric space $\bar{V}_n$.

Substituting (6) into the algebraic condition of semisymmetric space $\bar{V}_n$, which is analogical to (2), because that V_n is also semisymmetric space and we have the equations

$$[R_{hmjk}\psi_{il} + \psi_{m\alpha}R^{\alpha}_{ijk}g_{hl} + R_{h\overline{m}jk}\sigma_{il} + \sigma_{m\alpha}R^{\alpha}_{ijk}F_{hl}]_{[lm]}+$$

$$[R_{him[k}\psi_{j]l} + \psi_{\alpha(i}R^{\alpha}_{j)lm}g_{hk} + R_{hi\overline{m}[k}\sigma_{j]l} + \sigma_{\alpha(i}R^{\alpha}_{j)lm}F_{hk}]_{[jk]}+$$

$$[b_{\overline{m}(i}\sigma_{j)l}g_{hk} + b_{\overline{k}m}\sigma_{ij}g_{hl} + b_{m(i}\sigma_{j)l}F_{hk} + b_{km}\sigma_{ij}F_{hl}]_{[jk][lm]} = 0$$

where denotes $A_{\dots\bar{i}\dots} \equiv A_{\dots\alpha\dots}F^{\alpha}_i$. The expressions in square brackets are alternated according to the indices in small square brackets behind.

If $\det((n-1)\psi_{ij} - \sigma_{\overline{ij}} + F\sigma_{ij}) \neq 0$ we obtain from these conditions that Riemannian tensor has the following form

$$R_{hijk} = [A(g_{ij}F_{hk} + g_{hk}F_{ij}) + B(eg_{ij}g_{hk} + F_{hk}F_{ij})]_{[jk]} \tag{7}$$

where $F \equiv F^{\alpha}_{\alpha}$, A and B are functions.

Applying Bianci identity for Riemannian tensor

$$R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0$$

we obtain that A and B are const. Then V_n is a symmetric space.

The following theorem holds.

Theorem 1 *Semisymmetric (pseudo-) Riemannian space V_n admits special almost geodesic mapping π_2 onto semisymmetric (pseudo-) Riemannian space $\bar{V}_n$ under the condition* $\det((n-1)\psi_{ij} - \sigma_{\overline{ij}} + F\sigma_{ij}) \neq 0$ *if only if V_n is symmetric space with the special Riemannian tensor*

$$R_{hijk} = [A(g_{ij}F_{hk} + g_{hk}F_{ij}) + B(eg_{ij}g_{hk} + F_{hk}F_{ij})]_{[jk]}$$

where A and B are constant.

The necessity follows from the previous text. The sufficiency follows from the results on the special almost geodesic mappings π_2 symmetric spaces [8].

Remarks. The symmetric Riemannian spaces with non constant curvature do not admit nontrivial geodesic mappings onto Riemannian spaces [4], [6]. The symmetric spaces which are satisfying (7) and $A, B \neq 0$ have not constant curvature and these spaces admit special almost geodesic mappings π_2.

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