NOVI SAD J. MATH.
VOL. 29, NO. 3, 1999, 291–299
XII Yugoslav Geometric Seminar
Novi Sad, October 8–11, 1998

ON GEODESIC LINES OF METRIC SEMI-SYMMETRIC CONNECTION ON RIEMANNIAN AND HYPERBOLIC KAEHLERIAN SPACES

Nevena Pušić
Institute of Mathematics, Faculty of Science, Novi Sad, Yugoslavia
Email address: nevena@unsim.ns.ac.yu

Abstract

Geodesic lines on any metric space are autoparallel lines of its Levi-Civita connection. Here we give necessary and sufficient conditions for a metric semi-symmetric connection of a Riemannian space and ametric semi-symmetric connection of a hyperbolic Kaehlerian space to have some of their autoparallel lines in common with their Levi-Civita connection.

AMS Mathematics Subject Classification (1991): 53A30.
Key words and phrases: metric semi-symmetric connection, gradient vector field, generating vector field, isotropic vector field.

1 Introduction

The subject of the present paper is in frame of work of the author and M. Prvanović, about properties of a series of connections with their torsion tensor of a special forms given in advance, on Riemannian (maybe, with indefinite metrics), product, Kaehlerian and hyperbolic Kaehlerian spaces. But the geodesity as a geometric subject is a matter of special interest of the author herself, specially for semi-symmetric connections, which may be very helpful in working in physics of macro- and micro-phenomena.This is the reason why we are omitting the product spaces this time – there is no natural way to introduce a metric semi-symmetric F-connection. Also, as Kaehlerian spaces have strongly positive definite metrics, this time we also omitted them. But, as hyperbolic Kaehlerian spaces have indefinite metrics an can be divided in a natural way into two subspaces of equal dimension which are both totally geodesic, there must be much more geodesity and much more "shortents" then in Kaehlerian spaces, and, probably, then in Riemannian spaces.

2 Riemannian space

Let us consider a Riemannian manifold M, of dimension n, with metric tensor (g_{ij}). Let us denote the Christoffel symbols towards given metrics by $\{{}^{i}_{jk}\}$, the operator of covariant differentiation towards the Levi-Civita connection by $\overset{\circ}{\nabla}$ and components of its curvature tensor by $K^{i}{}_{jkl}$ (or by K_{ijkl} for its Riemann-Christoffel tensor).

The geodesic line, that means, an autoparallel line of Levi-Civita connection is characterized by the relation

$$v^k \overset{\circ}{\nabla}_k v^j = 0, \tag{1}$$

where v^k stands for a component of tangent vector field of the geodesic line.

The components of metric semi-symmetric connection are given by

$$\Gamma^a_{ik} = \{{}^{a}_{ik}\} + p_i \delta^a_k - p^a g_{ik}, \tag{2}$$

where p_i and p^a are covariant and contravariant components of a vector field. This vector field is called **the generator** or **the generating vector field** of metric semi-symmetric connection. The torsion tensor components of the metric semi-symmetric connection are equal

$$T^a_{ik} = p_i \delta^a_k - p^k \delta^a_i. \tag{3}$$

Let v^k be the component of tangent vector field of a geodesic line; then, this geodesic line will be autoparallel for metric semi-symmetric connection if and only if there holds

$$p_k v^k v^j = p^j v_k v^k. \tag{4}$$

As we consider a Riemannian space and its metrics is positively definite, (3) happens if and only if

$$p_k = \alpha v_k, \tag{5}$$

where α is a scalar function.

If we explore properties of the scalar function α, as it is the proportionality coefficient between the tangent vector field of the geodesic line and generating vector field of the metric semi-symmetric connection, we shall get an answer to the following question: for a geodesic line on a Riemannian space, how many semi-symmetric metric connections have the same line as an autoparallel line.

Let us denote the operator of covariant differentiation towards metric semi-symmetric connection by ∇; it is easy to calculate

$$\nabla_k v_j = \overset{\circ}{\nabla}_k v_j - \alpha(v_k v_j - g_{kj})v,$$

$$\overset{\circ}{\nabla}_k p_j = \alpha_k v_j + \alpha \overset{\circ}{\nabla}_k v_j,$$

where v stands for scalar square of the vector v_k and $\alpha_k = \frac{\partial \alpha}{\partial x^k}$.

The components of Riemann-Christoffel tensor of the metric semi-symmetric connection can be expressed in this way:

$$R_{ijkl} = K_{ijkl} + g_{ik}p_{lj} - g_{il}p_{kj} + g_{jl}p_{ki} - g_{jk}p_{li}, \tag{6}$$

where K_{ijkl} denotes a Riemann-Christoffel tensor component of Levi-Civita connection and the abbreviation p_{kj} stands for the tensor

$$p_{kj} = \overset{\circ}{\nabla}_k p_j - p_j p_k + \frac{1}{2} p_s p^s g_{jk}. \tag{7}$$

The tensor p_{kj} is symmetric if and only if p_j is a gradient, that means

$$\alpha_k v_j - \alpha_j v_k = \alpha(\overset{\circ}{\nabla}_j v_k - \overset{\circ}{\nabla}_k v_j),$$

or, equivalently

$$\frac{\partial v}{\partial x^k} = \frac{2}{\alpha}(\varphi v_k - v\alpha_k), \tag{8}$$

where φ stands for $\alpha_k v^k$.

From the expression (5), we easily obtain

$$R_{jk} = K_{jk} + (2-n)p_{kj} - g_{jk}p^s_s,$$

$$g_{kj}p^s_s = K_{jk} - R_{jk} - (n-2)p_{jk},$$

$$np^s_s = K - R - (n-2)p^s_s,$$

$$p^s_s = \frac{K-R}{2(n-1)},$$

$$p_{kj} = \frac{K_{jk} - R_{jk}}{n-2} - \frac{K-R}{2(n-1)(n-2)} g_{jk}.$$

By K_{jk}, K and R_{jk}, R we denote the Ricci tensor and the curvature scalar for the Levi Civita connection and for metric semi-symmetric connection respectively.

Obviously, one can find a curvature-type tensor which is independent on the choice of generating vector field; it depends only on the fact that the generator is a gradient and it would be invariant for all such semi-symmetric connections. But it is not the aim of this paper.

We want our curvature tensor (5) to satisfy the all algebraic properties which are most common for curvature tensors: to be skew-symmetric in first two indices, to be invariant under change of places of first and second pair of indices and to satisfy the first Bianchi identity. All these properties are satisfied if and only if the generating vector field is a gradient.

There holds

$$v^i \overset{\circ}{\nabla}_i v_k = 0$$

and

$$p^i \overset{\circ}{\nabla}_i p_k = p^i \overset{\circ}{\nabla}_k p_i = \varphi p_k = \varphi \alpha v_k.$$

Now, we apply the Ricci identity for the metric semi-symmetric connection to the generator and we obtain

$$v\alpha_j - \varphi v_j = 0. \tag{9}$$

Then there yields, in view of (7),

$$\frac{\partial v}{\partial x^k} = 0,$$

and the tangent vector of the geodesic line is of constant length.

From (8), we have

$$\alpha_k = \frac{\varphi}{v} v_k,$$

or

$$\alpha_k = f p_k.$$

This means that all three vectors are mutually proportional. Then,

$$p_s p^s = \alpha^2 v.$$

Besides,

$$\frac{\partial (p_s p^s)}{\partial x^k} = p^s \overset{\circ}{\nabla}_k p_s + p_s \overset{\circ}{\nabla}_k p^s = p^s \alpha_k p_s + \alpha p^s \overset{\circ}{\nabla}_k v_s =$$

$$= 2\alpha_k p_s p^s + 2\alpha p^s \overset{\circ}{\nabla}_k v_s = 2\alpha_k p_s p^s + 2\alpha^2 v^s \overset{\circ}{\nabla}_k v_s =$$

$$= 2p_s p^s \alpha_k = 2\alpha^2 \alpha_k v.$$

As

$$\overset{\circ}{\nabla}_k \, v^s v_s = 0$$

and, consequently

$$v^s \, \overset{\circ}{\nabla}_k \, v_s = 0,$$

then, on the other side

$$\frac{\partial(p_s p^s)}{\partial x^k} = 2p^s \, \overset{\circ}{\nabla}_k \, p_s = 2\alpha_k p^s p_s = 2\alpha^2 v \alpha_k = 2\alpha v \alpha_k.$$

Comparing two results for $\frac{\partial(p_s p^s)}{\partial x^k}$, we obtain $\alpha = 1$ or $\alpha_k = 0$. Then

(1) (p_k) and (v_k) are equal, both gradients, both of constant length; or

(2) (p_k) and (v_k) are collinear vectors, both of constant length and both gradients.

We proved

Theorem 1 *On a Riemannian manifold, the curvature tensor of metric semi-symmetric metric connection satisfies the all most common algebraic properties for any curvature tensor if and only if the generating vector field is a gradient. Then the Levi-Civita connection and metric semi-symmetric connection have some of their autoparallel lines in common if the generating vector field of metric semi-symmetric connection and tangent vector field of the geodesic line are collinear gradients of constant length,*

3 Hyperbolic Kaehlerian space

A hyperbolic Kaehlerian space is an even-dimensional pseudo-Riemannian space, endowed with a nondegenerate structure tensor (F^i_j) satisfying

$$F^i_j F^j_k = \delta^i_k, \; F_{ij} = -F_{ji}, \; \overset{\circ}{\nabla} F_{ij} = 0. \tag{10}$$

We can see that a hyperbolic Kaehlerian space is in fact a product space, but its covariant structure tensor is skew-symmetric. Besides, the structure tensor has n linearly independent eigen vectors; by the fact of skew-symmetry of the structure, it sends any vector into an orthogonal vector and eigen vectors of the structure are, consequently, self-orthogonal. In any point of a hyperbolic Kaehlerian manifold, its tangent space can be spanned by these self-orthogonal vectors; it is its adapted basis. It is obvious that there are two eigen subspaces of equal dimension, for two structure's eigenvalues, 1 and -1. On both these invariant subspaces, the metric tensor vanishes. Actually, a hyperbolic Kaehlerian space in any point is a product of two totally geodesic subspaces. This is the reason to investigate geodesic lines of this kind of space. Here we may have such a geodesic lines which are minimizing the distance between two different points up to zero; from one point, we may reach another point instantly along such a geodesic line.

A semi-symmetric connection on a hyperbolic Kaehlerian space has the torsion

$$T_{ij}^{k} = p_i\delta_j^k - p_j\delta_i^k + q_iK_j^k - q_jF_i^k, \tag{11}$$

where (p_i) and (q_i) are components of certain vector fields. If we, moreover, want this connection to be a metric one, then it has components

$$\Lambda_{ik}^{a} = \left\{ {a \atop ik} \right\} + p_i\delta_k^a - p^a g_{ik} - q_kF_i^a. \tag{12}$$

The connection ∇ is an F-connection, that means that $\nabla F = 0$. Then

$$q_j = -\frac{n}{2}p_aF_j^a,\ p_aF_j^a = -\frac{2}{n}q_j. \tag{13}$$

Then we can denote

$$\Lambda_{ik}^{a} = \Gamma_{ik}^{a} - q_kF_i^a, \tag{14}$$

where Γ_{ik}^{a} is a component of Riemannian part of metric semi-symmetric F-connection, which is itself a component of a metric semi-symmetric connection on the adjoint pseudo-Riemannian space, satisfying conditions of the **Theorem 1.**

Now we can calculate the coefficients of curvature tensor of connection (12)

$$\bar{R}_{ijkl} = R_{ijkl} - F_{ji}(\overset{\circ}{\nabla}_l\, q_k - \overset{\circ}{\nabla}_k\, q_l) +$$

$$+q_k(p_jF_{li} + \frac{2}{n}q_jg_{li} + \frac{2}{n}q_ig_{lj} + p_iF_{jl}) -$$

$$-q_l(p_jF_{ki} + \frac{2}{n}q_jg_{ki} + \frac{2}{n}g_{kj}q_i) + p_iF_{jk}.$$

By R_{ijkl} we denote a component of curvature tensor of metric semi-symmetric connection, satisfying conditions of **Theorem 1.**

One can easily notify that the tensor $\bar{R}_{ijkl}$ is skew-symmetric in first two indices.

$\bar{R}_{ijkl}$ is invariant under changing places of first and second pair of indices if and only if the tensor $p_l q_k + q_l p_k$ is skew-symmetric. Then

$$p_k p^k q_l = -p^k q_k p_l.$$

As the vectors p^k and q^k are mutually orthogonal, there yields $p_k p^k = 0$. This means that the generator of metric semi-symmetric connection, that is, the Riemannian part of metric semi-symmetric F-connection is an isotropic gradient, which is in accordance with the statement of **Theorem 1.**Then the vector q_k is also an isotropic vector.

The autoparallel lines of the connection (12) are geodesic lines of the adjoint pseudo-Riemannian space if and only if the condition

$$p_j \delta^i_k v^j v^k - p^i q_j v^j v^k - q_k F^i_j v^j v^k = f v^i, \tag{15}$$

where v^i is the tangent vector field of a geodesic line and f stands for $\frac{2}{n} p_j v^j$. Then

$$(p_j v^j - v_j v^j - f) v^i = -q_k v^k u^i, \tag{16}$$

where

$$u^i = -F^{ji} v_j = F^{ij} v_j.$$

Then, from (16),

$$u^i = \alpha v^i, \tag{17}$$

or

$$q_k v^k = 0.$$

If (16) holds, then the tangent vector field is eigen for the structure, for one of its eigen values, 1 or -1. Then v_k is a self-orthogonal, or isotropic vector field. The scalar product $q_k v^k$ then equals to

$$\frac{n-2}{n} p_j v^j - v_j v^j = \frac{n-2}{n} p_j v^j.$$

If (17) holds, we express the vectors in the adapted basis:

$$p = p^a l_a + p^{\hat{b}} l_{\hat{b}}, \tag{18}$$

where $l_{\hat{b}}$ are also eigen vectors, for the eigen value -1. Then

$$q = -\frac{2}{n} p^a l_a + \frac{2}{n} p^{\hat{b}} l_{\hat{b}}$$

and

$$v = v^a l_a + v^{\hat{b}} l_{\hat{b}}.$$

Then (17) gives

$$q_k v^k = \frac{2}{n}(p^{\hat{b}} v^a - p^a v^{\hat{b}}) g_{a\hat{b}} = 0; \tag{19}$$

it is satisfied if and only if v is proportional to p. Anyway, the tangent vector field of the geodesic–autoparallel line is isotropic.

As for the hyperbolic Kaehlerian space the generating vector field of the metric semi-symmetric F-connection having some of its autoparallel lines in common with Levi-Civita connection is isotropic, then the tensor (6) looks this way

$$p_{kj} = \overset{\circ}{\nabla}_k p_j - p_k p_j \tag{20}$$

and

$$\overset{\circ}{\nabla}_s p^s = p^s_s = \frac{K - R}{2(n-1)}. \tag{21}$$

Contracting the tensor $\overset{\circ}{\nabla}_k q_l - \overset{\circ}{\nabla}_l q_k$ with the tensor F^l_b, we obtain

$$-\frac{2}{n} \overset{\circ}{\nabla}_k p_b + \frac{2}{n} F^l_b F^a_k \overset{\circ}{\nabla}_l p_a = -\frac{n}{2} \overset{\circ}{\nabla}_k p_b + F^l_b \overset{\circ}{\nabla}_l q_k$$

$$\frac{n-4}{2n} \overset{\circ}{\nabla}_k p_b = F^l_b (\overset{\circ}{\nabla}_l q_k - \frac{2}{n} F^a_k \overset{\circ}{\nabla}_l p_a).$$

Contracting the last relation with g^{kb}, we obtain

$$\frac{n-4}{2n} \overset{\circ}{\nabla}_a p^a = -\frac{2}{n} \overset{\circ}{\nabla}_a p^a + \frac{2}{n} p_a p^a = 0.$$

If $n > 4$, then

$$\overset{\circ}{\nabla}_a p^a = 0. \tag{22}$$

Then, by (21), $K = R$.

However, from (14) and the form of curvature tensor, we can easily obtain that

$$\bar{R} = R + F^{lk} (\overset{\circ}{\nabla}_l q_k - \overset{\circ}{\nabla}_k q_l) \tag{23}$$

or

$$\bar{R} = R + \frac{2}{n} (\overset{\circ}{\nabla}_l p^l + \overset{\circ}{\nabla}_l p^l) = R + \frac{4}{n} \overset{\circ}{\nabla}_l p^l.$$

If we wanted our curvature tensor to satisfy the first Bianchi identity, using expression for the curvature tensor, we also obtain $\overset{\circ}{\nabla}_a p^a = 0$.

We proved

Theorem 2 *The curvature tensor of a metric semi-symmetric F-connection on the hyperbolic Kaehlerian space is invariant under changing places of first and second pair of indices and satisfies the first Bianchi identity if and only if the generators of the connection are isotropic and* $\overset{\circ}{\nabla}_a p^a = 0$. *Then the all geodesic lines whose tangent vectors are proportional to the generators or eigen for the structure are autoparallel for the metric semi-symmetric F-connection, and conversely. Also, under this condition, the curvature scalars of Levi-Civita connection, the Riemannian part of metric semi-symmetric connection and metric semi-symmetric F-connection are mutually equal.*

References

[1] S. I. Goldberg, Note on projectively Euclidean Hermitian spaces, Proc. Nat. Acad. Sci. U. S. A. (1956), 128–130.

[2] A. Gray, Pseudo-Riemannian almost product Manifolds and submersions, Mich. Math. J. 16 (1967) 715–737.

[3] H. Mizusawa and S. Koto, Holomorphically projective curvature tensors in certain almost Kaehlerian spaces, J. Fac. Sci. Niigata Univ. 2 (1960) 33–43.

[4] M. Prvanović and N. Pušić, On Manifolds admitting some semi-symmetric metric connection, Indian Journal of Math. 37, 1 (1995) 37–67.