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# ON GRADED ALGEBRA BUNDLES *

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**Abstract**

The aim of this paper is to study the associative graded algebra bundles which have the fibre a real or complex associative graded algebra. Some examples are considered. We define an analogous of the tangent bundle and the corresponding vector fields associated with.

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## 1 Vector bundles over associative graduate algebras

In the sequel we consider associative graduate **k**-algebras (*a.g.a.*) $\mathcal{A} = \{\mathcal{A}^n\}_{n\in\mathbf{Z}}$ which are finite dimensional as vector spaces over the field **k**, which is $I\!R$ or $\mathbf{C}$, and have an unity. Denoting by "$\wedge$" the product in $\mathcal{A}$, we say that it is:

1. *commutative*, if $a^p \wedge b^q = (-1)^{pq} b^q \wedge a^p$, for every $a^p \in \mathcal{A}^p$ and $b^q \in \mathcal{A}^q$, and

2. *anticomutative*, if $a^p \wedge b^q = (-1)^{pq+1} b^q \wedge a^p$, for every $a^p \in \mathcal{A}^p$ and $b^q \in \mathcal{A}^q$.

An element $a^p \in \mathcal{A}^p$ is called *homogeneous* of degree $p$.

The *center* of $\mathcal{A}$ is denoted as $\mathbf{Z}(\mathcal{A}) = \bigoplus_{p\in\mathbf{Z}} \mathbf{Z}^p(\mathcal{A})$, where $\mathbf{Z}^p(\mathcal{A}) = \{a^p \in \mathcal{A}^p \mid a^p \wedge b^q = (-1)^{pq} b^q \wedge a^p, (\forall) b^q \in \mathcal{A}^q\}$. We say that $\mathcal{A}$ is a *commutative a.g.a. (c.a.g.a.)* if $\mathcal{A} = \mathbf{Z}(\mathcal{A})$.

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It is obvious that if $\mathcal{A}$ is an a.g.a., then $\mathbf{Z}(\mathcal{A})$ is a c.a.g.a.

For every $r \in \mathbf{Z}$, we denote by $\mathcal{D}er^r(\mathcal{A})$ the set of $r$-derivation of the a.g.a. $\mathcal{A}$, i.e. the $\mathbf{k}$-linear maps $D : \mathcal{A} \to \mathcal{A}$ which satisfy $D(a^p \wedge b^q) = D(a^p) \wedge b^q + (+1)^{rp} a^p \wedge D(b^q)$. In the sequel we consider morphisms of a.g.a. $f : \mathcal{A}' \to \mathcal{A}$ which preserve the center, i.e. $f(\mathbf{Z}(\mathcal{A}')) \subset \mathbf{Z}(\mathcal{A})$. Defining the commutator of two derivations $D^r \in \mathcal{D}er^r(\mathcal{A})$ and $D^s \in \mathcal{D}er^s(\mathcal{A})$ as $[D^r, D^s] = D^r \circ D^s - (-1)^{rs} D^s \circ D^r$, then $\mathcal{D}er(\mathcal{A}) = \sum_{n=-\infty}^{n=+\infty} \mathcal{D}er^r(\mathcal{A})$ is a graded Lie super-algebra, i.e. $[D^r, D^s] \in \mathcal{D}er^{r+s}(\mathcal{A})$, $[D^r, D^s] = (-1)^{rs}[D^s, D^r]$ and $(-1)^{rt}[[D^r, D^s], D^t] + (-1)^{sr}[[D^s, D^t], D^r] + (-1)^{ts}[[D^t, D^r], D^s] = 0$.

If $X \in \mathcal{D}er(\mathcal{A})$ and $a \in \mathcal{A}$ we use the notations $X(a) = [X, a]$. We use also the notations $\mathcal{A}ut(\mathcal{A})$ for the Lie group of the automorphisms of the algebra $\mathcal{A}$ and $\mathcal{E}nd(\mathcal{A})$ for the a.g.a. of the endomorphisms on $\mathcal{A}$.

**Example 1** Every associative algebra $\mathcal{A}$ can be considered as a graded algebra $\mathcal{A}^*$ taking $\mathcal{A}^0 = \mathcal{A}$ şi $\mathcal{A}^p = \{0\}$, $p > 0$. We call this graded algebra as a *null graded algebra*. If we denote by $\mathbf{Z}(\mathcal{A}) = \{z \in \mathcal{A} \mid za = az, (\forall)a \in \mathcal{A}\}$ and $\mathcal{D}er(\mathcal{A}) = \{D : \mathcal{A} \to \mathcal{A} \mid D \textit{ is } \mathbf{k}\textit{-linear and } D(a \cdot b) = D(a) \cdot b + a \cdot D(b)\}$ the center and the derivations of $\mathcal{A}$ we have $\mathbf{Z}^*(\mathcal{A}^*) = \mathbf{Z}^0(\mathcal{A}^0) = \mathbf{Z}(\mathcal{A})$ *and* $\mathcal{D}er^*(\mathcal{A}^*) = \mathcal{D}er^0(\mathcal{A}^0) = \mathcal{D}er(\mathcal{A})$.

**Example 2** The most known example of a c.a.g.a. is the *Grassmann algebra* of a vector space. If $V$ is a finite dimensional $\mathbf{k}$-vector space and $\{e_1, \ldots e_n\} \subset V$ is a base, then the Grassmann algebra $\Lambda = \Lambda_*(V)$ is the smallest commutative graded algebra which satisfy the condition that the elements of $V$ have the degree 1. The algebra $\Lambda$ is generated by $1 \in I\!\!R$ and the base $\{e_1, \ldots e_n\}$: For $p < 0$ and $p > n$ we have $\Lambda_p = \{0\}$, $\Lambda_0 = \mathbf{k}$, $\Lambda_1 = V$, and for $1 \le p \le n$ we have

$$\{e_{i_1} \wedge \cdots \wedge e_{i_p}\}_{1 \le i_1 < \cdots < i_p \le n}$$

as a base in $\Lambda_p$. So, $\Lambda_p$ has the dimension $C_n^p$, and $\Lambda^*(V)$ has the dimension $2^n$.

**Example 3** For $V$ as above, one can consider the Grassmann algebra $\Lambda^*(V) = \Lambda_*(V^*)$. For $p < 0$ and $p > n$ we have $\Lambda^p = \{0\}$, $\Lambda_0 = \mathbf{k}$, $\Lambda^1 = V^*$, and if $\{e^1, \ldots e^n\}$ is a base in $V^*$, for $1 \le p \le n$ we have the base

$$\left\{e^{i_1} \wedge \cdots \wedge e^{i_p}\right\}_{1 \le i_1 < \cdots < i_p \le n}$$

in $\Lambda^p$ . As above, $\Lambda^p$ has dimension $C_n^p$ and $\Lambda^*(V)$ has dimension $2^n$. One can define a duality between $\Lambda^*(V)$ and $\Lambda_*(V)$, thus $\Lambda^*(V) = (\Lambda_*(V))^*$.

In the sequel all differentiable manifolds are in the class $C^1$ at least.

**Definition 1** Let $\xi = (E, \pi, M)$ be a $\mathbf{k}$-vector bundle which has the fibre type $\mathcal{A}$. We say that $\xi$ is an *associative graded algebra bundle* (a.g.a.b.) if $\mathcal{A}$ is a $\mathbf{k}$-a.g.a. such that there is a cover $\{U_i\}_{i\in I}$ of the base $M$ and a cocycle system $\{g_{ij}\}$ which has the property that if $U_i \cap U_j \neq \emptyset$, then $g_{ij} : U_i \cap U_j \to Aut(\mathcal{A})$.

We denote as $End(\xi)$ the a.g.a.b. of the endomorphisms of the a.g.a.b. $\xi$. It has the fibre type the a.g.a. $\mathcal{E}nd(\mathcal{A})$.

Let $\xi' = (E', \pi', M')$ be an other a.g.a.b., $f : M' \to M$, and $f : \xi' \to \xi$ be a morphism of vector bundles which induces on each fibre a morphism of a.g.a.. Then we say that $f$ is a *morphism of associative graded algebra bundle (m.a.g.a.b.)*.

If $\xi' \xrightarrow{i} \xi$ is the inclusion morphism of a vector subbundle, such that a subalgebra structure is induced on each fibre, then we say that $\xi'$ is an *subbundle of the a.g.a.b.* $\xi$ (a.g.a.s.b.).

For every a.g.a.b. $\xi = (E, \pi, M)$ as above, we have:

1. Each fibre $E_x = \pi^{-1}(x)$, $(\forall) x \in M$, has an intrinsic $\mathbf{k}$-algebra structure, such that for every local vector chart $(U, \varphi)$, $x \in U$, $\varphi : \pi^{-1}(U) \to U \times \mathcal{A}$, if we denote as $t_x : \mathcal{A} \to \pi^{-1}(x)$, defined by $t_x(a) = \varphi^{-1}(x, a)$, then $t_x$ is an isomorphism of $\mathbf{k}$-algebras;

2. The set of sections $\Gamma(\xi)$ has an associative graded $\mathbf{k}$-algebra structure;

3. It has as a.g.a.s.b. $\mathbf{Z}(\xi) = (\mathbf{Z}(E), \pi_Z, M)$, where $\mathbf{Z}_x(E) = \mathbf{Z}(E_x)$, $(\forall) x \in M$. We call the a.g.a.b. $\mathbf{Z}(\xi)$ the *center* of $\xi$. It has as the fibre type the c.a.g.a. $\mathbf{Z}(\mathcal{A})$. If $\mathcal{A}$ is commutative, then $\xi = \mathbf{Z}(\xi)$.

**Example 4** *Examples of associative algebra bundles (null graded algebras in our setting), can be found in [6].*

**Example 5** *As non-trivial a.g.a.b.s are the Grassmann vector bundles $\Lambda_*(\xi)$ şi $\Lambda^*(\xi)$ associated with a vector bundle $\xi$. If the fibre type of the vector bundle $\xi$ is the vector space $V$, then the c.a.g.a. $\Lambda_*(V)$ is the fibre type of the c.a.g.a.b. $\Lambda_*(\xi)$, and the c.a.g.a. $\Lambda^*(V)$ is the fibre type of the c.a.g.a.b. $\Lambda^*(\xi)$.*

A lot of results proved in that follows are valid when the fibre type is an infinite dimensional vector space. In order to avoid any complications, we deal only with finite dimensional algebras.

## 2 The tangent space in a point of the base

If $\xi = (E, \pi, M)$ is an a.g.a.b. and $x_0 \in M$, then an equivalence relation can be defined on the local sections defined around $x_0$, using the relation $s_1 \overset{x_0}{\sim} s_2$ iff there is an open set $U \ni x_0$, such that $s_{1|U} = s_{2|U}$. An equivalence class $[s]$ is called a germ and $s(x_0) \in E_{x_0}$ can be associated with. The set of germs $\tilde{E}_{x_0} = \Gamma / \overset{x_0}{\sim}$ is an a.g.a.b..

**Definition 2** If $\xi = (E, \pi, M)$ is an a.g.a.b. and $x_0 \in M$, then we say that $\tilde{E}_{x_0}$ is the *germ algebra* in $x_0$.

An $x_0$-*derivation* on $\tilde{E}_{x_0}$ of degree $r \in \mathbf{Z}$ is a map $D : \tilde{E}_{x_0} \to E_{x_0}$ which has the properties:

1. $D$ is a $\mathbf{k}$-linear map and if $s$ is homogenous of degree $n$, then $D([s])$ is homogenous and has the degree $n + r$;
2. $D([s] \cdot [t]) = D([s]) \cdot t(x_0) + (-1)^{r \cdot \deg[s]} s(x_0) \cdot D([t])$, where $[s]$ and $[t]$ are homogenous, and $\deg[s]$ is the degree of $[s]$.

We denote as $\mathcal{D}^1_r(x_0, \xi)$ the vector space of $x_0$-derivations of degree $r$ in $x_0$ and we call it as the *tangent space of degree* $r$ in $x_0$ to $\xi$.

We denote as $\mathcal{D}^1(x_0, \xi) = \bigoplus_{r \in \mathbf{Z}} \mathcal{D}^1_r(x_0, \xi)$ and we call it as the *tangent space* in $x_0$ to $\xi$.

We give now two examples of derivations, which are involved in the structure of the tangent space.

1. Let $(U, \varphi)$ be a vector chart, $x_0 \in U$. For every $x \in U$, we consider the canonical isomorphism of a.g.a. $t_x : \mathcal{A} \to E_x$, $t_x(a) = \varphi^{-1}(x, a)$. If $D^0 \in \mathcal{D}er^r(\mathcal{A})$ is an $r$-degree derivation on $\mathcal{A}$, then the formula:

$$D^0_{x_0}([s]) = (t_x \circ D^0(t_x^{-1} \circ s \circ t_x) \circ t_x^{-1})(x_0) \tag{1}$$

defines $D^0_{x_0} \in \mathcal{D}^1_r(x_0, \xi)$. We denote by $\widetilde{\mathcal{D}}^1_r(x_0, \xi)$ the vector subspace of $\mathcal{D}^1_r(x_0, \xi)$, which consists of the $x_0$-derivations which have this form.

If $D \in \mathcal{D}^1_r(x_0, \xi)$, then if we define $D^0_{x_0}$ using the formula:

$$D^0_{x_0}([s]) = t_{x_0}^{-1} \circ D\left([s(x_0)]\right),$$

we have that $D^0_{x_0} \in \widetilde{\mathcal{D}}^1_r(x_0, \xi)$, is just that which correspond to the derivation $D^0 \in \mathcal{D}er^r(\mathcal{A})$, given by the formula:

$$D^0(a) = t_{x_0}^{-1}\left(D\left(\left[t_{x_0}(a)\right]\right)\right). \tag{2}$$

It follows that the correspondence $D^0 \longleftrightarrow D^0_{x_0}$ is an isomorphism of the $\mathbf{k}$-vector spaces $\mathcal{D}er^r(\mathcal{A})$ and $\widetilde{\mathcal{D}}^1_r(x_0,\xi)$. The correspondence $D \to D^0_{x_0}$ is a surjective morphism of the vector spaces $\mathcal{D}^1_r(x_0,\xi) \to \widetilde{\mathcal{D}}^1_r(x_0,\xi)$.

2. Let $(U,\varphi)$, $x, x_0 \in U$ and $t_x$ be as above such that there is a local chart $\psi : U \to \psi(U) \subset I\!R^m$ on $M$ which has $(x^i)_{i\in\overline{1,m}}$ as coordinate, $m = \dim M$. Consider the formula:

$$\left(\frac{\tilde{\partial}}{\partial x^i}\right)_{x_0}([s]) = \left(t_{x_0}\left(\left(\frac{\partial}{\partial x^i}\right)_{x_0}\left(t_x^{-1}(s_x)\right)\right)\right),$$

for $i \in \overline{1,m}$, where $\left(\frac{\partial}{\partial x^i}\right)_{x_0}$ is the partial differentiation operator defined locally according to the local chart $(U,\psi)$, which applies to a function with values in a vector space. It defines $\left(\frac{\tilde{\partial}}{\partial x^i}\right)_{x_0} \in \mathcal{D}^1_0(x_0,\xi)$ (of zero degree). If $a \in \mathbf{Z}(\mathcal{A})$, then the formula $\left(a\cdot\left(\frac{\tilde{\partial}}{\partial x^i}\right)_{x_0}\right)([s]) = a\cdot\left(\frac{\tilde{\partial}}{\partial x^i}\right)_{x_0}([s])$ defines the derivation $a\cdot\left(\frac{\tilde{\partial}}{\partial x^i}\right)_{x_0} \in \mathcal{D}^1_0(x_0,\xi)$, which we can associate with $a\otimes_{I\!R}\left(\frac{\partial}{\partial x^i}\right)_{x_0} \in Z(E_{x_0})\otimes_{I\!R} T_{x_0}M$.

**Proposition 1** *Let $\xi$ be an a.g.a.b. which has as fibre type the a.g.a. $\mathcal{A}$, $r \in \mathbf{Z}$ and $x_0 \in M$. Then there is a short exact sequence of vector spaces:*

$$0 \to \widetilde{\mathcal{D}}^1_r(x_0,\xi) \xrightarrow{i} \mathcal{D}^1_r(x_0,\xi) \xrightarrow{\Pi} \mathbf{Z}(E_{x_0})\otimes_{I\!R} T_{x_0}M \to 0. \tag{3}$$

*If $(U,\varphi)$ is a vector chart on $E$ and $(U,\psi)$ is a local chart on the base $M$, $x_0 \in U$, then there are isomorphisms $\mathcal{D}^1_r(x_0,\xi) \cong \mathcal{D}er^r(\mathcal{A})$ and $\mathbf{Z}(E_{x_0})\otimes_{I\!R} T_{x_0}M \simeq \mathbf{Z}(\mathcal{A})\otimes_{I\!R} I\!R^m$. Every splitting of the short exact sequence (3) defines an isomorphism $\mathcal{D}^1_r(x_0,\xi) \simeq \mathcal{D}er^r(\mathcal{A}) \oplus (\mathbf{Z}(\mathcal{A})\otimes_{I\!R} I\!R^m)$.*

The proof of the Proposition uses similar computations as in the case of a null a.g.a.b. (see [6]), using especially the formula:

$$D([s]) = D\left(\left[t_x\left(t_{x_0}^{-1}(s(x_0))\right)\right]\right) + D\left(\left[x^i e_x\right]\right)\left(\frac{\tilde{\partial}}{\partial x^i}\right)_{x_0}([s]). \tag{4}$$

If we consider a local change of vector charts from $(U,\varphi)$ to $(U',\varphi')$ and we denote as $\bar{t}_x : \mathcal{A} \to E_x$ and $\left(\frac{\tilde{\partial}'}{\partial x^i}\right)_{x_0}$ for the chart $(U',\varphi')$, then the following formula holds:

$$\left(\frac{\tilde{\partial}'}{\partial x^i}\right)_{x_0}([s]) = \bar{t}_{x_0}\left(\left(\frac{\partial}{\partial x^i}\right)_{x_0}\left(\bar{t}_x^{-1} \circ t_x^{-1}\left(t_{x_0}^{-1}\left(s\left(x_0\right)\right)\right)\right)\right) + \left(\frac{\tilde{\partial}}{\partial x^i}\right)_{x_0}([s]). \tag{5}$$

A local change of the local chart $(U,\psi)$ in the chart $(U,\psi')$, leaving the local vector chart $(U,\varphi)$ unchanged, has as effect the formula:

$$\frac{\tilde{\partial}}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\tilde{\partial}}{\partial x^{i'}}. \tag{6}$$

**Definition 3** Let $\xi' = (E',\pi',M')$ and $\xi = (E,\pi,M)$ be a.g.a.b.s which have as fibre type the a.g.a.s $\mathcal{A}'$ and $\mathcal{A}$ respectively and $f_0 : M \to M'$. An *$f_0$-comorphism of associative graded algebra bundle* ($f_0$-c.m.a.g.a.b.) is a comorphism of vector bundle $f : \xi' \to \xi$ which induces a morphism of a.g.a. $f_x : E'_{f_0(x)} = \pi'^{-1}(f_0(x)) \to \pi^{-1}(x) = E_x$, $(\forall)x \in M$.

An $f_0$-c.m.a.g.a.b. $f : \xi' \to \xi$ defines a morphism of a.g.a. $f_* : \Gamma(\xi') \to \Gamma(\xi')$, where $f_*(s') = f \circ s' \circ f_0$.

**Example 6** Let $f_0 : M \to M'$ be a differentiable map, $\xi' = (E',\pi',M')$ and $\xi = (E,\pi,M)$ be two vector bundles and $\xi \stackrel{(f_0,f)}{\to} \xi'$ be a morphism of vector bundle. Then the $f_0$-comorphism of vector bundle $f^* : \Lambda^*(\xi') \to \Lambda^*(\xi)$ is a $f_0$-c.m.a.g.a.b. and the $f_0$-morphism of vector bundle $f_* : \Lambda_*(\xi') \to \Lambda_*(\xi)$ is a *$f_0$-m.a.g.a.b.*.

**Definition 4** Let $f : \xi' \to \xi$ be an $f_0$-c.m.a.g.a.b. and $g : \xi \to \xi'$ be an $f_0$-morphism of vector bundles which is a left inverse for $f$ on each fibre, i.e. $g_x \circ f_x = id_{E_x}$, $(\forall)x \in M$. Then we say that $f$ is an $f_0$-comorphism of associative graded algebra bundle which splits (*$f_0$-c.m.f.a.g.a.b.s.*). If the condition $g_x(f_x(u) \cdot v) = u \cdot g_x(v)$, $g_x(v \cdot f_x(u)) = g_x(v) \cdot u \quad (\forall)x \in M$, $u \in E_x$, $v \in E'_{f_0(x)}$, is satisfied, then we say that $g$ is a *regular splitting* or that $f$ splits *regularly* .

If $f : \xi' \to \xi$ is an $f_0$-c.m.a.g.a.b., then we say that it is a *regular comorphism* if it enjoys the property that for every $X \in \mathcal{D}er(\Gamma(\xi))$, $s' \in \Gamma(\xi')$ *and* $x \in M$ *there is* $a \in E$ *such that* $X(f \circ s' \circ f_0)(x) = f(a)$.

We list now some important particular cases:

1. If $M = M'$ and $f_0 = id_M$, then $f$ is a m.a.g.a.b..

2. If the $f_0$-comorphism $f$ is an isomorphism on fibres, one can take $g_x = f_x^{-1}$. It follows that $f$ is a regular $f_0$-comorphism and it has $g$ as a regular splitting.

3. Let $\mathcal{A}$ be an associative graded **k**-algebra, $\xi$ and $\xi'$ be the trivial a.g.a.b.s which have as total spaces $M \times \mathcal{A}$ and $M' \times \mathcal{A}$ respectively and the bases $M$ and $M'$ respectively. Taking the $f_0$-morphism $f : \xi \to \xi'$ and the $f_0$-comorphism $g : \xi' \to \xi$ which are the identities of the fibres, it follows that $f$ is a regular $f_0$-comorphism and it has $g$ as a regular splitting. Moreover, the associations $(M \to \xi)$ and $(f_0 \to f)$ is functorial from the category of differentiable manifolds in the category of a.g.a.b.s. In the particular cases when $\mathcal{A}=I\!R$ or $\mathcal{A} = \mathbf{C}$, on obtain the usual construction of real or complex algebra of functions on a real or on a complex manifold.

4. If $f_0 : M \to M'$ is a differentiable map, $\xi = (E, \pi, M)$ and $\xi' = (E', \pi', M')$ are vector bundles and $f : \xi \to \xi'$ is an $f_0$-morphism of vector bundle which has the $f_0$-comorphism $g : \xi' \to \xi$ as a right splitting, then $f^* : \Lambda^*(\xi') \to \Lambda^*(\xi)$ is an $f_0$-c.m.a.g.a.b. which has the $f_0$-morphism of vector bundles $g^* : \Lambda^*(\xi) \to \Lambda^*(\xi')$ as a regular splitting.

**Proposition 2** *Let $\xi' = (E', \pi', M')$ and $\xi = (E, \pi, M)$ be two a.g.a.b.s, $f_0 : M \to M'$ and $f : \xi' \to \xi$ be an $f_0$-c.m.a.g.a.b. which has the $f_0$-morphism of vector bundles $g : \xi \to \xi'$ as a left splitting.*

*If $g$ is a regular splitting or if $f$ is a regular comorphism, then the formula:*

$$\tau_{x_0}(D)([s']) = g \circ D\left([f \circ s' \circ f_0]\right) \tag{7}$$

*defines a linear map $\tau_{x_0} g = \tau_{x_0} : D_r^1(x_0, \xi) \to D_r^1(f(x_0), \xi')$ which enjoys the property $\tau_{x_0}\left(\widetilde{\mathcal{D}}_r^1(x_0, \xi)\right) \subset \widetilde{\mathcal{D}}_r^1(f(x_0), \xi')$.*

The proof is a straightforward computation.

# 3 Generalized vector fields

If $\xi$ is an a.g.a.b. then $\Gamma(\xi)$ is an a.g.a., which has the center $Z(\Gamma(\xi)) = \Gamma(Z(\xi))$. The *generalized vector fields* are the elements of the graded $Z(\Gamma(\xi))$-module $\mathcal{D}er(\Gamma(\xi)) = \bigoplus_{r \in \mathbf{Z}} \mathcal{D}er^r(\Gamma(\xi))$, where $\mathcal{D}er^r(\Gamma(\xi))$ is the set of the

$r$-derivations. The graduation means that for $a^p \in Z^p(\Gamma(\xi))$ and $D^r \in \mathcal{D}er^r(\Gamma(\xi))$, then $a^p \cdot D^r \in \mathcal{D}er^{p+r}(\Gamma(\xi))$.

In the sequel the differentiability class $C^\infty$ is assumed for of all differentiable manifolds and vector bundles.

**Proposition 3** *If $X \in \mathcal{D}er^r(\Gamma(\xi))$ and $x_0 \in M$, then $X_{x_0} : \tilde{E}_{x_0} \to E_{x_0}$ given by the formula $X_{x_0}([s]) = X(s)(x_0)$, defines an $r$-degree $x_0$-derivation.*

*The map $X \to X_{x_0}$ is a surjective morphism of* $\mathbf{k}$*-vector spaces $\mathcal{D}er^r(\Gamma(\xi)) \to \mathcal{D}^1_r(x_0, \xi)$.*

The Lie group of automorphism of the algebra $\mathcal{A}$, denoted as $\mathcal{A}ut(\mathcal{A})$, acts at left on $\mathcal{D}er(\mathcal{A})$ according to the formula:

$$\begin{array}{c} \mathcal{A}ut(\mathcal{A}) \times \mathcal{D}er(\mathcal{A}) \to \mathcal{D}er(\mathcal{A}) \\ (g, D) \to gDg^{-1}. \end{array} \tag{8}$$

Since the algebra $\mathcal{A}$ is finite dimensional, it follows that $\mathcal{D}er(\mathcal{A}) \subset \mathcal{E}nd(\mathcal{A})$ and $Z(\mathcal{A}) \subset \mathcal{A}$ are finite dimensional vector spaces.

**Proposition 4** *If $\xi$ is an a.g.a., then $\widetilde{\mathcal{D}}^1(E) = \bigcup_{x \in M} \widetilde{\mathcal{D}}^1(x, \xi)$, $D^1(E) = \bigcup_{x \in M} D^1(x, \xi)$ and $Z(E) = \bigcup_{x \in M} Z(E_x)$ are the total spaces of the vector bundles denoted as $\widetilde{\mathcal{D}}^1(\xi)$, $\mathcal{D}^1(\xi)$ and $Z(\xi)$ respectively, over the same base as $\xi$. The vector bundle $\widetilde{\mathcal{D}}^1(\xi)$ is a Lie super-algebra bundle which has the fibre type the Lie super-algebra $\mathcal{D}er(\mathcal{A})$ over* $\mathbf{k}$. *There is a short exact sequence of Lie super-algebra bundles over the base $M$:*

$$0 \to \widetilde{\mathcal{D}}^1(\xi) \xrightarrow{i} \mathcal{D}^1(\xi) \xrightarrow{\Pi} (Z(\xi) \otimes \tau(M)) \to 0. \tag{9}$$

**Corrolary 1** *Every splitting of the exact sequence of vector bundles (9) defines the canonical isomorphism of vector bundles:*

$$\mathcal{D}^1(\xi) \simeq \widetilde{\mathcal{D}}^1(\xi) \oplus (Z(\xi) \otimes \tau(M))$$

*and the isomorphism of $\Gamma(Z(\xi))$-module:*

$$\Gamma(D^1(\xi)) \cong \Gamma(\widetilde{\mathcal{D}}^1(\xi)) \oplus (\Gamma(Z(\xi) \otimes_{\mathcal{F}(M)} \mathcal{X}(M))), \tag{10}$$

*where $\mathcal{X}(M) = \Gamma(\tau M)$.*

The formula (4) extends to a formula which is valid for local fields defined on the domains of a vector chart $(U, \varphi)$ on $E$ and a local chart $(U, \psi)$ on $M$.

**Definition 5** Let $\xi$ be an a.g.a.b., $X \in \mathcal{D}er^r(\Gamma/\xi))$ *and* $Y \in \mathcal{D}er^s(\Gamma/\xi))$, $r, s \in \mathbf{Z}$. We define the *bracket* $[X, Y] \in \mathcal{D}er(\Gamma/\xi))$ as the commutator $[X, Y] = X \circ Y - (-1)^{rs} Y \circ X$.

If $X \in \mathcal{D}er^r(\Gamma(\xi))$ and $(U, \varphi)$, $(U, \psi)$ are a vector chart on $E$ and a local chart on $M$ respectively, then we call a *local decomposition* of $X$ in the local charts $(U, \varphi)$ and $(U, \psi)$ as the local decomposition $X = X_1 + X_2$ given by the formula (4).

**Proposition 5** *If* $X, Y \in \mathcal{D}er(\Gamma(\xi))$ *and* $X = X_1 + X_2$, $Y = Y_1 + Y_2$ *are their locally decompositions in the same local charts, then the commutator* $[X.Y]$ *admits the local decomposition* $[X.Y] = [X_1, Y_1] + [X_2, Y_2]$.

The proof is essentially the same as in the case of a non-graded algebra [6] and it suffices to show that $X = X_1$ and $Y = Y_2$, then $[X, Y] = 0$, which follows by a straightforward computation.

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