NOVI SAD J. MATH.
VOL. 29, NO. 3, 1999, 249-255
XII Yugoslav Geometric Seminar
Novi Sad, October 8-11, 1998

PROJECTABLE NONLINEAR CONNECTIONS *

Marcela Popescu **Paul Popescu**
Department of Mathematics, University of Craiova,
13, Al.I.Cuza st., Craiova 1100, Romania
Email address: paul@udjmath2.sfos.ro

Abstract

The definition of a projectable non-linear connection of a vector bundle on a subbundle is given. The relations with the left and right Finsler splittings, defined by the first author, is studied.

AMS Mathematics Subject Classification (1991): 53C60, 53C40.
Key words and phrases: left and right Finsler splittings, projectable connection.

Definition 1 Let $\xi' \xrightarrow{f} \xi$ be a morphism of vector bundles, where $\xi' = (E', \pi', M)$ and $\xi = (E, \pi, M)$ have the same base M. A *left* (*right*) *Finsler splitting* is a left (right respectively) splitting S of the induced morphism $(\pi')^*\xi' \xrightarrow{(\pi')^* f} (\pi')^*\xi$.

Proposition 1 *Let $\xi \xrightarrow{P''} \xi''$ be a epimorphism of vector bundles $\xi = (E, \pi, M)$. Consider also a non-linear connection C'' on ξ'' and a right splitting S of the epimorphism $\tau E \xrightarrow{\tau P''} (P'')^*\tau E''$ of vector bundles over the base E.*

Then there is a unique non-linear connection C on the vector bundle ξ which projects by $\tau P''$ the fibres of the horizontal bundle isomorphically on the fibres of the horizontal subbundle of C''. The splitting S induces a right Finsler splitting S'' of the epimorphism P''.

Proof. The vertical subbundle $V\xi''$ and the horizontal subbundle $H\xi''$ of the non-linear connection C'' are mapped by P'' in the subbundles $(P'')^*V\xi'' \overset{not.}{=} V''\xi''$ and respectively $(P'')^*H\xi'' \overset{not.}{=} H''\xi''$, of $(P'')^*\tau E''$. The dimension of the fibres of $H''\xi''$ is $\dim M$ and these fibres are mapped injectively by the morphism S, in the fibres of a vector subbundle $H\xi$, of τE. This

*Supported in part by the grant MEN-CNCSIS 196/1999

subbundle is a supplementary vector bundle to the vertical subbundle, since the subbundle $V\xi$ is mapped surjectively by $\tau P''$ on $V''\xi''$. It follows that the subbundle $H\xi$ is the horizontal subbundle of a non-linear connection C on ξ. The first assertion follows from the construction of the non-linear connection.

Taking into account of the canonical isomorphisms as $V\xi \cong \pi^*\xi$ and denoting as $S'' = S_{|(P'')^*VE''}$, which is on its turn a splitting, it follows a right Finsler splitting of the epimorphism P'', defined by: $(\pi)^*\xi'' \cong (P'')^*(\pi'')^*\xi'' \cong (P'')^*V\xi'' \xrightarrow{S''} V\xi \cong \pi^*\xi$, thus the second assertion follows.□

Let us make explicit the construction of the non-linear connection C.

The non-linear connection C'' is defined by the right splitting D'' of the epimorphism $\tau E'' \xrightarrow{\tau\pi''} (\pi'')^*\tau M$, thus $(\pi'')^*\tau M \xrightarrow{D''} \tau E''$ şi $\tau\pi'' \circ D'' = id$. The morphism of vector bundles

$$(P'')^*(\pi'')^*\tau M \xrightarrow{(P'')^*D''} (P'')^*\tau E''$$

is induced. According to the isomorphism $(P'')^*(\pi'')^*\tau M \cong (\pi'' \circ P'')^*\tau M = \pi^*\tau M$ it can be regarded as

$$\pi^*\tau M \xrightarrow{(P'')^*D''} (P'')^*\tau E''.$$

Let us denote as $D = S \circ (P'')^*D''$. We have, for every $(e, X) \in \pi^*\tau M$ (where $\pi(e) = p(X)$, where $p : TM \to M$) is the canonical projection): $\tau\pi \circ D(e, X) = \tau_e\pi'' \circ \tau P'' \circ S((P'')^*D''(e, X)) = \tau_e\pi''(P''(e), D''(P''(e), X)) = (e, X)$. Thus it follows that D is a right splitting for the epimorphism $\tau E \xrightarrow{\tau\pi''} (\pi'')^*\tau M$. Its construction make clear that it defines the non-linear connection given by Proposition 1.

Definition 2 If $\xi \xrightarrow{f} \xi''$ is an epimorphism of vector bundles, we say that a non-linear connection C on ξ is *projectable* on ξ'' if there exist a non-linear connection C'' on ξ'' and a right splitting of the induced morphism $\tau E'' \xrightarrow{(P'')^*f} (P'')^*\xi''$ which induce, according to Proposition 1, the non-linear connection C.

Remark 1 There is a vector bundle $\eta = (E, P'', E'')$, which has as vertical bundle the vector bundle $\ker \tau P'' \overset{not.}{=} \mathcal{V}'\xi$. Denote as $\xi' = \ker P''$. Considering an adapted system of vectorial coordinates, the structural functions of the vector bundles $\pi^*\xi'$, $\mathcal{V}'\xi$ and $(P'')^*\eta$ are the same, thus we have canonical isomorphisms $\mathcal{V}'\xi \cong (P'')^*\eta$ şi $\mathcal{V}'\xi \cong \pi^*\xi'$.

Remark 2 Denote as $\mathcal{V}'\xi \xrightarrow{I} \tau E$ the inclusion morphism. A right splitting S, given by Proposition 1, is equivalent with a left splitting T of the inclusion I.

If $\xi' = (E', \pi', M)$ is a vector subbundle of the vector bundle $\xi = (E, \pi, M)$ and we take as $\xi'' = \xi/\xi'$, we are in the above case, thus there is a canonical injective morphism $\pi^*\xi' \xrightarrow{i'} \tau E$.

Proposition 2 *Let $\xi' = (E', \pi', M)$ be a vector subbundle of the vector bundle $\xi = (E, \pi, M)$, T be a splitting of the canonical injective morphism $\pi^*\xi' \xrightarrow{i'} \tau E$, and C'' be a non-linear connection on the vector bundle $\xi'' = \xi/\xi'$.*

Then an unique non-linear connection C on ξ, projectable on ξ'', which has the horizontal bundle included in the kernel of the splitting T, can be induced.

Proof. Using the first part of Proposition 1 and the Remarks 1 and 2, the conclusion follows.□

Using the same notations, we have:

Proposition 3 *Let $\xi' = (E', \pi', M)$ be a vector subbundle of the vector bundle $\xi = (E, \pi, M)$, T be a left splitting of the canonical injective morphism $\pi^*\xi' \xrightarrow{i'} \tau E$. Then:*

a) An unique non-linear connection C' is induced, provided that the fibres of the horizontal bundle are included in the fibres of the kernel of T, in every point of E'.

b) A canonical Finsler left splitting T' of the inclusion $\xi' \xrightarrow{j'} \xi$ is induced.

Proof. a) The left splitting T of i', is uniquely defined by a right splitting S of $\tau P''$, i.e. by a non-linear connection in the vector bundle $\eta = (E, P'', E'')$. Using the notations in Remark 1, there is a reduction of τE as $\mathcal{V}'\xi \oplus \mathcal{H}'\xi$ thus for every $u \in E$ we have $T_u E = \mathcal{V}'_u E \oplus \mathcal{H}'_u E$. For every $u' \in E'$ we have $\mathcal{V}'_{u'} E = \tau i(V_{u'} E')$, but the intersection $\mathcal{H}'_{u'} E \cap T_{u'} E' \stackrel{not.}{=} \mathcal{H}E'_{u'}$ has a dimension $\dim(\mathcal{H}'_{u'} E) + \dim(T_{u'} E') - \dim(\mathcal{H}'_{u'} E + T_{u'} E') = k_1 + m + k_2 + m - (k_1 + k_2 + m) = m$ thus $\mathcal{H}E'$ defines a non-linear connection C' on ξ'.

We prove now b). The splitting T induces a splitting T'_1 of the inclusion $\mathcal{V}'\xi \xrightarrow{I'} V\xi$, defining the projection which define it by $\ker T'_{1,e} = V_e E \cap \mathcal{H}'_e E$, for every $e \in E$. Indeed $\dim(V_eE \cap \mathcal{H}'_eE) = \dim(V_eE) + \dim(\mathcal{H}'_eE) - \dim(V_eE \cup \mathcal{H}'_eE) = k_1 + k_1 - k_2 + m - (k_1 + m) = k_1 - k_2$ thus $V_eE = (V_eE \cap \mathcal{H}'_eE) \oplus \mathcal{V}'_eE$. Keeping notice of Remark 1 it follows that the splitting T'_1 induces a left splitting T'_2 of the injective morphism of vector bundles $\pi^*\xi' \xrightarrow{\pi^*j'} \pi^*\xi$. Restricting to ξ', a Finsler splitting T' of the inclusion $(\pi')^*\xi' \xrightarrow{(\pi')^*} (\pi')^*\xi$ is obtained.□

The results in Proposition 3 can be improved as follows:

Proposition 4 *let $\xi' = (E', \pi', M)$ be a vector subbundle of the vector bundle $\xi = (E, \pi, M)$, T be a left splitting of the canonical injective morphism $(\pi')^*\xi' \xrightarrow{i'} \tau E_{|E'}$. Then:*

a) A unique non-linear connection C' is induced, provided that the fibres of the horizontal bundle are included in the fibres of the kernel of T, in every point of E'.

b) A canonical Finsler left splitting T' of the inclusion $\xi' \xrightarrow{j'} \xi$ is induced.

c) A non-linear connection C' on ξ' and a Finsler left splitting T' of the inclusion $\xi' \xrightarrow{j'} \xi$ define together a left splitting T of the canonical injective morphism $(\pi')^*\xi' \xrightarrow{i'} \tau E_{|E'}$.

Proof. We repeat the proof of Proposition 3, taking vector bundles over the base E' instead of those of base E, and using analogous notations and conventions.

a) The left splitting T of i', is uniquely defined by a right splitting S of $\tau P''_{|E'}$. Let us denote as $\mathcal{H}'_0\xi = \ker S$ and $\mathcal{V}'_0\xi = \ker \tau P''_{|E'}$. There is a reduction of $\tau E_{|E'}$ as $\mathcal{V}'_0\xi \oplus \mathcal{H}'_0\xi$ thus for every $u \in E$ we have $T_uE = \mathcal{V}'_{0u}E \oplus \mathcal{H}'_{0u}E$. For every $u' \in E'$ we have $\mathcal{V}'_{u'}E = \tau i(V_{u'}E')$. The intersection $\mathcal{H}'_{0u'}E \cap T_{u'}E' \overset{not.}{=} \mathcal{H}E'_{u'}$ has the dimension $\dim(\mathcal{H}'_{0u'}E) + \dim(T_{u'}E') - \dim(\mathcal{H}'_{0u'}E + T_{u'}E') = k_1 + m + k_2 + m - (k_1 + k_2 + m) = m$ thus $\mathcal{H}E'$ defines a non-linear connection C' on ξ'.

We prove now b). The splitting T induces a splitting T'_1 of the inclusion $\mathcal{V}'_0\xi \xrightarrow{I'} V\xi_{|E'}$, defining the suitable projection by $\ker T'_{1,e'} = V_{e'}E \cap \mathcal{H}'_{e'}E$, for every $e' \in E'$. Indeed $\dim(V_{e'}E \cap \mathcal{H}'_{e'}E) = \dim(V_{e'}E) + \dim(\mathcal{H}'_{e'}E') - \dim(V_{e'}E \cup \mathcal{H}'_{e'}E) = k_1 + k_1 - k_2 + m - (k_1 + m) = k_1 - k_2$ thus $V_{e'}E = V_{e'}E \cap \mathcal{H}'_{e'}E \oplus \mathcal{V}'_{e'}E$. Keeping notice of the canonical isomorphisms $\mathcal{V}'_0\xi \cong (\pi')^*\xi'$ and $V\xi_{|E'} \cong (\pi')^*\xi$, (which can be obtained in a canonical adapted system of coordinates), the splitting T_1' induces a left splitting T_2' of the inclusion morphism $(\pi')^*\xi' \stackrel{(\pi')^*}{\longrightarrow} (\pi')^*\xi$, which is in fact the asked Finsler splitting.

In order to prove c), the kernel of the splitting T is defined by the Whitney sum of the kernels of the splitting C' and T'. □

We give now an explicit form of the constructions made above, using local coordinates, adapted to the vector bundle structures.

Consider some coordinates (x^i, y^α, y^u) on E:, (x^i, y^u) on E'', (x^i, y^α) on E', $(x^i, y^\alpha, y^u, X^j, Y^\beta, Y^v)$ on TE, (x^i, y^u, X^j, Y^v) on TE'', $(x^i, y^\alpha, y^u, Y^\beta)$ on $V'E$, around the corresponding points. The functions considered above have the local forms:

$$(x^i, y^\alpha, y^u) \stackrel{P''}{\to} (x^i, y^u), (x^i, y^\alpha, y^u, X^j, Y^\beta, Y^v) \stackrel{TP''}{\to} (x^i, y^u, X^j, Y^v),$$

$$(x^i, y^u, X^j) \stackrel{D''}{\to} (x^i, y^\alpha, X^j, -\tilde{N}_j^v(x^i, y^u)X^j),$$

$$(x^i, y^\alpha, y^u, X^j, Y^v) \stackrel{T}{\to}$$

$$(x^i, y^\alpha, y^u, X^j, -T_v^\alpha(x^i, y^\alpha, y^u)Y^v - T_j^\alpha(x^i, y^\alpha, y^u)X^j, Y^v),$$

thus

$$(x^i, y^\alpha, y^u, X^j, Y^v) \stackrel{D}{\to} (x^i, y^\alpha, y^u, X^j, T_v^\alpha(x^i, y^\alpha, y^u)$$
$$\tilde{N}_j^v(x^i, y^u))X^j - T_j^\alpha(x^i, y^\alpha, y^u)X^j, -\tilde{N}_j^v(x^i, y^u)X^j),$$

or, for the corresponding left splitting:

$$(x^i, y^\alpha, y^u, X^j, Y^\beta, Y^v) \stackrel{C}{\to} (x^i, y^\alpha, y^u, X^j, Y^\alpha - T_v^\alpha(x^i, y^\alpha, y^u)\cdot$$
$$\tilde{N}_j^v(x^i, y^u))X^j + T_j^\alpha(x^i, y^\alpha, y^u)X^j, Y^v + \tilde{N}_j^v(x^i, y^u)X^j).$$

It follows that the relations between the local components of the non-linear connections C and C'' and those of the splitting T are

$$N_j^\alpha(x^i, y^\alpha, y^u) = -T_v^\alpha(x^i, y^\alpha, y^u)\tilde{N}_j^v(x^i, y^u)) + T_j^\alpha(x^i, y^\alpha, y^u) \quad (1)$$

$$N_j^u(x^i, y^\alpha, y^u) = \tilde{N}_j^u(x^i, y^u) \quad (2)$$

Proposition 5 *Let $\xi \stackrel{P''}{\longrightarrow} \xi''$ be an epimorphism of vector bundles and C, C'' be non-linear connections on ξ, respectively on ξ''. Let us suppose that in an adapted system of coordinates, as above, the relations between the local components of the non-linear connections C and C'' there is a relation (2).*

Then the non-linear connection C is projectable on the non-linear connection C' by an arbitrary left splitting U of the inclusion $\mathcal{V}'\xi' \stackrel{I'}{\to} V\xi$ (this is equivalent with a left splitting of the induced injective morphism $\pi^*\xi' \stackrel{\pi^*j'}{\longrightarrow} \pi^*\xi$).

Proof. The local form of U is

$$(x^i, y^\alpha, y^u, Y^\beta, Y^v) \xrightarrow{U} (x^i, y^\alpha, y^u, Y^\beta + U^\beta_v(x^i, y^\alpha, y^u)Y^v).$$

The splitting U and the non-linear connection C define a left splitting T of the inclusion $\mathcal{V}'\xi' \xrightarrow{I} \tau E$, or, in an equivalent way, a right splitting S of the epimorphism $\tau E \xrightarrow{\tau P''} (P'')^*\tau E''$. According to Proposition 1, using the splitting S and non-linear connection C'' a non-linear connection $\bar{C}$ on ξ is induced. Let us show that in fact the $\bar{C}$ is the same as C.

Indeed, the local form of the splitting T is:

$$(x^i, y^\alpha, y^u, X^j, Y^\beta, Y^v) \xrightarrow{S} (x^i, y^\alpha, y^u, Y^\beta + N^\beta_j X^j + U^\beta_v(Y^v + N^v_j X^j)).$$

It follows that $T^\beta_j = -N^\beta_j - U^\beta_v N^v_j$ şi $T^\beta_v = U^\beta_v$. Using the formulas (1) and (2) for the components of the induced connection $\bar{C}$, we have

$$\overline{N}^\alpha_j(x^i, y^\alpha, y^u) = -U^\alpha_v \tilde{N}^v_j + N^\alpha_j + U^\alpha_v N^v_j = N^\alpha_j(x^i, y^\alpha, y^u).$$

The non-linear connection C' on ξ', given by Proposition 3 a) has the local form

$$(x^i, y^\alpha, X^j, Y^\beta) \xrightarrow{S} (x^i, y^\alpha, Y^\alpha + T^\alpha_j(x^i, y^\alpha, 0)X^j).$$

The splitting T' given by Proposition 3 b) has a local form

$$(x^i, y^\alpha, Y^\alpha, Y^u) \xrightarrow{T'} (x^i, y^\alpha, Y^\alpha + T^\alpha_v(x^i, y^\alpha, 0)Y^v). \square$$

References

[1] Miron R., Anastasiei M. Vector bundles. Lagrange spaces. Applications to the theory of relativity, Ed. Academiei, Bucuresti, 1987.

[2] Miron R., Anastasiei M. The Geometrie of Lagrange Spaces: Theory and Applications, Kluwer Acad. Publ., 1994.

[3] Popescu Marcela, Connections on Finsler bundles, An. St. Univ. "Ovidius" Constanţa, Ser. mat., **III**:2 (1995), 97–101;

[4] Popescu Marcela, On Induced Non-linear Connections, An. Univ. "Al.I.Cuza" Iaşi, Ser. Mat., **XLII**:1, (1996), 119–126.

[5] Popescu Marcela, On vectorial Finsler connections, Proc. of The second international workshop on global Anal. Diff.Geom. Lie Alg. 13-16 dec. 1995, Thessaloniki (Greece), Balkan Press Publ., Bucharest, (1997), 61–64;

[6] Popescu Marcela, Semisprays Induced on Submanifolds, New Developments in Differential Geometry, Ed. J. Szenthe, Kluwer Academic Publ. (1998), 317–328.