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SURFACE - SURFACE INTERSECTION: AUXILIARY SPHERES

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Abstract

In this paper a mathematical model for determination of intersecting curve between two surfaces of revolution was formed. A generating meridian and an axis of revolution give each surface of revolution where meridian and axis lie in one plane. Projecting rays are orthographic to the plane defined by the axis and the meridian, and in this case this meridian is one contour generatrix of the surface.

Based on combined transformation [4] (translation, rotation and reflection) the other contour generatrix is determined, and the axis of rotation is coincident with axis of symmetry. Using purely descriptive geometric method of auxiliary spheres the intersecting curve between two surfaces of revolution can be easily determined. This method is used when axes of surfaces meet each other [1]. The centre of all auxiliary spheres is intersecting point of these axes. Each sphere intersects both surfaces at two parallel circles. These four circles are on their common sphere so that circles either meeting each other or not. Circles, which intersect each other, define real points of the space curve. Spheres with minimal and maximal diameter, which define particular points of the intersecting curve, are determined as well.

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1 Introduction

There are many algorithms for determination of intersects of two surfaces. In case when intersect surfaces of revolution and when their axes intersect we can used a descriptive geometric method of auxiliary spheres. Section 2 presents the mathematical model for determination of reflecting curve through an arbitrary axis. Sphere with maximal radius is the topic of Section 3. A connection between two coordinate systems is represented in Section 4. Minimal (tangent) sphere is determined in Section 5. The last section summarises the conclusion and gives some directions for future work.

2 Contour generatrices of surfaces of revolution - reflection through the arbitrary axis

Curve and axis of rotation are given in plane xOz. It is necessary to determine reflecting curve through an arbitrary axis. This problem can be solved by combining two-dimensional transformations [4], through the following steps (Fig.1):

I. Translate the axis and the object (curve) so that the axis passes through the origin, where axis is given by equation $z = mx + n$ (Fig.1b).

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -n & 1 \end{bmatrix}$$

II. Next step is rotate the axis and the object until the axis is coincident with one coordinate axis (Fig.1c), in this case with x axis, angle of rotation is $\theta = arctg\left|\frac{n}{x_0}\right|$, where $x_0 = x\{z = 0\}$ and matrix of rotation is

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

III. Reflection through the x coordinate axis (Fig.1d)

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

IV. Apply the inverse rotation about the origin (Fig.1e)

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

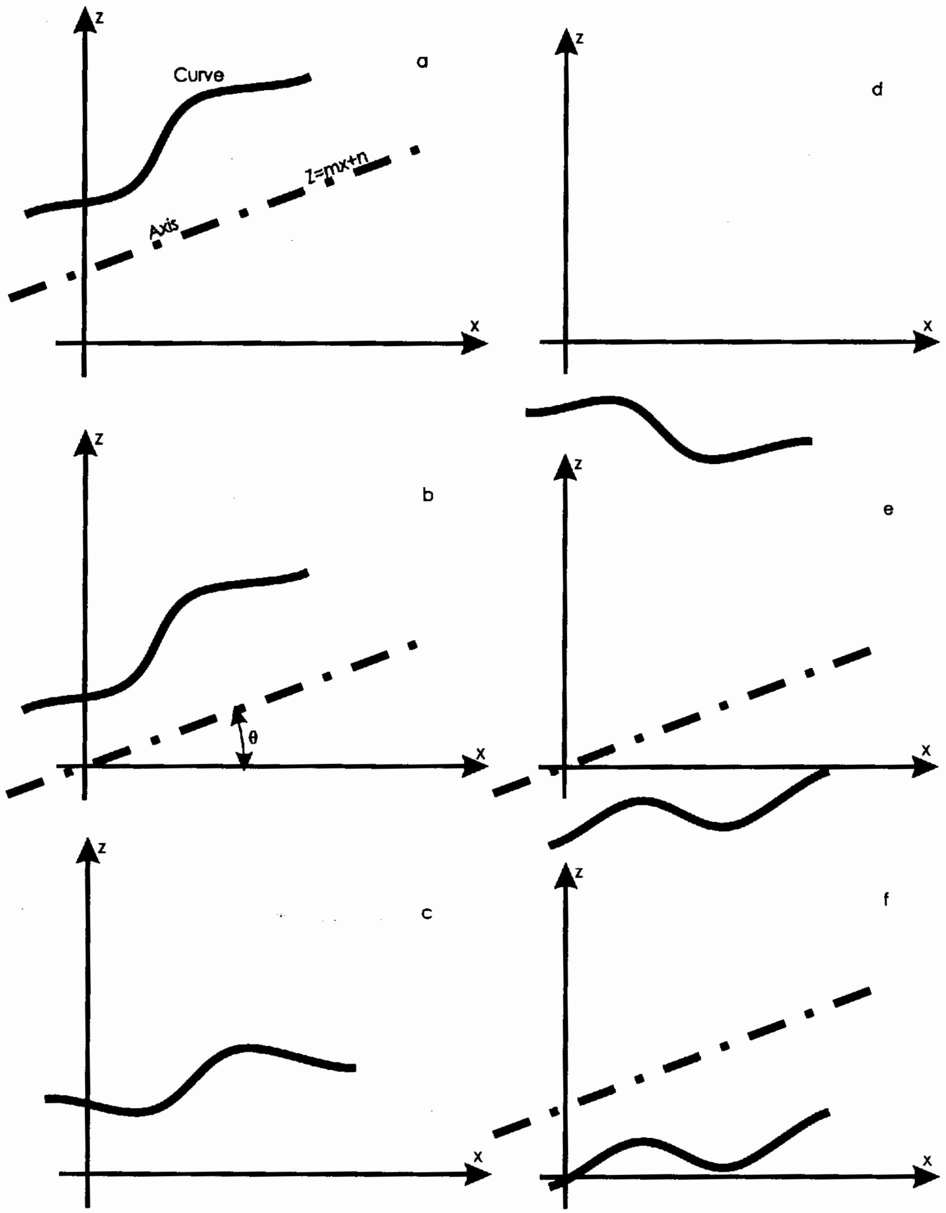

Figure 1. Reflection of the curve through the arbitrary axis (a) Original position; (b) translation through origin; (c) rotation to x-axis; (d) reflection about x-axis; (e) undo rotation; (f) undo translation;

V. Translate back to the origin location (Fig.1f)

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & n & 1 \end{bmatrix}$$

Reflection through an arbitrary axis is combined transformation, where the global transformation is represented by next matrix

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -n & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & n & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 2\cos^2\theta - 1 & 2\sin\theta\cos\theta & 0 \\ 2\sin\theta\cos\theta & 1 - 2\cos^2\theta & 0 \\ -2n\sin\theta\cos\theta & 2n\cos^2\theta & 1 \end{bmatrix}$$

Now, reflection through an arbitrary axis is determined by equation

$$[X^*] = \begin{bmatrix} x^* & z^* & 1 \end{bmatrix} = \begin{bmatrix} x & z & 1 \end{bmatrix} [T] = \begin{bmatrix} x & z & 1 \end{bmatrix} \begin{bmatrix} 2\cos^2\theta - 1 & 2\sin\theta\cos\theta & 0 \\ 2\sin\theta\cos\theta & 1 - 2\cos^2\theta & 0 \\ -2n\sin\theta\cos\theta & 2n\cos^2\theta & 1 \end{bmatrix}$$

$$[X^*] = \begin{bmatrix} 2x\cos^2\theta + 2(y-n)\sin\theta\cos\theta - x & 2(n-y)\cos^2\theta + 2x\sin\theta\cos\theta + z & 1 \end{bmatrix}$$

3 Sphere with maximal radius

A curve and an axis of revolution define one surface of revolution (Fig.2) where meridian and axis lie in plane xOz. First surface of revolution is given with their equations of plane curve and axis of revolution

$$x_1 = x_1(t) \qquad z_1 = z_1(t)$$
$$z = m_1 x + n_1$$

The second surfaces is given with

$$x_2 = x_2(t) \qquad z_2 = z_2(t)$$
$$z = m_2 x + n_2$$

Now, it is easily to determine equations for reflecting curves trough an arbitrary axes, and results are noted as x_1^*, z_1^* and x_2^*, z_2^*.

Intersecting points of meridians are shown in the table:

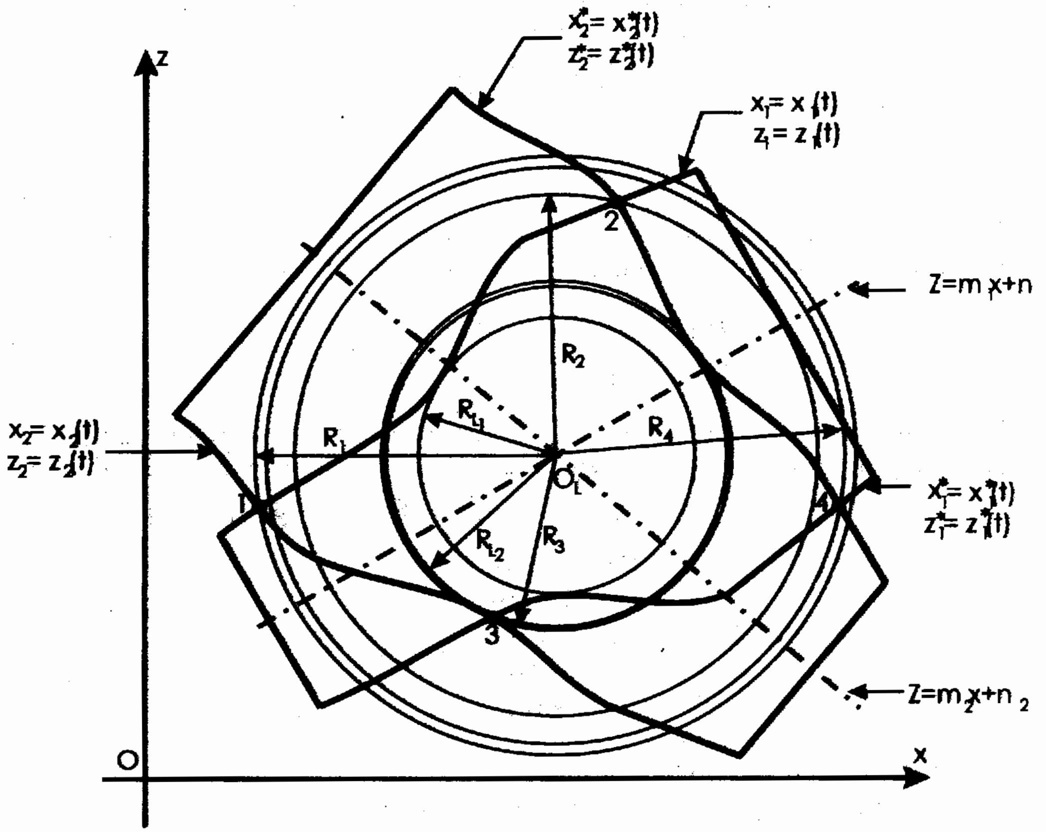

Figure 2. Surfaces of revolution, spheres with minimal and maximal radiuses.

Point	$1(x_1, z_1)$	$1(x_2, z_2)$	$1(x_3, z_3)$	$1(x_4, z_4)$
Position	$\begin{Bmatrix} x_1=x_1(t) \\ z_1=z_1(t) \end{Bmatrix} \cap \begin{Bmatrix} x_2=x_2(t) \\ z_2=z_2(t) \end{Bmatrix}$	$\begin{Bmatrix} x_1=x_1(t) \\ z_1=z_1(t) \end{Bmatrix} \cap \begin{Bmatrix} x_2^*=x_2^*(t) \\ z_2^*=z_2^*(t)) \end{Bmatrix}$	$\begin{Bmatrix} x_1^*=x_1^*(t) \\ z_1^*=z_1^*(t)) \end{Bmatrix} \cap \begin{Bmatrix} x_2=x_2(t) \\ z_2=z_2(t)) \end{Bmatrix}$	$\begin{Bmatrix} x_1^*=x_1^*(t) \\ z_1^*=z_1^*(t)) \end{Bmatrix} \cap \begin{Bmatrix} x_2^*=x_2^*(t) \\ z_2^*=z_2^*(t)) \end{Bmatrix}$

Intersecting point O_L of two axes is determined by combining next two equations

$$z = m_1 x + n_1 \qquad \text{and} \qquad z = m_2 x + n_2.$$

Point O_L (x_L, z_L) has coordinates

$$x_L = \frac{n_2 - n_1}{m_1 - m_2}$$

$$z_L = \frac{m_1 n_2 - m_2 n_1}{m_1 - m_2}.$$

Radius of spheres (they are circles in plane xOz) from the common centre O_L to the points $1, \ldots 4$ are defined with

$$R[i] = \sqrt{(x_L - x_i)^2 + (z_L - z_i)^2}$$
$$i = 1, \ldots, 4$$

There are four radiuses and it is necessary to determine maximal radius. Next step in procedure is separation this sphere which has maximal radius:

$R_{\max} := R[1]$
For $i = 2$ **to** 4 **do**
Begin
If $R[i] > R_{\max}$ **then** $R_{\max} = R[i]$
Else
End;

Now it is necessary to determine radiuses of spheres which are tangent to surfaces of revolution and have common centre O_L.

4 Transformation of coordinate system

It is necessary to make a connection between the global and the local coordinate systems xOz and uCv [2]. Axis v of the local coordinate system (Fig.3) is coincident with axis of rotation $z = mx + n$, and because axis u is orthogonal to the axis v, equation of line, which is coincident with u axis, is

$$z = -\frac{1}{m} x + k.$$

On axis u one point of meridian is located whose coordinates are $(x(t_{\min}), z(t_{\min}))$ in global system xOz. Substituting coordinates of this point in previous equation, yields

$$k = z(t_{\min}) + \frac{1}{m} x(t_{\min}).$$

Point C is origin of coordinate system uCv, and their coordinates in system xOz are determined combining next two equations

$$z = mx + n$$
$$z = -\frac{1}{m} x + k$$

Point $\mathrm{C}(x_C, z_C)$

$$x_C = \frac{k-n}{m^2+1} m$$
$$z_C = m\, x_C + n$$

Translation and rotation of system xOz can form local coordinate system uCv. First, system xOz is translated for vector OC, and then it is rotated for angle

$$\theta = \alpha - \frac{\pi}{2}$$
$$\alpha = \operatorname{arctg}(m)$$

Now, we can use two parametric functions

$$p(t) = x(t) - x_C$$
$$q(t) = z(t) - z_C$$

Results are two equations which connect points in global and local coordinate system, see Fig.3

$$u(t) = p(t)\cos\theta + q(t)\sin\theta$$
$$v(t) = -p(t)\sin\theta + q(t)\cos\theta$$

5 Determining of tangent sphere

It is necessary to determine sphere with centre O_L on axis v of local coordinate system, if radius of sphere is determined from condition that sphere is tangent to meridian. Centre of sphere in system uCv is determined with

$$\mathrm{O}_L = (0, v_L)$$
$$v_L = \sqrt{(x_L - x_C)^2 + (z_L - z_C)^2}$$

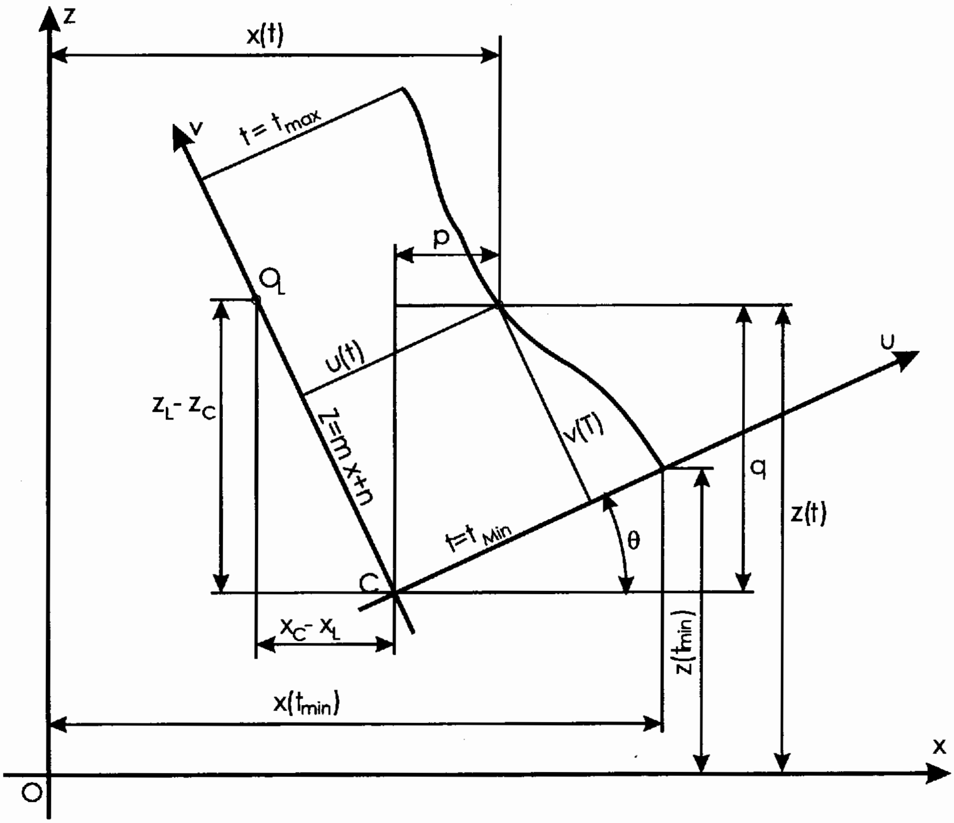

Figure 3. Transformation of coordinate system

For the centre of auxiliary sphere [5]

$$v_L = v(t) - u(t)\frac{n_v(t)}{n_u(t)}$$

$$\mathrm{DS} = v(t) - u(t)\frac{n_v(t)}{n_u(t)}$$

$$n_u(t) = -\frac{dv(t)}{dt}$$

$$n_v(t) = \frac{du(t)}{dt}$$

Since $t_{\min} \leq t \leq t_{\max}$, and v_L are known, then radius of auxiliary sphere with centre O_L which is tangent to meridian, is given by using substitution of value for t (from $t_{\min}$ and if it necessary to $t_{\max}$) in equations which represent value for component of normal vector $n_u(t), n_v(t)$. Then right side of equation DS is solved and if $v_L = \mathrm{DS}$, then value for parameter t for this sphere is determined. If value for parameter t for tangent sphere is marked as t_L then:

$$R_L = \sqrt{u(t_L)^2 + (v_L - v(t_L))^2}$$

For one surface of revolution (i.e. for one meridian and axis of rotation) radius of tangent sphere R_{L1} is given and for the other surface the join radius R_{L2}. Now, we need to check if these spheres give real points of intersecting curve. If sphere with radius R_{L1} which is tangent to sphere for meridian with parametric equations $x_1 = x_1(t)$, $z_1 = z_1(t)$ intersects or tangents the other meridian with equations $x_2 = x_2(t)$, $z_2 = z_2(t)$ then this sphere gives real points of intersecting curve. Thus, equations are to be combined

$$(x - x_L)^2 + (z - z_L)^2 = R_{L1}^2$$

$$\left\{ \begin{array}{c} x_2 = x_2(t) \\ z_2 = z_2(t) \end{array} \right\}$$

In general case solution can be real and different, and let that solution to give parameters t_{P1}, t_{P2}. Intersecting points are

$$\left\{ \begin{array}{c} x_1 = x_1(t) \\ z_1 = z_1(t) \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} x_2 = x_2(t) \\ z_2 = z_2(t) \end{array} \right\}$$

Similarly, we can solve intersection of next equations $(x - x_L)^2 + (z - z_L)^2 = R_{L2}^2$ and $\left\{ \begin{array}{c} x_1 = x_1(t) \\ z_1 = z_1(t) \end{array} \right\}$, and intersecting points are $\left\{ \begin{array}{c} x_1(t_{P3}) \\ z_1(t_{P3}) \end{array} \right\}$ and $\left\{ \begin{array}{c} x_1(t_{P4}) \\ z_1(t_{P4}) \end{array} \right\}$. From these six spheres one with minimal radius is

$$R_{\min} = \{R_{L1} \text{ or } R_{L2}\} = \max\{R_{L1}, R_{L2}\}$$

Where with or that tangent sphere to one surface which intersects the other surface is defined.

6 Conclusions and future work

In this paper using combine transformation and descriptive geometric method of auxiliary spheres a mathematical method for determination of intersecting curve of two surfaces of revolution was determined. In next work will be interesting to determine intersecting curve of two surfaces of revolution when their axes do not intersect each other using method of auxiliary planes [3] or some new adapting method of auxiliary spheres.

References

[1] L. Dovniković, Technical Drawing, University of Novi Sad, Novi Sad, 1990.

[2] J. Hoschek, D. Lasser, Fundamentals of Computer Aided Geometric Design, A K Peters, Wellesley, Massachusetts, 1993.

[3] R. Obradović, Application of Descriptive Geometrical Methods in Geometric Design by Use of Computer: Determination of Parabolic Quadrics - Cones and Cylinders Intersection, Master thesis, University of Novi Sad, Novi Sad, 1997.

[4] D.F. Rogers, J.A. Adams, Mathematical Elements for Computer Graphics, McGraw-Hill International Editions, New York, 1990.

[5] R. Štulić, Descriptive Geometric Methods in computer Graphics: Contour and Isophots of Surfaces of revolution, Master thesis, University of Belgrade, Belgrade, 1994.