Novi Sad J. Math.
Vol. 29, No. 3, 1999, 187–197
XII Yugoslav Geometric Seminar
Novi Sad, October 8–11, 1998

THE GIESEKING MANIFOLD AND ITS SURGERY ORBIFOLDS *

E. Molnár **I. Prok** **J. Szirmai**

Technical University of Budapest
Institute of Mathematics, Department of Geometry
H-1521 Budapest (Hungary), Egry. J. u. 1. H.II.22

Abstract

We determine all Dehn-surgeries (see [1], [6], [7]) of Gieseking hyperbolic simplex manifold $\tilde{\mathcal{S}}$, leading to compact orbifolds $\mathcal{S}(k)$ $k = 2$, $3, \dots$ with singularity geodesics of rotation order k. We compute also the volume of $\mathcal{S}(k)$, tending to 0 if $k \to \infty$ (see our Table 2.1 and Figures 2.1–8).

1. Introduction

The famous Gieseking manifold (1912) is the regular simplex with face angles $\pi/3$ in the Bolyai-Lobachevskian hyperbolic space $\mathbf{H}^3$ with ideal vertices at the absolute, equipped by face pairing isometries $\mathbf{z}_1$, $\mathbf{z}_2$ as horospherical glide reflections (Fig. 2.1). These induce one equivalence class of the 6 edges and so a ball-like neighbourhood for any point of an edge, and so for any point of the identified simplex $\tilde{\mathcal{S}}$. If we vary the face angles α_1, α_2, α_3 at the opposite edges of $\tilde{\mathcal{S}}$, then it will be no more a manifold, but for special angles there exist a natural parameter $1 < k \in \mathbb{N}$ such that $\tilde{\mathcal{S}}(k)$ will represent (seemingly new) compact hyperbolic orbifold with a closed singularity geodesics of rotation order k. We shall use the halfspace model of $\mathbf{H}^3$ and compute the volume $\mathbf{V}(k)$ of $\mathcal{S}(k)$ as well by means of the Lobachevski function $\Lambda(x)$ in (2.12). It may be surprising that $\mathbf{V}(k) \to 0$ if $k \to \infty$. This research is a byproduct of [3], [4], [5] showing some new phenomena.

---

*Supported by Hungarian NFSR (OTKA) T 020498 (1996–99).
Dedicated to our Friends in Yugoslavia in this sad time on April 04 Easter 1999.

2. Gieseking Manifold and its Surgeries

We start with the ideal simplex of $\mathbf{H}^3$ in the half-space model, where its ideal vertices at infinity are represented by

$$0, 1, z, \infty \in \mathbb{C} \cup \{\infty\} =: \mathbb{C}_\infty \quad \text{(Fig. 2.1-2)} \tag{2.1}$$

of the complex projective line. This will be an identified ideal simplex $\tilde{S}$ with face pairing mappings $\mathbf{z}_1$ and $\mathbf{z}_2$ as "horospherical glide reflections"

$$\begin{aligned}
&\mathbf{z}_1 : \infty z 1 =: [\mathbf{z}_1^{-1}] \to \infty 1 0 =: [\mathbf{z}_1], \quad \text{i.e} \\
&\mathbf{z}_1 : u \mapsto \frac{\bar{u}-1}{\bar{z}-1} \quad \text{or} \quad (u,1) \mapsto (\overline{u,1}) \begin{pmatrix} 1 & 0 \\ -1 & \bar{z}-1 \end{pmatrix}, \\
&\mathbf{z}_2 : 0 z \infty =: [\mathbf{z}_2^{-1}] \to 0 1 z =: [\mathbf{z}_2], \quad \text{i.e} \\
&\mathbf{z}_2 : u \mapsto \frac{\bar{u} z}{\bar{u} - \bar{z}(1-z)} \quad \text{or} \quad (u,1) \mapsto (\overline{u,1}) \begin{pmatrix} z & 1 \\ 0 & -\bar{z}(1-z) \end{pmatrix}.
\end{aligned} \tag{2.2}$$

Going round e.g. the edge $\overrightarrow{\infty z}$ from the starting identity simplex, we meet first the face $[\mathbf{z}_1^{-1}]$, then follows, on the other side, the image face $[\mathbf{z}_1]^{\mathbf{z}_1^{-1}}$ $(= [\mathbf{z}_1^{-1}])$ at the edge $(\infty 1)^{\mathbf{z}_1^{-1}}$ in the $\mathbf{z}_1^{-1}$-image simplex. Then the image face $[\mathbf{z}_1^{-1}]^{\mathbf{z}_1^{-1}}$ and, on the other side, the face $[\mathbf{z}_1]^{\mathbf{z}_1^{-1}\mathbf{z}_1^{-1}}$ $(= [\mathbf{z}_1^{-1}]^{\mathbf{z}_1^{-1}})$ come at edge $(\infty 0)^{\mathbf{z}_1^{-1}\mathbf{z}_1^{-1}}$ in the $\mathbf{z}_1^{-1}\mathbf{z}_1^{-1}$-image simplex. Then we meet the image face $[\mathbf{z}_2^{-1}]^{\mathbf{z}_1^{-1}\mathbf{z}_1^{-1}}$ and, on the other side, the image simplex $\mathbf{z}_2^{-1}\mathbf{z}_1^{-1}\mathbf{z}_1^{-1}$ by the conjugate mapping $\mathbf{z}_1\mathbf{z}_1\mathbf{z}_2^{-1}\mathbf{z}_1^{-1}\mathbf{z}_1^{-1}$ of $\mathbf{z}_2^{-1}$. Thus [2], we obtain the cycle transformation $\mathbf{z}_2\mathbf{z}_1\mathbf{z}_2^{-1}\mathbf{z}_2^{-1}\mathbf{z}_1^{-1}\mathbf{z}_1^{-1}$, and we prescribe the trivial rotation order $\nu = 1$ for the unique edge class containing 6 edges. Finally we get the cycle relation

$$\mathbf{z}_1\mathbf{z}_1\mathbf{z}_2\mathbf{z}_2\mathbf{z}_1^{-1}\mathbf{z}_2^{-1} = \mathbf{1} \tag{2.3}$$

in equivalent form, in conformity with the fact that the dihedral angles of a regular ideal simplex are $\pi/3$, $6 \cdot (\pi/3) = 2\pi$ will guarantee ball-like neighbourhood of any point at simplex edges. However, the relation (2.3) with (2.2) leads to equation

$$|z-1|^2 = |z| \tag{2.4}$$

with more general ideal simplex, not necessarily regular one.

Now, we turn to the ideal vertex class forming a cusp (Fig. 2.3–4). This will be represented by gluing corresponding images of the 4 vertex domains to that of ∞. The side face pairing of $\tilde{S}$ induces the pairing of the sides of a 2-dimensional polygon, denoted by $\tilde{\mathfrak{s}}$ in Fig. 2.3–4, say, on a horosphere

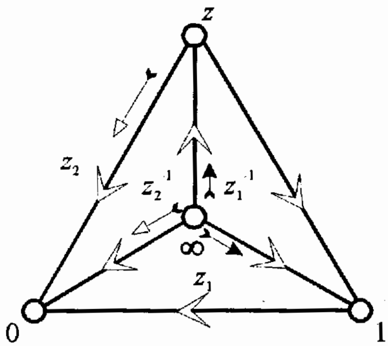

Figure 2.1

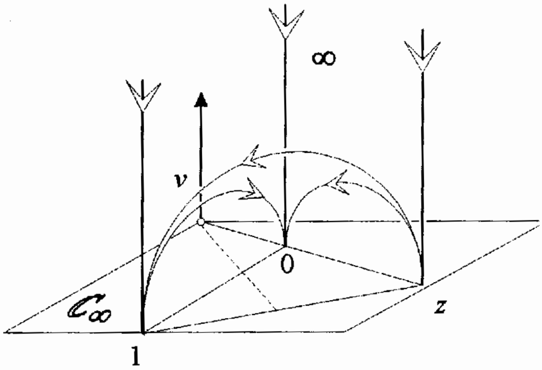

Figure 2.2

centred in ∞. This is represented in our half-space model by a Euclidean plane parallel to the absolute, and it can also be described on the absolute by $\mathbb{C}_\infty$.

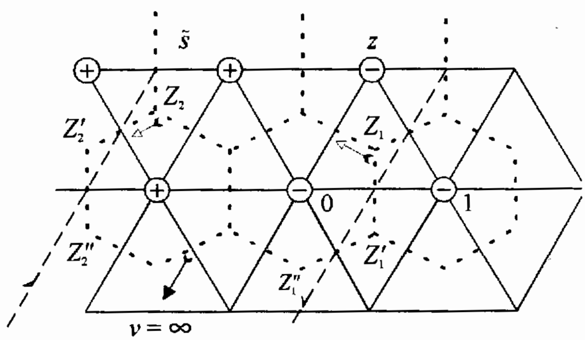

Figure 2.3

Topologically, the polygon $\tilde{s}$ is a Klein-bottle with fundamental group equivariant to the crystallographic plane group **pg**. This group, as the stabilizer $\mathcal{G}_\infty$ of ∞, is determined by the starting group $\mathcal{G}(\mathbf{z}_1, \mathbf{z}_2)$ in formula

(2.2). Fig. 2.4 shows that $\mathcal{G}_\infty$ is generated by pairing of $\mathfrak{s}$:

$$
\begin{aligned}
&\mathbf{z}_1 : [\mathbf{z}_1^{-1}] \to [\mathbf{z}_1] \quad \text{a “glide reflection” as before; then}\\
&\mathbf{p} \ : [\mathbf{z}_1^{-1}]^\star := [\mathbf{z}_1^{-1}]^{\mathbf{z}_2^{-1}} \to [\mathbf{z}_1]^\star := [\mathbf{z}_1]^{\mathbf{z}_2^{-1}\mathbf{z}_2^{-1}},\\
&\qquad \text{i.e.} \quad \mathbf{p} = \mathbf{z}_2\mathbf{z}_1\mathbf{z}_2^{-1}\mathbf{z}_2^{-1} \quad \text{a “translation”;} \qquad\qquad (2.5)\\
&\mathbf{z}_2^\star : [\mathbf{z}_2^{-1}]^\star := [\mathbf{z}_2^{-1}]^{\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\mathbf{z}_2^{-1}} \to [\mathbf{z}_2]^\star := [\mathbf{z}_2]^{\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\mathbf{z}_2^{-1}},\\
&\qquad \text{i.e.} \quad \mathbf{z}_2^\star = \mathbf{z}_2\mathbf{z}_2\mathbf{z}_1\mathbf{z}_2\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\mathbf{z}_2^{-1} \quad \text{a “glide reflection”}
\end{aligned}
$$

again, it is conjugated to $\mathbf{z}_2$. We see that $\mathbf{p}$ is a “translation”, it is $\mathbf{z}_2^{-1}$-conjugated to $\mathbf{z}_1\mathbf{z}_2^{-1}$. This group $\mathcal{G}_\infty$ is **pg** itself (on the Euclidean plane represented by $\mathbb{C}$) if $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then Fig. 2.3 shows the exact situation. We have obtained the Gieseking manifold with one cusp.

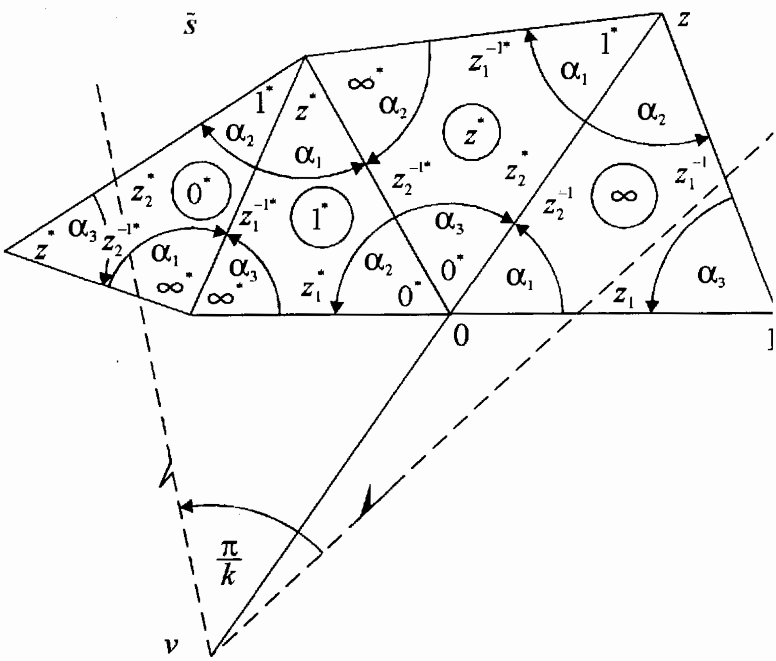

Figure 2.4

Other z, as a complex parameter, makes the stabilizer $\mathcal{G}_\infty$ to a conformal group with fixed points

$$\infty \quad \text{and} \quad v = \frac{z}{1 - |z|}.$$

This line $v\infty$ will not be covered by the $\mathcal{G}_\infty$-images of the simplex $\tilde{\mathcal{S}}$ in $\mathbf{H}^3$. In the model half-space the translations of $\mathcal{G}_\infty$ in (2.5) by (2.2) will be similarities with fixed points v, ∞. E.g. $\mathbf{z}_1$ in (2.2) and $\mathbf{z}_2^\star$ in (2.5) are similarity-reflections indicated in Fig. 2.2–4. For $\mathbf{z}_2^\star$ we can write by (2.2)

$$\mathbf{z}_2^\star : (u,1) \to (\overline{u,1}) \begin{pmatrix} 1 & 1 \\ 0 & |1-z|^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & \bar{z}-1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \bar{z} & 1 \\ 0 & -z(1-\bar{z}) \end{pmatrix} \begin{pmatrix} z-1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |1-z|^2 & -1 \\ 0 & 1 \end{pmatrix} = \tag{2.6}$$

$$\stackrel{(2.4)}{=} (\overline{u,1}) \begin{pmatrix} z(1-\bar{z})^2 & 0 \\ |z|^2\{|z|(\bar{z}-1) - (|z|+1)\} & -\bar{z}(1-z) \end{pmatrix}.$$

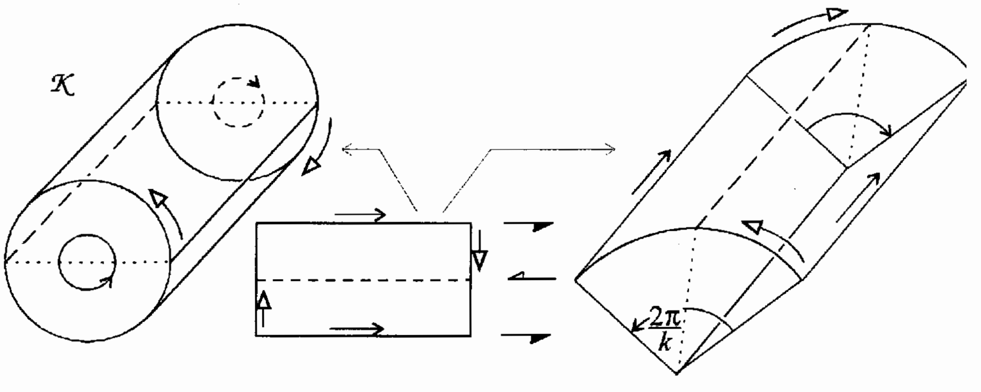

Figure 2.5

Now, we turn to the critical transform $\mathbf{z}_1\mathbf{z}_2^\star$. We obtain

$$\mathbf{z}_1\mathbf{z}_2^\star : (u,1) \to (u,1) \begin{pmatrix} z(1-\bar{z})^2 & 0 \\ \{|z|+1\}[|z|^2 - z^2] & \bar{z}(1-z)^2 \end{pmatrix}, \tag{2.7}$$

fixing ∞ and v of course. We see by (2.4) that $\mathbf{z}_1\mathbf{z}_2^\star$ describe a rotation of the model half-space about the line ∞v with angle

$$\arg\left[\frac{z(1-\bar{z})^2}{\bar{z}(1-z)^2}\right] = 2\arg z - 4\arg(1-z) \qquad \text{mod } 2\pi. \tag{2.8}$$

If we require the stabilizer $\mathcal{G}_\infty$ to act discontinuously on the model half-space, then this angle has to be $2\pi/k$, i.e.

$$\frac{z}{(1-z)^2} = e^{-i\pi/k} \quad \text{with} \quad 1 < |z-1| < |z|, \quad \operatorname{Im} z > 0, \quad k = 2,3,\ldots \tag{2.9}$$

can be assumed. Then k is the periodicity of the rotation $\mathbf{z}_1\mathbf{z}_2^*$ and we get a compact orbifold with a closed geodesic line of k-singularity. Our freedom in (2.9) is obvious. The other assumption $|z| < |z-1| < 1$ by (2.4) would lead to $e^{i\pi/k}$ on the right hand side of (2.9) and to the same angles of the ideal simplex $\tilde{\mathcal{S}}$.

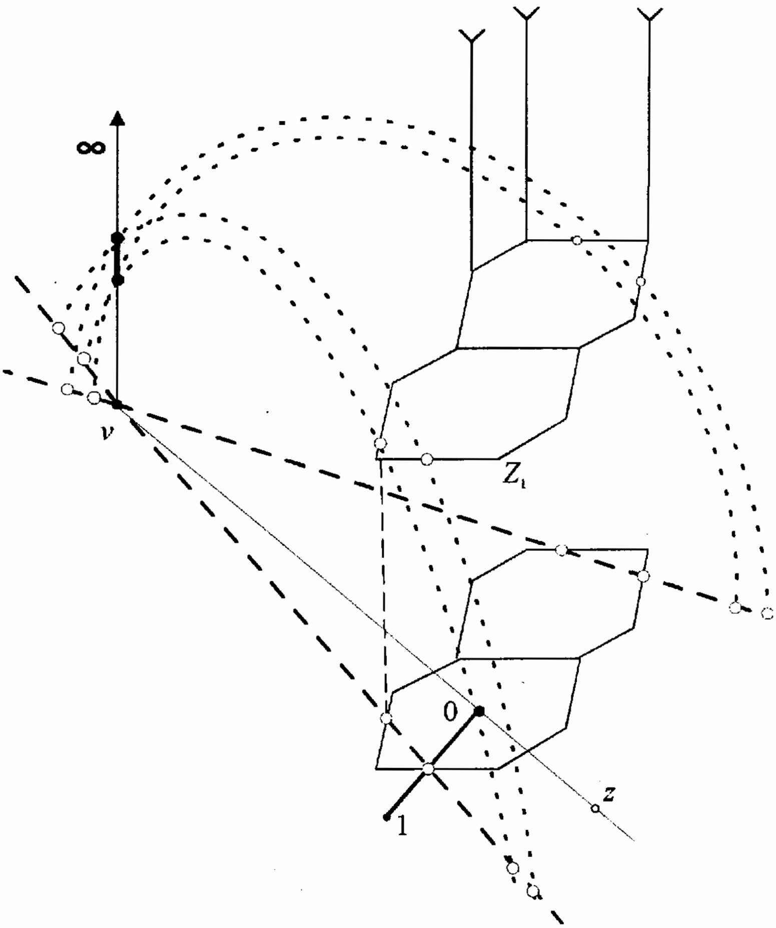

Figure 2.6

All data can be computed from (2.9), especially the face angles of $\tilde{\mathcal{S}}$, equal at the opposite edges (Fig. 2.2–4)

$$\begin{Bmatrix} \overline{\infty 0} \\ \overline{z1} \end{Bmatrix} : \alpha_1 = \arg z; \qquad \begin{Bmatrix} \overline{\infty z} \\ \overline{10} \end{Bmatrix} : \alpha_2 = \arg \frac{z-1}{z}; \qquad \begin{Bmatrix} \overline{\infty 1} \\ \overline{z0} \end{Bmatrix} : \alpha_3 = \arg \frac{1}{1-z} \tag{2.10}$$

In this case (2.9) can be solved explicitly

$$z = 1 + \frac{1}{2} e^{i\pi/k} \Big(1 - \sqrt{1 + 4e^{-i\pi/k}} \Big), \qquad k = 2, 3, \dots \tag{2.11}$$

however, the computer gives more guarantees. In Table 2.1 we have computed by MAPLE 5.0 the volume of $\tilde{\mathcal{S}}$ as well for some values of k. We know [7] that the Lobachevski function

$$\Lambda(x) = -\int_0^x \ln |2 \sin \xi| d\xi \quad \text{with} \quad \operatorname{Vol} \tilde{\mathcal{S}} = \Lambda(\alpha_1) + \Lambda(\alpha_2) + \Lambda(\alpha_3) \tag{2.12}$$

provides the volume of the ideal simplex with above angles. Of course, the group $\mathcal{G}(\mathbf{z}_1, \mathbf{z}_2, k)$ above has a unified presentation

$$\mathcal{G}(k) = \Big(\mathbf{z}_1, \mathbf{z}_2 — \mathbf{1} = \mathbf{z}_1\mathbf{z}_1\mathbf{z}_2\mathbf{z}_2\mathbf{z}_1^{-1}\mathbf{z}_2^{-1} = (\mathbf{z}_1\mathbf{z}_2\mathbf{z}_2\mathbf{z}_1\mathbf{z}_2\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\mathbf{z}_2^{-1})^k \Big). \tag{2.13}$$

For $k = 2, 3, \dots$ we indicate by Fig. 2.3–7 how to construct a compact fundamental domain $\tilde{\mathcal{F}}_{\mathcal{G}(k)}$ (in Fig. 2.7) by deforming an ideal vertex domain to a compact one. We introduce an edge (e) on the line ∞v and its $\mathbf{z}_1$-image with

$$(e) = (EE'), \qquad (e)^{\mathbf{z}_1} = (e)^{\mathbf{z}_2^{\star-1}} = (E'E''), \qquad (EE') \cap (E'E'') = \emptyset.$$

Then we choose a point Z_1 in the simplex $\tilde{\mathcal{S}} = \infty 01z$ and consider the segments

$$(EZ_1), \qquad (EZ_1)^{\mathbf{z}_1} = (E'Z_1'), \qquad (EZ_1)^{\mathbf{z}_1\mathbf{z}_1} = (E''Z_1'').$$

Similarly take Z_2, as the $\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\mathbf{z}_2^{-1}$-image of Z_1 at the cusp gluing

$$(EZ_2), \qquad (EZ_2)^{\mathbf{z}_2^{\star-1}} = (E'Z_2'), \qquad (EZ_2)^{\mathbf{z}_2^{\star-1}\mathbf{z}_2^{\star-1}} = (E''Z_2'').$$

Then the corresponding curved (bent) surfaces $[\mathbf{q}^{-1}]$ and its $\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}\mathbf{z}_2^{-1}$-image $[\mathbf{q}]$ will be constructed, transversally to the edges of $\tilde{\mathcal{S}}$ (see Fig. 2.3, 4, 6, 7 and also [?]).

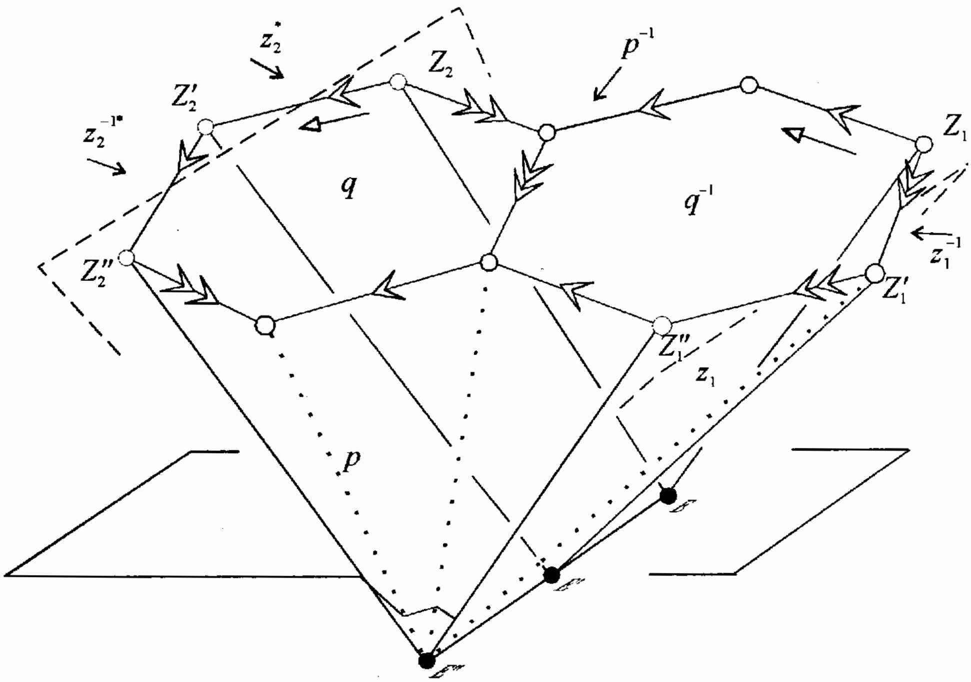

Figure 2.7. Fundamental domain scheme $\tilde{\mathcal{F}}_{\mathcal{G}(k)}$ for $k = 2, 3, 4, \ldots$

Finally, in Fig. 2.7 we get a compact fundamental domain $\tilde{\mathcal{F}}$, with piecewise linear bent faces, equipped by a pairing $\mathcal{I}(\mathbf{z}_1, \mathbf{z}_2^\star, \mathbf{p}, \mathbf{q})$ and defining relations to the corresponding edge classes:

$$\mathbf{z}_1\mathbf{z}_1\mathbf{p}^{-1} = \mathbf{z}_2^\star\mathbf{z}_2^\star\mathbf{p} = \mathbf{q}\mathbf{z}_2^{\star-1}\mathbf{q}^{-1}\mathbf{p}\mathbf{q}^{-1}\mathbf{p}^{-1} = \mathbf{z}_1\mathbf{q}\mathbf{q}\mathbf{p}^{-1}\mathbf{q}^{-1} = (\mathbf{z}_1\mathbf{z}_2^\star)^k = \mathbf{1}. \tag{2.14}$$

In Fig. 2.8 we have only pictured the very economic presentation with its combinatorial fundamental domain:

$$\mathcal{G}(k) = \Big(\mathbf{z}_1, \mathbf{z}_2 - \mathbf{1} = \mathbf{z}_1^2\mathbf{z}_2^2\mathbf{z}_1^{-1}\mathbf{z}_2^{-1} = (\mathbf{z}_1\mathbf{z}_2^2\mathbf{z}_1\mathbf{z}_2^{-1}\mathbf{z}_1^{-2}\mathbf{z}_2^{-1})^k\Big). \tag{2.15}$$

Of course, (2.15) equivalent with (2.13) and with (2.14) if

$$\mathbf{p} = \mathbf{z}_1^2, \qquad \mathbf{z}_2^\star = (\mathbf{z}_2^2\mathbf{z}_1)\mathbf{z}_2(\mathbf{z}_1^{-1}\mathbf{z}_2^{-2}), \qquad \mathbf{q} = \mathbf{z}_1^{-1}\mathbf{z}_2^{-2}. \tag{2.16}$$

Remark 2.1 Observe that our compactification procedure works also for $k = 1$ as Fig. 2.5 indicates. The cusp of our ideal simplex $\tilde{S}$ as a Klein-bottle can be glued by a "solid Klein-bottle" $\mathcal{K}$. Then the splitting effect

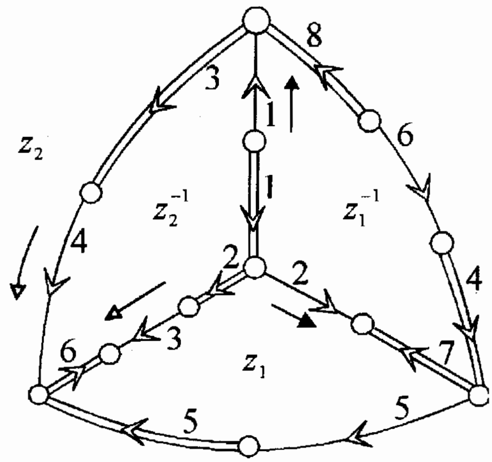

Figure 2.8. A combinatorial fundamental domain for the Gieseking manifold and orbifold to defining relations $\longrightarrow$ $\mathbf{z}_1\mathbf{z}_1\mathbf{z}_2\mathbf{z}_2\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}$ and $\Longrightarrow$ $\mathbf{z}_1\mathbf{z}_2\mathbf{z}_2\mathbf{z}_1\mathbf{z}_2^{-1}\mathbf{z}_1^{-1}\mathbf{z}_1^{-1}\mathbf{z}_2^{-1}$ in (2.15).

occurs. The cusp of $\tilde{S}$ will be cut along a Klein-bottle surface to get a boundary. Then we glue to this boundary the boundary of $\mathcal{K}$, considered as $\mathbf{S}^2\times\mathbb{R}$-manifold with boundary, as follows

$$\mathcal{K} := \mathbf{S}^2\times\mathbb{R}/\langle\mathbf{g}\rangle, \quad \mathbf{g} : (P,r) \mapsto \left(P^{\mathbf{m}};\ r+\frac{1}{2}\right), \quad (P,r) \in \mathbf{S}^2\times\mathbb{R} \qquad (2.17)$$

The generator $\mathbf{g}$ is a product of a reflection $\mathbf{m}$, say, in the equator of $\mathbf{S}^2$, combined by a 1/2-translation in $\mathbb{R}$. The boundary Klein-bottle can be obtained by cutting out one half-sphere, say, at longitudes 0 and π, of $\mathbf{S}^2$ with complete $\mathbb{R}$-fibers.

We summarize our results in

Theorem 2.1 *The surgery procedure, gluing the cusp Klein-bottle of Gieseking simplex $\tilde{S}$ by a solid Klein-bottle $\mathcal{K}$, leads to nongeometrizable compact nonorientable Gieseking manifold, because of the splitting into a $\mathbf{H}^3$- and $\mathbf{S}^2\times\mathbb{R}$-pieces in case $k=1$.*

For other cases $k = 2, 3, \ldots,$ the surgery yields compact nonorientable hyperbolic orbifolds, with underlying Gieseking manifold before, where a closed geodesic line exists with singularity of order k. The orbifold can be realized by a deformed ideal simplex $S(k)$ by (2.1–2) with algebraic complex parameter $z(k)$ by (2.9) that uniquely determines all metric data in our figures and tables. The volume of $S(k)$ tends to 0 if $k \to \infty$. ∎

k	z	$\alpha_1, \quad \alpha_2, \quad \alpha_3$	Volume
1	$\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} = 60°$	1.014 941 606 410
2	1.624 810 533 844 $+i \cdot$ 1.300 242 590 220	0.674 888 845 586 $\approx$ 38.67° 0.447 953 740 604 $\approx$ 25.67° 2.018 750 067 399 $\approx$ 115.67°	0.696 701 139 104
3	2.121 964 426 952 $+i \cdot$ 1.053 755 774 241	0.460 919 465 741 $\approx$ 26.41° 0.293 139 042 728 $\approx$ 16.80° 2.387 534 145 121 $\approx$ 136.80°	0.486 617 604 149
4	2.327 485 420 368 $+i \cdot$ 0.844 915 596 541	0.348 223 418 295 $\approx$ 19.95° 0.218 587 372 551 $\approx$ 12.52° 2.574 781 862 743 $\approx$ 147.52°	0.370 676 286 965
5	2.428 586 742 493 $+i \cdot$ 0.696 962 110 900	0.279 471 848 013 $\approx$ 16.01° 0.174 423 341 352 $\approx$ 9.99° 2.687 697 464 224 $\approx$ 153.99°	0.298 642 003 414
6	2.485 138 400 483 $+i \cdot$ 0.590 504 158 346	0.233 287 894 672 $\approx$ 13.37° 0.145 155 440 463 $\approx$ 8.32° 2.763 149 318 455 $\approx$ 158.32°	0.249 816 177 026
7	2.519 800 954 125 $+i \cdot$ 0.511 214 705 460	0.200 162 293 137 $\approx$ 11.47° 0.124 318 328 688 $\approx$ 7.12° 2.817 112 031 765 $\approx$ 161.41°	0.214 617 084 964
8	2.542 527 170 976 $+i \cdot$ 0.450 210 237 700	0.175 255 336 321 $\approx$ 10.04° 0.108 721 872 689 $\approx$ 6.23° 2.857 615 444 580 $\approx$ 163.73°	0.188 067 287 612
9	2.558 212 860 705 $+i \cdot$ 0.401 960 317 976	0.155 851 202 654 $\approx$ 8.93° 0.096 607 323 872 $\approx$ 5.54° 2.889 134 127 063 $\approx$ 165.54°	0.167 339 803 689
10	2.569 485 167 172 $+i \cdot$ 0.362 909 442 274	0.140 310 125 517 $\approx$ 8.04° 0.086 924 569 921 $\approx$ 4.98° 2.914 357 958 152 $\approx$ 166.98°	0.150 714 483 022
500	2.618 014 408 803 $+i \cdot$ 0.007 356 441 963	0.002 809 924 413 $\approx$ 0.16° 0.001 736 630 447 $\approx$ 0.10° 3.137 046 098 730 $\approx$ 179.73°	0.003 023 539 371
$\to \infty$	$z \to (3+\sqrt{5})/2 \approx$ $\approx 2.618\,0339$ $v \to -(1+\sqrt{5})/2 \approx$ $\approx -1.618\,0339$	$\alpha_1 \to 0°$ $\alpha_2 \to 0°$ $\alpha_3 \to 180°$	$\to 0$

Table 2.1. Gieseking manifold and its surgery orbifolds

The last assertion follows from (2.11–12), but this may be surprising at the first moment.

References

[1] A. T. Fomenko and S. V. Matveev, Isoenergetic surfaces of Hamiltonian systems, account of three-dimensional manifolds in order of their complexity and computation of volumes of closed hyperbolic manifolds, Uspehi mat. nauk **43** (1988), 5–22, in Russian.

[2] E. Molnár, Polyhedron complexes with simply transitive group actions and their realizations, Acta Math. Hung. **59(1-2)** (1992), 175–216.

[3] E. Molnár and I. Prok, Classification of solid transitive simplex tilings in simply connected 3-spaces, Part I. The combinatorial description by figures and tables, Colloquia Math. Soc. János Bolyai **63.** Intuitive Geometry, Szeged (Hungary), 1991 (Amsterdam–Oxford–New York), North–Holland Publ. Comp., 1994, pp. 311–362.

[4] E. Molnár, I. Prok, and J. Szirmai, Classification of solid transitive simplex tilings in simply connected 3-spaces, Part II. Metric realizations of the maximal simplex tilings, Periodica Math. Hung. **35(1–2)** (1997), 47–94.

[5] I. Prok, Data structures and procedures for a polyhedron algorithm, Periodica Polytechnica Ser. Mech. Eng. **36(3-4)** (1992), 299–316.

[6] W. Thurston, The geometry and topology of 3-manifolds, Lecture notes, Princeton University, 1978.

[7] E. B. Vinberg and O. V. Shvartsman, Discrete transformation groups of spaces of constant curvature, Geometriya 2 VINITI Itogi Nauki i Techniki, Sovr. Probl. Mat. Fund. Napr. **29** (1988), 147–259, in Russian, English translation: [8].

[8] ———, Geometry II, Encyclopedia of Math. Sci. (Berlin–Heidelberg) (E. B. Vinberg, ed.), Springer, 1993.