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THE JACOBI FIELDS FOR A SPRAY ON THE TANGENT BUNDLE *

Ioan Bucataru
Faculty of Mathematics
"Al.I.Cuza" University Iaşi 6600 Iaşi, Romania
Email address: bucataru@uaic.ro

Abstract

For a spray S on the total space of the tangent bundle we consider a variation of integral curves. The vector field of variation satisfies a kind of Jacobi equation in that appears the curvature of the nonlinear connection induced by S. A global form for the Berwald connection associated to a nonlinear connection is given. This is useful in study of the horizontal curves and in variation of such curves. When S is provided by a linear connection ∇ the vector field of variation is just the Jacobi vector field for ∇.

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Introduction

It is well known that a second order differential equation (SODE) or a semispray S determines a nonlinear connection N on the tangent bundle of a manifold M by means of an almost product structure [5] or by a vertical projector. For a nonlinear connection we can consider a linear connection on the tangent bundle, usually denoted *Berwald connection*. A global expression of it is proposed. A global equation which determine the Berwald connection was given by Martinez, Carinena and Sarlet in [9]. The horizontal curves of the nonlinear connection N associated to a semispray S are integral curves for S if and only if S is a homogeneous vector field with respect to velocity (in that case S is called a spray). In the homogeneous case is proved that a curve $\widetilde{c}$ on the tangent bundle is horizontal for the

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nonlinear connection if and only if $\widetilde{c}$ is a geodesic for the Berwald connection. If the semispray S is Lagrangian, that means there exists a Lagrangian $L : TM \to I\!R$ such that the canonical semispray associated to it is just S, then the geometry of the tangent bundle TM can be derived from L and the geometrical object associated to it.

In this paper we provide a study of the integral curves of a spray, especially when this is no Lagrangian. The vector field of variation for an integral curve of a spray, equivalent for a geodesic of the Berwald connection, satisfies two types of equations that generalise the classical *Jacobi equation.*

1 Preliminaries

Let M be a real, smooth, n-dimensional manifold. The tangent bundle of the manifold M will be denoted by $(\mathrm{TM}, \pi, \mathrm{M})$. Denote by $\widetilde{TM} = TM \backslash \{0\}$, where 0 denotes the null section of the tangent bundle. For a local chart $(U, \varphi = (x^i))$ in $p \in M$ its lifted local chart in $u \in \pi^{-1}(p)$ will be denoted by $(\pi^{-1}(U), \Phi = (x^i, y^i))$.

There is a vertical subbundle $V \subset TTM$, provided by the kernel of the differential of the natural submersion $\pi : TM \to M$. Let $\{\frac{\partial}{\partial x^i}|_u, \frac{\partial}{\partial y^i}|_u\}$ be the natural frame of the tangent space T_uTM in a point $u \in TM$. It is easy to check that $\{\frac{\partial}{\partial y^i}|_u\}$ is a local frame for $V(u)$. Denote by $\Gamma(V)$ the $\mathcal{F}(TM)$ module of vector fields that belong to V.

The tensor field $J = \frac{\partial}{\partial y^i} \otimes \mathrm{dx}^i$, is globally defined. It is called the natural almost tangent structure. One has: 1. $J^2 = O$, 2. rank $J = n$, 3. Im $J =$ Ker $J = V$.

A vector field $S \in \chi(TM)$ is said *to be a semispray* on TM if $JS = I\!\!C$, where $I\!\!C = y^i \frac{\partial}{\partial y^i}$ is the *Liouville vector field.* The local expression of a semispray is: $S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$. A subbundle N of the tangent bundle (TTM, τ, TM) which is supplementary to the vertical subbundle V, i.e. the following Whitney sum holds:

$$TTM = N \oplus V,$$

is called a *nonlinear connection* on TM. A nonlinear connection determines a n-dimensional distribution $N : u \in TM \to N(u) \subset T_uTM$.

Definition 1.1 *An $\mathcal{F}(TM)$-linear map $v : \chi(TM) \to \chi(TM)$ for which we have:*

$$J \circ v = 0, \quad v \circ J = J \tag{1.1}$$

will be called a vertical projector.

Note that a vertical projector can be regarded as a morphism of vector bundles. As $v(\frac{\partial}{\partial y^i}) = v(J(\frac{\partial}{\partial x^i})) = J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$, from the first condition (1.1) it results that $v(\chi(TM)) = \Gamma(V)$. Obviously rank $v = n$. We have that every vertical projector v induces a nonlinear connection and conversely every nonlinear connection N induces a vertical projector v such that the nonlinear connection determined by v is just N. In local coordinates a vertical projector can be written as follows:

$$v = N^i_j \frac{\partial}{\partial y^i} \otimes dx^j + \frac{\partial}{\partial y^i} \otimes dy^i.$$

The functions N^i_j are called the coefficients of the nonlinear connection N. For every $u \in TM$, $\{\frac{\delta}{\delta x^i}|_u := \frac{\partial}{\partial x^i}|_u - N^j_i(u)\frac{\partial}{\partial y^j}|_u, \frac{\partial}{\partial y^i}|_u\}$ is a basis for T_uTM adapted to the horizontal and the vertical distribution.

Proposition 1.1 *Let v be a vertical projector. There is an unique vector field $S \in \chi(TM)$ such that*

$$\begin{cases} J(S) = \mathbb{C}, \\ v(S) = 0. \end{cases} \tag{1.2}$$

This vector field is called *the canonical semispray of the nonlinear connection N* induced by v.

2 Sprays and integral curves

Proposition 2.1 *Let S be a semispray on TM. Then the map $v : \chi(TM) \to \chi(TM)$ defined by*

$$v(X) = \frac{1}{2}(X + [S, JX] + J[X, S]) \tag{2.1}$$

is a vertical projector.

In local coordinates: $v = \frac{\partial G^i}{\partial y^j}\frac{\partial}{\partial y^i} \otimes dx^j + \frac{\partial}{\partial y^i} \otimes dy^i$.

Proposition 2.2 *Let S be a semispray on $\widetilde{TM}$ and v be the vertical projector associated to it like in Proposition 2.1. Then S is the canonical semispray of the nonlinear connection determined by v if and only if*

$$\mathcal{L}_{\mathbb{C}}S = [\mathbb{C}, S] = S. \tag{2.2}$$

A semispray verifying (2.2) (which means that S is a homogeneous vector field of degree two with respect to velocity) is called a *spray*. In local coordinates (2.2) becomes:

$$2G^i = \mathbb{C}(G^i) = y^j \frac{\partial G^i}{\partial y^j}. \tag{2.2'}$$

The condition (2.2)' means that the functions G^i are homogeneous of degree 2.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field on M. Then the complete lift of X to TM follows:

$$X^c = (X^i \circ \pi)\frac{\partial}{\partial x^i} + \frac{1}{1!}S(X^i \circ \pi)\frac{\partial}{\partial y^i} \tag{2.3}$$

and it does not depend of the choice of the semispray S. The vertical lift of X will be denoted by X^v. We have imediatelly $X^v = J(X^c)$.

Let $\widetilde{c} : t \in I \subset \mathbb{R} \to \widetilde{c}(t) \in \widetilde{TM}$ be an integral curve of a semispray S. Then $c = \pi \circ \widetilde{c}$ is called a *path of the semispray S*.

Next we consider a spray S on $\widetilde{TM}$.

Proposition 2.3 *Let $c : I \subset \mathbb{R} \to M$ be a smooth curve and $X(t) = \frac{dc}{dt}$ be the tangent vector field along c. Then c is a path for the spray S if and only if*

$$v(X^c) = 0. \tag{2.4}$$

Let N be the nonlinear connection associated to the spray S. Denote by X^h the horizontal lift of a vector field $X \in \chi(M)$ and by h the horizontal projector induced by N. We have $X^h = h(X^c)$. Taking into account (2.4) we have that a curve c is a path for a spray S if and only if

$$X^c = X^h = h(X^c). \tag{2.4'}$$

For a horizontal curve $\widetilde{c} : I \to \widetilde{TM}$ with respect to a nonlinear connection N its projection $c = \pi \circ \widetilde{c}$ is called a *h-path of N*.

Proposition 2.4 *Let $c : I \subset I\!R \to M$ be a smooth curve and $X(t) = \dfrac{dc}{dt}$ be the tangent vector field along c. Then c is a h-path of the nonlinear connection N associated to the spray S if and only if c is a path of S.*

Denote by Ω the $\Gamma(V)$-valuated two form of curvature of the nonlinear connection N, $\Omega : \chi(M) \times \chi(M) \to \Gamma(V)$,

$$\Omega(X,Y) = [X^h, Y^h] - [X,Y]^h.$$

Definition 2.1 *Let c be a path for s spray S on $\widetilde{TM}$. A vector field Y on M along c is called a Jacobi vector field along c if it satisfies:*

$$(JE) \qquad [v(Y^c), X^c] + \Omega(Y,X) = 0,$$

where X is the tangent vector field along c.

Theorem 2.1 *Let $c : I \subset I\!R \to M$ be a path of a spray S. Consider a variation by paths of S, $\alpha : (-\varepsilon,\varepsilon) \times I \to M$, i.e. $\alpha(0,t) = c(t), \forall t \in I$ and $\alpha_s(t) := \alpha(s,t)$ are paths of S for every $s \in (-\varepsilon,\varepsilon)$. Let $X = \dfrac{\partial\alpha}{\partial t}|_{s=0}$ and $Y = \dfrac{\partial\alpha}{\partial s}|_{s=0}$. Then Y is a Jacobi vector field along c.*

Proof. First, we observe that $[X,Y] = 0$. Since $v(X^c) = 0$ then $[X^c, Y^c] = [X,Y]^c = 0$ and $\Omega(X,Y) = [X^h, Y^h] = [X^c - v(X^c), Y^c - v(Y^c)] = -[X^c, v(Y^c)]$. Thus Y satisfies (JE). **q.e.d.**

Proposition 2.5 *For a path c of a spray, the tangent vector field $X = \dfrac{dc}{dt}$ and $\widetilde{X}(t) = tX(t)$ are Jacobi vector fields along c.*

Proof. It is easy to check that X satisfies (JE). For $\widetilde{X}$ we have: $\widetilde{X}^c = (tX)^c = tX^c + X^v$ and $v(\widetilde{X}^c) = tv(X^c) + X^v = X^v$. Then $[v(\widetilde{X}^c), X^c] + \Omega(\widetilde{X}, X) = [X^v, X^c] + t\Omega(X,X) = [X^v, X^c] = 0$. **q.e.d.**

Proposition 2.6 *The solutions Y of the Jacobi equations (JE) along a path c are completely determined by the initial condition $Y(t_0) = Y_0 \in T_{c(t_0)}M$ and $v(Y^c)(t_0) = V_0 \in V(\widetilde{c}(t_0))$.*

Proof. For a vector field $X = X^i \dfrac{\partial}{\partial x^i}$ along a curve $c : t \in I \subset I\!R \mapsto c(t) = (x^i(t))$ we have:

$$(2.5) \qquad v(X^c) = (\frac{\partial X^i}{\partial x^j}(x) y^j + \frac{\partial G^i}{\partial y^j}(x,y) X^j)\frac{\partial}{\partial y^i}, y^j = \frac{dx^j}{dt}.$$

We observe that the equation $v(X^c) = 0$ is linear with respect to the components X^i of the vector fields X. So, with the initial condition $X(t_0) = X_0 \in T_{c(t_0)}M$ the equation $v(X^c) = 0$ has a unique solution. This solution is called *a parallel vector field along the curve c*.

Let $t_0 \in I$ and $\{E_i^0; i = \overline{1,n}\}$ be a basis for the tangent vector space $T_{c(t_0)}M$. Then there exist n parallel vector fields $\{E_1, ..., E_n\}$ along c with the initial conditions $E_i(t_0) = E_i^0$.

Next let $X = a^i E_i$ be the tangent vector of c and Y be a vector field along c. Then $Y = f^i E_i$. We obtain $Y^c = (f^i \circ \pi)E_i^c + \frac{df^i}{dt}E_i^v$. Since $v(E_i^c) = 0$ we have that $v(Y^c) = \frac{df^i}{dt}E_i^v$. The Jacobi Equation (JE) becomes

$$[\frac{df^i}{dt}E_i^v, X^c] + f^i\Omega(E_i, X) = 0,$$

which is equivalent with

$$\frac{d^2f^i}{dt^2}E_i^v + \frac{df^i}{dt}[E_i^v, X^c] + f^i\Omega(E_i, X) = 0.$$

Set $\Omega(E_i, E_j) = \Omega_{ij}^k E_k^v$, and $[E_i^v, X^c] = b_i^j E_j^v$. The last equation becomes:

$$\frac{d^2f^i}{dt^2} + \frac{df^j}{dt}b_j^i + f^j a^k \Omega_{jk}^i = 0. \tag{2.6}$$

The equation (2.6) is linear in $f = (f^i)$ and the given initial conditions determine in a unique way the vector field Y. **q.e.d.**

Corollary 2.1 *The set of Jacobi vector fields along a path c is a real linear space of dimension* $2n$.

Proof. We observe that if Y and Z are two vector fields along c and a, b are two real numbers the vector field $aY + bZ$ satisfies also (JE). Taking into account the above Proposition we have the statement.

3 Sprays and linear connection

Let S be a spray on $\widetilde{TM}$. We consider the nonlinear connection N induced by S, the vertical projector v given by (2.1), the horizontal projector $h = Id - v$ and the bundle isomorphism $\theta : V \to N$, $\theta(X^i\frac{\partial}{\partial y^i}) = X^i\frac{\delta}{\delta x^i}$.

Proposition 3.1 *The map $D : \chi(TM) \times \chi(TM) \to \chi(TM)$ given by:*
(3.1)
$$D_X Y = v[hX, vY] + h[vX, hY] + J[vX, (\theta \circ v)Y] + (\theta \circ v)[hX, JY],$$

is a linear connection on the tangent bundle, compatible with the nonlinear connection N, that means: $Dh = Dv = 0$.

Proof. By a straightforward computation one verifies:
1)$D_{fX}Y = fD_X Y, D_X fY = X(f)Y + fD_X Y$;
2)$D \circ J = 0, D \circ h = 0$.
This linear connection is called *the Berwald connection* and it appears in many papers [8], [1] in local coordinates. A global characterisation of this connection was given in [9], too.

Proposition 3.2 *In the basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ adapted to the nonlinear connection N, the Berwald connection D has the following form:*

$$D_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = F^k_{ji} \frac{\delta}{\delta x^k}, \; D_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = F^k_{ji} \frac{\partial}{\partial y^k}, \; F^k_{ji} = \frac{\partial N^k_i}{\partial y^j},$$
$$D_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = D_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0.$$

Next, the Berwald connection will be indicated by the set $B\Gamma = (N^i_j, F^k_{ji}, 0)$.

Remark 3.1 *For the Berwald connection we have the following formula:*

$$v(D^H_X Y) - h(D^V_Y X) = [h(X), v(Y)], \tag{3.2}$$

where:
$$D^H_X Y := D_{h(X)} Y; \; D^V_Y X := D_{v(Y)} X$$

are the h- and v-covariant derivatives of the Berwald connection. See also [9].

Proposition 3.3 *Let $c : I \subset \mathbb{R} \to M$ be a smooth curve, $X = \frac{dc}{dt}$ its tangent vector field and $\widetilde{c} : I \to TM$, $\widetilde{c}(t) = (c(t), \frac{dc}{dt})$. Then c is a path for S if and only if $\widetilde{c}$ is a geodesic for the Berwald connection D.*

Proof. We must prove that the conditions $v(X^c) = 0$ and $D_{X^c} X^c = 0$ are equivalent.

Assuming $v(X^c) = 0$, by (3.1) we have: $D_{X^c}X^c = (\theta \circ v)[X^c, J(X^c)] = (\theta \circ v)[X^c, X^v] = 0$.

Let now $D_{X^c}X^c = D_{X^c}h(X^c) + D_{X^c}v(X^c) = 0$. Since D preserve the horizontal and the vertical distributions then $D_{X^c}h(X^c) = D_{X^c}v(X^c) = 0$. But

$$D_{X^c}h(X^c) = h[v(X^c), h(X^c)] + (\theta \circ v)[h(X^c), J(X^c)] = 0.$$

It is very easy to check that $h[v(X^c), h(X^c)] = 0$. So we have $(\theta \circ v)[h(X^c), J(X^c)] = 0$. Using $h(X^c) = X^c - v(X^c)$, $J(X^c) = X^v$ and $[X^c, X^v] = 0$ we obtain $(\theta \circ v)[v(X^c), X^v] = 0$. Along the curve $\tilde{c}$ we have $\mathbb{C} = X^v$ and since $v(X^c)$ is a homogeneous vector field of degree two, $\mathcal{L}_{\mathbb{C}}(v(X^c)) = v(X^c)$. Thus $0 = -(\theta \circ v)(v(X^c)) = -\theta(v(X^c))$. It follows $v(X^c) = 0$. **q.e.d.**

Now, let c be a path for S, X be the tangent vector field and Y be a vector field along c. Since $h(X^c) = X^c$, $[X^c, Y^v] = [X, Y]^v$ and $[X^c, v(Y^c)]$ is a vertical vector field we have:

$$D_{X^c}Y^c = [X^c, v(Y^c)] + [X, Y]^h. \tag{3.2}$$

Let us consider the Nijenhuis tensor field $N_v(X, Y)$ of the vertical projector v. We have:

$$N_v(X^c, Y^c) = \Omega(X, Y).$$

Theorem 3.1 *Let $c : I \subset \mathbb{R} \to M$ be a geodesic for the Berwald connection D. Consider a variation of c, $\alpha : (-\varepsilon, \varepsilon) \times I \to M$, such that $\alpha(0, t) = c(t), \forall t \in I$ and $\alpha_s(t) := \alpha(s, t)$ are geodesics for D for every $s \in (-\varepsilon, \varepsilon)$. Let $X = \frac{\partial \alpha}{\partial t}|_{s=0}$ and $Y = \frac{\partial \alpha}{\partial s}|_{s=0}$. Then Y satisfies the following equation:*

$$v(D_{X^c}Y^c) + N_v(X^c, Y^c) = 0. \tag{3.3}$$

Proof. Taking into account (3.2) and $[X, Y] = 0$ we get $D_{X^c}Y^c = [X^c, v(Y^c)]$. Since c is a geodesic for D, according to Proposition 3.3, c is a path for S or equivalent a h-path for N. In these conditions Y is a Jacobi vector field and $N_v(X^c, Y^c) = \Omega(X, Y) = -[X^c, v(Y^c)]$, so (3.3) is proved. **q.e.d.**

Theorem 3.1 shows that the Jacobi equation (JE) and (3.3) are equivalent.

4 Examples

1. Let M be a manifold and S be a spray on TM. Then the local coefficients of S are given by $G^i(x, y) = \gamma^i_{jk}(x)y^j y^k$, where $\gamma^i_{jk}(x)$ are the

local coefficients of a symmetric linear connection ∇ on M. Then $S = y^i\frac{\partial}{\partial x^i} - \gamma^i_{jk}(x)y^jy^k\frac{\partial}{\partial y^i}$ is a spray on TM. A vector field that is a Jacobi vector field in the sense of Definition 2.1 is a Jacobi vector field in the classical sense for the linear connection ∇.

For the considered spray S the 2-form of curvature of the nonlinear connection associated to it is given by:

$$\Omega(X^i(x)\frac{\partial}{\partial x^i}|_x, Z^j(x)\frac{\partial}{\partial x^j}|_x) = R_i{}^k{}_{jp}(x)X^i(x)Z^j(x)y^p\frac{\partial}{\partial y^k}|_{(x,y)},$$

where R is the (1,3) curvature tensor of ∇. We consider c a geodesic, X the tangent vector field and Y the transverse vector field. A path for S is a geodesic for ∇. We have $v(Y^c) = (\nabla_X Y)^v$ and $v(D_{X^c}Y^c) = v(D_{X^h}Y^h + D_{X^h}v(Y^c)) = D_{X^h}(\nabla_X Y)^v = (\nabla^2_X Y)^v$. Then (JE) which is equivalent with (3.3) becomes

$$\nabla^2_X Y + R(Y,X)X = 0.$$

The last equations is the classical Jacobi equation for the linear connection ∇.

2. Suppose that the spray S is Lagrangian. This means that there is a map $L : \widetilde{TM} \to I\!R$ for which the matrix with the entries

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$$

is nondegenerate and the local coefficients G^i of the spray S are given by

$$2G^i = \frac{1}{2}g^{ij}(\frac{\partial^2 L}{\partial y^i \partial x^m}y^m - \frac{\partial L}{\partial x^j}).$$

Put $N^i_j = \frac{\partial G^i}{\partial y^j}$ and $F^i_{jk} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}$. Let $B\Gamma = (N^i_j, F^i_{jk}, 0)$ be the Berwald connection and R its curvature. For a geodesic c we consider the tangent vector field X and Y the transverse vector field. We observe that X is fixed. If we consider the X-covariant derivative D^X and the X-curvature R^X, the equation (3.3) reduces to that obtained by Z.Shen in ([10]):

$$(D^X)^2_X Y + R^X(Y) = 0.$$

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