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HELMHOLTZ TYPE CONDITION FOR MECHANICAL INTEGRATORS

Ion D. Albu **Dumitru Opriş***
West University of Timişoara, Romania

Abstract

The Helmholtz (self-adjoint) conditions for differential systems of second and first order have been heighlighted by many authors. On the other hand the discretization of systems of differential equations of first and second order leads to mechanical integrators which are described with difference equations. The paper establishes the Helmholtz conditions for variational integrators associated to discrete evolutive systems.

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1 Introduction

The mechanical integrators are numerical integration methods for mechanical systems simulation. They preserve some of the invariants of the mechanical system as energy, momentum or symplectic structure. The construction of the integrators uses time-stepping algorithms to approximate the continuous equations of motion. Concerning their classification there are explicit integrators and implicit integrators according to the nature of the numerical algorithms used in construction. The implicit integrators obtained by discretizing Hamilton's principle are called variational integrators and they are studied by many authors: Moser-Veselov [3], Wendlandt-Marsden [7], Marsden-Patrick-Shkoller [2], Albu-Opriş [5], Crăciun-Opriş [1], etc.

A variational integrator Φ satisfies some relations of the form:

$$A_i \circ \Phi(q_{k+1}, q_k) + B_i(q_{k+1}, q_k) = 0$$

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$$\Phi : Q^2 \to Q^2 \quad , \quad \Phi(q_{k+1}, q_k) = (q_{k+2}, q_{k+1})$$

as we shall see in the sequel.

The aim of this paper is to present necessary and sufficient conditions in order to an integrator Φ verifying the above relations be variational. The obtained conditions represent the analogous of the Helmholtz conditions for second order differential systems $F_i(x, \dot{x}, \ddot{x}) = 0$, $i = \overline{1,n}$, systematically studied by many authors among which Obădeanu-Marinca [4].

2 Discrete Variational Principle (DVP)

Let Q be the n-dimensional configuration manifold and consider a function $L : Q^2 \to \mathbf{R}$. We associate to L the corresponding action $S : Q^{N+1} \to \mathbf{R}$ defined by

$$S(q_0, ..., q_N) = \sum_{k=0}^{N-1} L(q_{k+1}, q_k) \tag{1}$$

where $q_k \in Q, k \in \mathbf{Z}$.

The discrete variational principle (DVP) seeks the sequences $(q_0, ..., q_N)$ for which the action S is stationary for all variations of $(q_0, ..., q_N)$ with q_0 and q_N fixed. This yields the *discrete Euler-Lagrange equations* (DEL):

$$\frac{\partial L}{\partial q_k^i}(q_{k+1}, q_k) + \frac{\partial L}{\partial q_k^i}(q_k, q_{k-1}) = 0 \ , \ k = \overline{1, N-1} \ , \ i = \overline{1,n} \tag{2}$$

or

$$\frac{\partial L}{\partial q_k^i} \circ \Phi(q_k, q_{k-1}) + \frac{\partial L}{\partial q_k^i}(q_k, q_{k-1}) = 0 \tag{3}$$

where the mapping $\Phi : Q^2 \to Q^2$ is defined implicitly by $\Phi(q_k, q_{k-1}) = (q_{k+1}, q_k)$. Φ is called a *variational integrator* of the evolutive system gouverned by L.

Example 1. Let $Q = \mathbf{R}$ and $L : \mathbf{R}^2 \to \mathbf{R}$ given by $L(q_{k+1}, q_k) = \frac{1}{2}(-1)^k(q_{k+1} - q_k)^2 + \frac{1}{2}(-1)^k q_k^2$. The equations 2.(2) lead to $q_{k+1} = q_k + q_{k-1}$. With $q_0 = 1$, $q_1 = 1$, the obtained equation represent the Fibonacci's sequence. The mapping $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ is given by $\Phi(q_k, q_{k-1}) = (q_k + q_{k-1}, q_k)$.

Besides DEL and variational integrator, DVP supplies a symplectic structure on Q^2.

Let θ be the canonical 1-form on T^*Q given in local coordinates by $\theta = p_i dq^i$. If we consider the fiber derivatives FL_1, $FL_2 : Q^2 \to T^*Q$

defined by

$$FL_1(q_{k+1}, q_k) = \left(q_k, \left(\frac{\partial L}{\partial q_k^i}(q_{k+1}, q_k)\right)\right) \tag{4}$$

$$FL_2(q_{k+1}, q_k) = \left(q_{k+1}, \left(\frac{\partial L}{\partial q_{k+1}^i}(q_{k+1}, q_k)\right)\right)$$

then we can define the 1-forms θ^+, θ^- on Q^2:

$$\theta^+ = FL_1^*(\theta) = \frac{\partial L}{\partial q_k^i}(q_{k+1}, q_k)dq_k^i \tag{5}$$

$$\theta^- = FL_2^*(\theta) = \frac{\partial L}{\partial q_{k+1}^i}(q_{k+1}, q_k)dq_{k+1}^i$$

by pulling back the canonical form θ on T^*Q.

Proposition 2.1. ([2], [7]). For L, Φ, θ^+, θ^- the following relations are satisfied:

$$\theta^+ + \theta^- = dL$$

$$\Phi^*(d\theta^+) = -d\theta^- \tag{6}$$

$$\Phi^*\omega = \omega \text{ , where } \omega = d\theta^-$$

Here

$$\omega = \frac{\partial^2 L}{\partial q_k^i \partial q_{k+1}^j}(q_{k+1}, q_k)dq_k^i \wedge dq_{k+1}^j$$

is the *symplectic form* on Q^2.

The last relation 2.(6) shows that Φ preserves the symplectic form i. e. Φ is *symplectic*.

Remark: We considered only discrete lagrangian systems for which L is time-independent. For the "nonautonomous" case a DVP and the variational integrator can be formulated by considering the fibered manifold $Q \to \mathbf{R}$ with dim $Q = n + 1$ ([1]).

3 Discrete Helmholtz Conditions for Integrators

We introduce a general class of discrete systems. We consider a 1-form $\bar{\theta}$ on Q^2 having the expression

$$\bar{\theta} = A_i(q_{k+1}, q_k)dq_k^i + B_i(q_{k+1}, q_k)dq_{k+1}^i \tag{1}$$

in a local coordinate system on Q^2.

We call an *evolutive system on* Q^2 *associated to* $\bar{\theta}$ the system

$$A_i(q_{k+1}, q_k) + B_i(q_k, q_{k-1}) = 0 \ , \ i = \overline{1,n} \tag{2}$$

A mapping $\Phi : Q^2 \to Q^2$, $\Phi(q_k, q_{k-1}) = (q_{k+1}, q_k)$ is said to be an *integrator associated to the system* 3.(2) if

$$A_i \circ \Phi(q_k, q_{k-1}) + B_i(q_k, q_{k-1}) = 0 \ , \ i = \overline{1,n} \tag{3}$$

The *inverse problem in the theory of the variational integrators* is to determine the necessary and sufficient conditions for the existence of a function $L : Q^2 \to \mathbf{R}$ such that the evolutive system 3.(3) derive from a DVP.

Theorem 3.1. The necessary and sufficient conditions for an evolutive system associated to the form $\bar{\theta}$ in order to come from a DVP are the following:

$$\begin{aligned}
\frac{\partial A_i}{\partial q_k^j}(q_{k+1}, q_k) &= \frac{\partial A_j}{\partial q_k^i}(q_{k+1}, q_k) \\
\frac{\partial B_i}{\partial q_{k+1}^j}(q_{k+1}, q_k) &= \frac{\partial B_j}{\partial q_{k+1}^i}(q_{k+1}, q_k) \\
\frac{\partial A_i}{\partial q_{k+1}^j}(q_{k+1}, q_k) &= \frac{\partial B_j}{\partial q_k^i}(q_{k+1}, q_k)
\end{aligned} \tag{4}$$

Proof: If 3.(2) comes from a DVP then there is a function $L : Q^2 \to \mathbf{R}$ such that

$$A_i(q_{k+1}, q_k) = \frac{\partial L}{\partial q_k^i}(q_{k+1}, q_k) \ , \ B_i(q_{k+1}, q_k) = \frac{\partial L}{\partial q_{k+1}^i}(q_{k+1}, q_k) \tag{5}$$

The functions given by 3.(5) satisfy the condition 3.(4).

Conversely if the relations 3.(4) hold then the 1-form $\bar{\theta}$ given by 3.(1) is closed. According to the Poincaré Lemma there exists locally a function $L : Q^2 \to \mathbf{R}$ such that $\bar{\theta} = dL$. L is given by

$$L(q_{k+1}, q_k) = \int_0^1 [q_k^i A_i(tq_{k+1}, tq_k) + q_{k+1}^i B_i(tq_{k+1}, tq_k)]dt \tag{6}$$

and 3.(2) comes from a DVP.

The relations 3.(4) are called the *discrete Helmholtz conditions* (DHC).

Example 2. Let $\bar{\theta} = q_{k+1} \ sin \ q_k dq_k - cos \ q_k dq_{k+1}$ be a 1-form on $\mathbf{R}^2$. The associated evolutive equation is $q_{k+1} \ sin \ q_k - cos \ q_{k-1} = 0$ and the corresponding integrator is $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$, $\Phi(q_k, q_{k-1}) = \left(\frac{cos \ q_{k-1}}{sin \ q_k}, q_{k-1}\right)$. The equation comes from a DVP because the functions $A(q_{k+1}, q_k) = q_{k+1} sin\ q_k$, $B(q_{k+1}, q_k) = -cos\ q_k$ satisfy 3.(4).

Example 3. Let $\bar{\theta} = (aq_{k+1} + bq_k)dq_k + (dq_{k+1}^2 + cq_k)dq_{k+1}$ be a 1-form on $\mathbf{R}^2$, where $a, b, c, d \in \mathbf{R}$, $a \neq 0$, $c \neq 0$. The associated evolutive equation is

$$aq_{k+1} + bq_k + cq_{k-1} + dq_k^2 = 0$$

and the integrator associated to this equation is $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$, $\Phi(q_k, q_{k-1}) = \left(-\frac{b}{a}q_k - \frac{c}{a}q_{k-1} - \frac{d}{a}q_k^2, q_k\right)$. The equation comes from a DVP if and only if $a = c$. In this case $L : \mathbf{R}^2 \to \mathbf{R}$ is given by $L(q_{k+1}, q_k) = aq_kq_{k+1} + \frac{1}{2}bq_k^2 + \frac{1}{3}dq_{k+1}^3$.

By considering the fiber derivatives FL_1, $FL_2 : Q^2 \to T^*Q$ associated to $\bar{\theta}$ and given by

$$FL_1(q_{k+1}, q_k) = (q_k, A_i(q_{k+1}, q_k))) \tag{7}$$

$$FL_2(q_{k+1}, q_k) = (q_{k+1}, B_i(q_{k+1}, q_k)))$$

we define the 1-forms on Q^2:

$$\theta^+ = FL_1^*(\theta)\ ,\ \theta^- = FL_2^*(\theta) \tag{8}$$

where θ is the canonical 1-form on T^*Q.

Proposition 3.2. The following relations are satisfied:

$$\theta^+ + \theta^- = \bar{\theta} \tag{9}$$

$$\Phi^*(\theta^+) = -\theta^-$$

Proof: From 3.(7) and 3.(8) we obtain:

$$\theta^+ = A_i(q_{k+1}, q_k)dq_k^i\ ,\ \theta^- = B_i(q_{k+1}, q_k)dq_{k+1}^i \tag{10}$$

which yield $\theta^+ + \theta^- = \bar{\theta}$. Let $\Phi(y, x) = (u, v)$ where $y = v = q_{k+1}$, $x = q_k$, $u = q_{k+2}$. Taking into account of 3.(3) we have

$$\Phi^*\theta^+ = \Phi^*(A_i(u, v)dv^i) = A_i \circ \Phi(y, x)d(v^i(y, x)) = -B_i(y, x)dy^i = -\theta^-$$

Proposition 3.3. The integrator $\Phi : Q^2 \to Q^2$ is symplectic if and only if DHC are satisfied.

Proof: From 3.(9) it results $\theta^+ - \Phi^*(\theta^+) = \bar{\theta}$ therefore $d\theta^+ - \Phi^*(d\theta^+) = d\bar{\theta}$. If the DHC 3.(4) are satisfied then $d\bar{\theta} = 0$ and $d\theta^+ = \Phi^*(d\theta^+)$. $d\theta^+$ is the symplectic form given by

$$d\theta^+ = \frac{\partial A_i}{\partial q_{k+1}^j}(q_{k+1}, q_k)dq_{k+1}^j \wedge q_k^i \tag{11}$$

Conversely if Φ is symplectic then $d\theta^+ = \Phi^*(d\theta^+)$ and $d\bar{\theta} = 0$ which implies DHC 3.(4).

4 Discrete Helmholtz Conditions for Integrators satisfying Decomposable Evolutive Systems

An evolutive system in the most general form is described by a subset of $Q^2 \times Q^2$ which is the graph of a system of equations:

$$F_i((q_{k+1}, q_k), (q_k, q_{k-1})) = 0 \ , \ i = \overline{1,n} \tag{1}$$

where $F_i : Q^2 \times Q^2 \to \mathbf{R}$. A mapping $\Phi : Q^2 \to Q^2$, $\Phi(q_k, q_{k-1}) = (q_{k+1}, q_k)$ is an *integrator associated to the system* 4.(1) if

$$F_i(\Phi(q_k, q_{k-1}), (q_k, q_{k-1})) = 0 \ , \ i = \overline{1,n} \tag{2}$$

The evolutive system 4.(1) is called *decomposable with respect to* Φ if there are some functions A_i, $B_i : Q^2 \to \mathbf{R}$, $i = \overline{1,n}$, such that

$$F_i = A_i \circ \Phi + B_i \quad , \quad i = \overline{1,n} \tag{3}$$

We shall say that *an evolutive system* of the form 4.(1) *comes from a DVP* if it is decomposable with respect to Φ and there exist a function $L : Q^2 \to \mathbf{R}$ such that

$$A_i(q_{k+1}, q_k) = \frac{\partial L}{\partial q_k^i}(q_{k+1}, q_k) \ , \ B_i(q_k, q_{k-1}) = \frac{\partial L}{\partial q_k^i}(q_k, q_{k-1})$$

Proposition 4.1. For $n > 1$ the system 4.(1) comes from a DVP if and only if

$$\frac{\partial F_i(k)}{\partial q_k^j} = \frac{\partial F_j(k)}{\partial q_k^i} \ , \ \frac{\partial F_i(k)}{\partial q_{k+1}^j} = \frac{\partial F_j(k+1)}{\partial q_k^i} \ , \ i, j = \overline{1,n} \tag{4}$$

where $F_i(k) \overset{not}{=} F_i((q_{k+1}, q_k), (q_k, q_{k-1}))$.

Proof: If 4.(1) comes from a DVP then it is decomposable; the left side has the form

$$F_i(k) = A_i(q_{k+1}, q_k) + B_i(q_k, q_{k-1}) = \frac{\partial L}{\partial q_k^i}(q_{k+1}, q_k) + \frac{\partial L}{\partial q_k^i}(q_k, q_{k-1}) \ , \ i = \overline{1,n} \tag{5}$$

The functions given by 4.(5) verify 4.(4).

Cosversely if 4.(4) hold then the first relations imply the existence of the functions $A, B : Q^2 \to \mathbf{R}$ such that

$$F_i(k) = \frac{\partial A}{\partial q_k^i}(q_{k+1}, q_k) + \frac{\partial B}{\partial q_k^i}(q_k, q_{k-1}) = A_i(q_{k+1}, q_k) + B_i(q_k, q_{k-1}) \quad (6)$$

From the other relations it results that the functions A_i, B_i verify that $A_i(q_{k+1}, q_k) = \frac{\partial L}{\partial q_k^i}(q_{k+1}, q_k)$, $B_i(q_k, q_{k-1}) = \frac{\partial L}{\partial q_k^i}(q_k, q_{k-1})$ and the system 4.(1) comes from a DVP.

Proposition 4.2. If $n = 1$ then the decomposable evolutive equation 4.(1) comes from a DVP if and only if

$$\frac{\partial F(k)}{\partial q_{k+1}} = \frac{\partial F(k+1)}{\partial q_k} \quad (7)$$

Example 4. Consider $F(k) = aq_{k+1} + bq_k + cq_{k-1}$ a function describing an evolutive system in the Samuelson-Hicks model. We have $F(k) = A(q_{k+1}, q_k) + B(q_k, q_{k-1})$, where $A(q_{k+1}, q_k) = aq_{k+1} + \frac{b}{2}q_k$, $B(q_{k+1}, q_k) = \frac{b}{2}q_k + cq_{k-1}$. This system is decomposable and it comes for a DVP if and only if $a = c$. Indeed $\frac{\partial F(k+1)}{\partial q_k} = \frac{\partial B}{\partial q_k}(q_{k+1}, q_k) = c$, $\frac{\partial F(k)}{\partial q_{k+1}} = \frac{\partial A}{\partial q_{k+1}}(q_{k+1}, q_k) = a$, therefore $a = c$.

We can associte to the functions F_i, $i = \overline{1, n}$, defining an evolutive system 4.(1), the Fréchet derivative

$$DF_i(k)(\eta) = \frac{\partial F_i(k)}{\partial q_{k+1}^j}\eta_{k+1}^j + 2\frac{\partial F_i(k)}{\partial q_k^j}\eta_k^j + \frac{\partial F_i(k)}{\partial q_{k-1}^j}\eta_{k-1}^j \quad (8)$$

and the adjoit operator

$$D^*F_i(k)(\eta) = \frac{\partial F_j(k-1)}{\partial q_k^i}\eta_{k-1}^j + 2\frac{\partial F_j(k)}{\partial q_k^i}\eta_k^j + \frac{\partial F_j(k+1)}{\partial q_k^i}\eta_{k+1}^j \quad (9)$$

where $\eta_k^j = \frac{d}{d\varepsilon}q_k^j(\varepsilon)\Big|_{\varepsilon=0}$, $i, j = \overline{1, n}$. [6]

An evolutive system is called self-adjoint if

$$D^*F_i(k)(\eta) = DF_i(k)(\eta)\ ,\ \forall\eta \quad (i = \overline{1, n}) \quad (10)$$

Proposition 4.3. [6] For $n > 1$ an evolutive system comes from a DVP if and only if it is self-adjoint.

Proposition 4.4. If $n = 1$ then a decomposable evolutive system comes from a DVP if and only if it is self-adjoint.

5 Discretization of Differential Systems

Let $\mathbf{L} : TQ \to \mathbf{R}$ be a Lagrange function; $\mathbf{L} = \mathbf{L}(x, \dot{x})$. The discretization of $\mathbf{L}$ can be maked by using the midpoint rule which consists in the substitution of x with $\frac{q_{k+1}+q_k}{2}$ and of $\dot{x}$ with $\frac{q_{k+1}-q_k}{h}$ where h is the time-step. One obtain a discrete Lagrangian $L : Q \times Q \to \mathbf{R}$,

$$L(q_{k+1}, q_k) = \mathbf{L}\left(\frac{q_{k+1}+q_k}{2}, \frac{q_{k+1}-q_k}{h}\right) \tag{1}$$

DVP for L yields an evolutive system:

$$F_i(k) \equiv \frac{1}{2}\left[\frac{\partial L}{\partial x^i}\left(\frac{q_{k+1}+q_k}{2}, \frac{q_{k+1}-q_k}{h}\right) + \frac{\partial L}{\partial x^i}\left(\frac{q_k+q_{k-1}}{2}, \frac{q_k-q_{k-1}}{h}\right)\right] + \tag{2}$$
$$+\frac{1}{h}\left[\frac{\partial L}{\partial \dot{x}^i}\left(\frac{q_k+q_{k-1}}{2}, \frac{q_k-q_{k-1}}{h}\right) - \frac{\partial L}{\partial \dot{x}^i}\left(\frac{q_{k+1}+q_k}{2}, \frac{q_{k+1}-q_k}{h}\right)\right] = 0 \ , \ i = \overline{1,n}$$

which is decomposable because

$$F_i(k) = A_i(q_{k+1}, q_k) + B_i(q_k, q_{k-1}) \ , \ i = \overline{1,n} \tag{3}$$

where

$$A_i(q_{k+1}, q_k) = \frac{1}{2}\frac{\partial L}{\partial x^i}\left(\frac{q_{k+1}+q_k}{2}, \frac{q_{k+1}-q_k}{h}\right) - \frac{1}{h}\frac{\partial L}{\partial \dot{x}^i}\left(\frac{q_{k+1}+q_k}{2}, \frac{q_{k+1}-q_k}{h}\right) \tag{4}$$
$$B_i(q_k, q_{k-1}) = \frac{1}{2}\frac{\partial L}{\partial x^i}\left(\frac{q_k+q_{k-1}}{2}, \frac{q_k-q_{k-1}}{h}\right) + \frac{1}{h}\frac{\partial L}{\partial \dot{x}^i}\left(\frac{q_k+q_{k-1}}{2}, \frac{q_k-q_{k-1}}{h}\right)$$

Generally by discretizing the Euler-Lagrange equations with the midpoint method we not obtain a decomposable evolutive system, consequently this system is not derived from a DVP. Moreover it differs from the system obtained by applying DVP to the discretized Lagrangian.

Example 5. Consider the Lagrangian of the harmonic osscilator $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}ax^2$, $a > 0$. The Euler-Lagrange equation is $\ddot{x} + ax = 0$. By discretizing this equation we obtain $(2 + ah^2)q_{k+1} + (ah^2 - 4)q_k + 2q_{k-1} = 0$ or $q_{k+1} = -\frac{ah^2-4}{2+ah^2}q_k - \frac{2}{2+ah^2}q_{k-1}$. By discretizing L we have $\mathbf{L}(q_{k+1}, q_k) = \frac{1}{2h^2}(q_{k+1} - q_k)^2 - \frac{a}{4}(q_{k+1} + q_k)^2$ which yields DEL: $\frac{2+ah^2}{2h^2}q_{k+1} - \frac{2-ah^2}{h^2}q_k + \frac{2+a}{2h^2}q_{k-1} = 0$ or $q_{k+1} = \frac{2(2-ah^2)}{2+ah^2}q_k - q_{k-1}$.

For the first order differentiable systems given by

$$\dot{x}^i - f^i(x) = 0 \quad , \quad i = \overline{1,n} \tag{5}$$

the midpoint rule is an usual method of discretization. From 5.(5) we obtain the discretized system:

$$q^i_{k+1} - q^i_k - hf^i\left(\frac{q_{k+1}+q_k}{2}\right) = 0\ ,\ i = \overline{1,n} \tag{6}$$

Consider the associated evolutive system:

$$F_i(k) \equiv \delta_{ij}\left(q^j_{k+1} - q^j_k - hf^j\left(\frac{q_{k+1}+q_k}{2}\right)\right) - \tag{7}$$

$$-\delta_{ij}\left(q^j_k - q^j_{k-1} - hf^j\left(\frac{q_k+q_{k-1}}{2}\right)\right) = 0$$

According to Proposition 4.1. we obtain

Proposition 5.1. The system 5.(6) comes from DVP if and only if the vector field $f = (f^i)$ is irotational that is $\frac{\partial f^i}{\partial x^j} = \frac{\partial f^j}{\partial x^i}$, $i, j = \overline{1,n}$ or 5.(6) is of gradient type.

Example 6. Consider the system $\dot{x}^1 = x^2$, $\dot{x}^2 = x^1$. By discretizing this system we have $q^1_{k+1} - q^1_k = \frac{h}{2}(q^2_{k+1} + q^2_k)$, $q^2_{k+1} - q^2_k = \frac{h}{2}(q^1_{k+1} + q^1_k)$ and the associated evolutive system is the following

$$q^1_{k+1} - 2q^1_k + q^1_{k-1} - \frac{h}{2}(q^2_{k+1} + 2q^2_k + q^2_{k-1}) = 0\ ,$$

$$q^2_{k+1} - 2q^2_k + q^2_{k-1} - \frac{h}{2}(q^1_{k+1} + 2q^1_k + q^1_{k-1}) = 0\ .$$

The corresponding Lagrangian is

$$L(q_{k+1}, q_k) = (q^1_{k+1} - q^1_k)^2 + (q^2_{k+1} - q^2_k)^2 + h(q^1_{k+1} + q^1_k)(q^2_{k+1} + q^2_k)$$

6 Integrating Factor

Generally the 1-form $\bar{\theta}$ on Q^2 given by 3.(1) is not closed which is an obstacle into decide if the evolutive system 3.(2) associated to $\bar{\theta}$ comes from a DVP. This is the reason for introducing the integrating factor method.

Recall that $\bar{\theta} = A_i(q_{k+1}, q_k)dq^i_k + B_i(q_{k+1}, q_k)dq^i_{k+1}$ and the evolutive system associated to $\bar{\theta}$ is $A_i(q_{k+1}, q_k) + B_i(q_{k+1}, q_k) = 0\ ,\ i = \overline{1,n}$.

A nondegenerate tensor field C of type $(1,1)$ on Q^2 is called an *integrating factor* for $\bar{\theta}$, respectively for the associated system, if the 1-form $\bar{\bar{\theta}} = \bar{\theta} \circ C$ is closed ($det\ C \neq 0$).

Such a tensor field C on Q^2 is locally given by

$$C = C^i_j(k)\frac{\partial}{\partial q^i_k}\otimes dq^j_k + E^i_j(k)\frac{\partial}{\partial q^i_k}\otimes dq^j_{k+1} + \tag{1}$$
$$+F^i_j(k)\frac{\partial}{\partial q^i_{k+1}}\otimes dq^j_k + G^i_j(k)\frac{\partial}{\partial q^i_{k+1}}\otimes dq^j_{k+1}$$

where $C^i_j(k) \overset{not}{=} C^i_j(q_{k+1},q_k)$, $E^i_j(k) \overset{not}{=} E^i_j(q_{k+1},q_k)$, $F^i_j(k) \overset{not}{=} F^i_j(q_{k+1},q_k)$, $G^i_j(k) \overset{not}{=} G^i_j(q_{k+1},q_k)$. It result that

$$\bar{\bar{\theta}} = (A_i(k)C^i_j(k) + B_i(k)F^i_j(k))dq^j_k + (A_i(k)E^i_j(k) + B_i(k)G^i_j(k))dq^j_{k+1} \tag{2}$$

By applying Theorem 3.1. we deduce

Proposition 6.1. The decomposable evolutive system 3.(2) admits an integrating factor C if and only if

$$C^i_h\left(\delta^h_j\frac{\partial A_i}{\partial q^l_k} - \delta^h_l\frac{\partial A_i}{\partial q^j_k}\right) + \left(\frac{\partial C^i_j}{\partial q^l_k} - \frac{\partial C^i_l}{\partial q^j_k}\right)A_i +$$
$$+F^i_h\left(\delta^h_j\frac{\partial B_i}{\partial q^l_k} - \delta^h_l\frac{\partial A_i}{\partial q^j_k}\right) + \left(\frac{\partial F^i_j}{\partial q^l_k} - \frac{\partial F^i_l}{\partial q^j_k}\right)B_i = 0\ ,$$
$$E^i_h\left(\delta^h_j\frac{\partial A_i}{\partial q^l_{k+1}} - \delta^h_l\frac{\partial A_i}{\partial q^j_{k+1}}\right) + \left(\frac{\partial E^i_j}{\partial q^l_{k+1}} - \frac{\partial E^i_l}{\partial q^j_{k+1}}\right)A_i + \tag{3}$$
$$+G^i_h\left(\delta^h_j\frac{\partial B_i}{\partial q^l_{k+1}} - \delta^h_l\frac{\partial A_i}{\partial q^j_{k+1}}\right) + \left(\frac{\partial G^i_j}{\partial q^l_{k+1}} - \frac{\partial G^i_l}{\partial q^j_{k+1}}\right)B_i = 0\ ;$$
$$\frac{\partial A_i}{\partial q^l_{k+1}}C^i_j - \frac{\partial A_i}{\partial q^l_k}E^i_j + \left(\frac{\partial C^i_j}{\partial q^l_{k+1}} - \frac{\partial E^i_j}{\partial q^l_k}\right)A_i +$$
$$+\frac{\partial B_i}{\partial q^l_{k+1}}F^i_j - \frac{\partial B_i}{\partial q^l_k}G^i_j + \left(\frac{\partial F^i_j}{\partial q^l_{k+1}} - \frac{\partial G^i_j}{\partial q^l_k}\right)B_i = 0\ .$$

In the concrete problems the integrating factor is sought such that the system 6.(3) be as simple as possible.

Example 7. Consider the evolutive equation given by $aq_{k+1} + bq_k + cq_{k-1} + dq_k^2 = 0$, $a \neq c$.

This system is not derived from a DVP. The associated 1-form is $\bar{\theta} = (aq_{k+1} + bq_k + dq_k^2)dq_k + cq_k dq_{k+1}$. We seek an integrating factor of the form $C\frac{\partial}{\partial q_k}\otimes dq_k + G\frac{\partial}{\partial q_{k+1}}\otimes dq_{k+1}$, where $C, G \in \mathbf{R}$. The conditions 6.(3) lead to $aC - cG = 0$ that is $G = \frac{a}{c}C$. It results that $\bar{\bar{\theta}} = (aq_{k+1} + bq_k + d_k^2)dq_k + aq_k dq_{k+1}$. $\bar{\bar{\theta}}$ is closed. A corresponding discrete Lagrangian is $L(q_{k+1},q_k) = aq_kq_{k+1} + \frac{b}{2}q_k^2 + \frac{d}{3}q_k^3$.

References

[1] D. Crăciun, D. Opriş, The Helmholtz conditions for first and second order infinite differential equations systems, An. Univ. Timişoara, 1997 (to appear).

[2] J. E. Marsden, G. W. Patrick, S. Shkoller, Variational Methods in Continuous and Discrete Mechanics and Field Theory, 1997.

[3] J. Moser, A. P. Veselov, Discrete Versions of some Classical Integrable Systems and factorization of Matrix Polynomials, 1989.

[4] V. Obădeanu, V. Marinca, Le problème inverse dans la Mécanique Newtonienne, Sem. Mecanică, Univ. Timişoara, Nr. 1 (1986), Nr. 2 (1987), Nr. 6 (1988).

[5] D. Opriş, I. D. Albu, On the discrete-continuous dynamical systems, An. Univ. "Al. I. Cuza", Iaşi, 1995.

[6] D. Opriş, I. Butulescu, Metode geometrice in studiul sistemelor de ecuaţii diferenţiale, Ed. Mirton, Timişoara, 1997.

[7] J. M. Wendlandt, J. E. Marsden, Mechanical Integrators Derived from a Discrete Variational Principle, 1997.