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ON FUZZY IDEALS IN HILBERT ALGEBRAS

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Abstract

The fuzzification of ideals introduced by I. Chajda and R. Halaš is presented. We show that any such ideal can be realized as a level of some fuzzy set and discuss the relation between fuzzy ideals and fuzzy deductive systems. The Cartesian product of fuzzy ideals is also considered.

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1. Introduction

The concept of Hilbert algebra was introduced in the early 50s by L. Henkin and T. Skolem for some investigations of implication in intuitionistic and other non-classical logics. In the 60s, these algebras were studied especially by A. Horn and A. Diego from the algebraic point of view. A. Diego proved (cf. [5]) that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by D. Busneag (cf. [2, 3]) and Y. B. Jun (cf. [9])

and their filters forming deductive systems were recognized recently. Fuzzy deductive systems are described by S. M. Hong and Y. B. Jun (cf. [8, 10]). I. Chajda and R. Halaš introduced in [4] the concept of ideals in Hilbert algebras and described connections between such ideals and congruences. In [7] is proved that every such ideal is a deductive system.

Our present paper is concerned with the fuzzification of ideals. We show that every ideal can be realized as a level ideal of some fuzzy ideal and discuss the relations between fuzzy ideals and fuzzy deductive systems. The Cartesian product of fuzzy relations is considered also.

Since there exist various modifications of the definition of Hilbert algebra, we use the one from [2].

Definition 1.1. A *Hilbert algebra* is a triplet $\mathcal{H} = (H;\cdot,\mathbf{1})$, where H is a nonempty set, $\cdot$ is a binary operation and $\mathbf{1}$ is a fixed element of H such that the following axioms hold for each $x, y, z \in H$:

(I) $x\cdot(y\cdot x) = \mathbf{1}$,

(II) $(x\cdot(y\cdot z))\cdot((x\cdot y)\cdot(x\cdot z)) = \mathbf{1}$,

(III) $x\cdot y = \mathbf{1}$ and $y\cdot x = \mathbf{1}$ imply $x = y$.

In the sequel, a binary multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. In this convention the above axioms will be written in the form:

(I) $x\cdot yx = \mathbf{1}$,

(II) $(x\cdot yz)\cdot(xy\cdot xz) = \mathbf{1}$,

(III) $xy = \mathbf{1}$ and $yx = \mathbf{1}$ imply $x = y$.

The following result was proved (cf. for example [5]).

Lemma 1.2. *Let $\mathcal{H} = (H;\cdot,\mathbf{1})$ be a Hilbert algebra and $x, y, z \in H$. Then*

(1) $xx = \mathbf{1}$,

(2) $\mathbf{1}x = x$,

(3) $x\mathbf{1} = \mathbf{1}$,

(4) $x \cdot yz = y \cdot xz$,

(5) $x \cdot yz = xy \cdot xz$.

It is easily checked that in a Hilbert algebra $\mathcal{H}$ the relation $\leq$ defined by

$$x \leq y \iff xy = \mathbf{1}$$

is a partial order on H with $\mathbf{1}$ as the largest element.

Definition 1.3. A subset D of a Hilbert algebra $\mathcal{H}$ is called *a deductive system* if it satisfies

(a) $\mathbf{1} \in I$,

(b) $x \in D$ and $xy \in I$ imply $y \in D$.

Definition 1.4. ([4]) A subset I of a Hilbert algebra $\mathcal{H}$ is called *an ideal* if it satisfies

(i) $\mathbf{1} \in I$,

(ii) $xy \in I$ for $x \in H$ and $y \in I$,

(iii) $(y_1 \cdot y_2 x)x \in I$ for $y_1, y_2 \in I$ and $x \in H$.

In a Hilbert algebra every ideal is a deductive system (cf. [7]). It is also a subalgebra. Moreover, every ideal may be written as a union of some deductive systems (cf. [7]).

2. Fuzzy ideals in Hilbert algebras

According to the general idea presented by L. A. Zadeh (cf. [11]), every function $\mu : X \to [0,1]$ is called a *fuzzy set* on X. The set $\mu_t = \{x \in X : \mu(x) \geq t\}$, where $t \in [0,1]$ is fixed, is called a *level subset of* μ. $Im(\mu)$ denotes the image set of μ. For any fuzzy sets μ and ρ in X, we define

$$\mu \subseteq \rho \iff \mu(x) \leq \rho(x) \text{ for all } x \in X.$$

In a Hilbert algebra $\mathcal{H}$ by H_μ we denote the set $\{x \in H : \mu(x) = \mu(\mathbf{1})\}$.

Definition 2.1. A fuzzy set μ in a Hilbert algebra $\mathcal{H}$ is called *a fuzzy ideal* if

(F1) $\mu(1) \geq \mu(x), \ \forall x \in H,$

(F2) $\mu(xy) \geq \mu(y), \ \forall x, y \in H,$

(F3) $\mu((y_1 \cdot y_2 x)x) \geq \min\{\mu(y_1), \mu(y_2)\}, \ \forall x, y_1, y_2 \in H.$

Observe that (F1) follows from (F2) and (1). Using (F2) we known that every fuzzy ideal is a fuzzy subalgebra in the sense of [6]. Moreover, putting $y_1 = y$ and $y_2 = 1$ in (F3) we obtain the following proposition.

Proposition 2.2. *If μ is a fuzzy ideal of a Hilbert algebra $\mathcal{H}$, then*

$$\mu(yx \cdot x) \geq \mu(y), \ \forall x, y \in H.$$

Corollary 2.3. *Every fuzzy ideal μ of a Hilbert algebra is order preserving, i.e. $\mu(x) \leq \mu(y)$ for $x \leq y$.*

Proof. Let $x, y \in H$ be such that $x \leq y$. Then

$$\mu(y) = \mu(1y) = \mu(xy \cdot y) \geq \mu(x),$$

ending the proof. □

Definition 2.4. A fuzzy set μ in a Hilbert algebra $\mathcal{H}$ is called a *fuzzy deductive system* if

(a) $\mu(1) \geq \mu(x), \ \forall x \in H,$

(b) $\mu(y) \geq \min\{\mu(xy), \mu(x)\}, \ \forall x, y \in H.$

Proposition 2.5. *A fuzzy ideal is a fuzzy deductive system.*

Proof. Let μ be a fuzzy ideal. Since (a) and (F1) are equivalent we must verify only (b). If $y_1 = xy$, $y_2 = x$, where $x, y \in H$, then by (1), (2) and (F3) we obtain

$$\mu(y) = \mu(1y) = \mu((xy \cdot xy)y) \geq \min\{\mu(xy), \mu(x)\},$$

which proves (b). Hence μ is a fuzzy deductive system. □

Proposition 2.6. *Let A be a nonempty subset of a Hilbert algebra $\mathcal{H}$ and let μ_A be a fuzzy set in $\mathcal{H}$ defined by*

$$\mu_A(x) = \begin{cases} t_1 & \text{if } x \in A, \\ t_2 & \text{otherwise,} \end{cases}$$

where $t_1 > t_2$ in $[0,1]$. Then μ_A is a fuzzy ideal of $\mathcal{H}$ if and only if A is an ideal of $\mathcal{H}$. Moreover, $H_{\mu_A} = A$.

Proof. Assume that μ_A is a fuzzy ideal of $\mathcal{H}$. Since $\mu_A(\mathbf{1}) \geq \mu_A(x)$ for all $x \in H$, we have $\mu_A(\mathbf{1}) = t_1$ and so $\mathbf{1} \in A$. Let $x \in H$ and $y \in A$. Then $\mu_A(xy) \geq \mu_A(y) = t_1$ and thus $\mu_A(xy) = t_1$. Hence $xy \in A$. For any $y_1, y_2 \in A$ and $x \in H$, we get $\mu_A((y_1 \cdot y_2x)x) \geq \min\{\mu_A(y_1), \mu_A(y_2)\} = t_1$, which implies that $\mu_A((y_1 \cdot y_2x)x) = t_1$. It follows that $(y_1 \cdot y_2x)x \in A$. Therefore A is an ideal of $\mathcal{H}$.

Conversely, suppose that A is an ideal of $\mathcal{H}$. Since $\mathbf{1} \in A$, it follows that $\mu_A(\mathbf{1}) = t_1 \geq \mu_A(x)$ for all $x \in H$. Let $x, y \in H$. If $y \in A$, then $xy \in A$ and so $\mu_A(xy) = t_1 = \mu_A(y)$. If $y \in H \setminus A$, then $\mu_A(y) = t_2$ and hence $\mu_A(xy) \geq t_2 = \mu_A(y)$. Finally, let $y_1, y_2, x \in H$. If $y_1 \in H \setminus A$ or $y_2 \in H \setminus A$, then $\mu_A(y_1) = t_2$ or $\mu_A(y_2) = t_2$. It follows that

$$\mu_A((y_1 \cdot y_2x)x) \geq t_2 = \min\{\mu_A(y_1), \mu_A(y_2)\}.$$

Assume that $y_1, y_2 \in A$. Then $(y_1 \cdot y_2x)x \in A$ and thus

$$\mu_A((y_1 \cdot y_2x)x) = t_1 = \min\{\mu_A(y_1), \mu_A(y_2)\}.$$

Hence μ_A is a fuzzy ideal of $\mathcal{H}$. □

Proposition 2.7. *A fuzzy set of a Hilbert algebra $\mathcal{H}$ is a fuzzy ideal if and only if for every $t \in [0,1]$, μ_t is either empty or an ideal of $\mathcal{H}$.*

Proof. If μ is a fuzzy ideal of $\mathcal{H}$ and $\mu_t \neq \emptyset$, then $\mathbf{1} \in \mu_t$ since $\mu(\mathbf{1}) \geq \mu(x)$ for every $x \in H$. Moreover, (F2) proves that $xy \in \mu_t$ for every $y \in \mu_t$ and $x \in H$. In a similar way, (F3) implies $(y_1 \cdot y_2x)x \in \mu_t$ for $y_1, y_2 \in \mu_t$. Thus μ_t is an ideal.

Assume now that every nonempty μ_t is an ideal. If $\mu(\mathbf{1}) \geq \mu(x)$ is not true, then there exists $x_0 \in H$ such that $\mu(\mathbf{1}) < \mu(x_0)$. But in this case for $s = \frac{1}{2}(\mu(\mathbf{1}) + \mu(x_0))$ we have $\mu(\mathbf{1}) < s < \mu(x_0)$. Thus $x_0 \in \mu_s$, i.e. $\mu_s \neq \emptyset$.

Since, by the assumption, μ_s is an ideal, then $\mu(\mathbf{1}) \geq s$, which is impossible. Therefore $\mu(\mathbf{1}) \geq \mu(x)$ for all $x \in H$.

If (F2) is false, then $\mu(x_0 y_0) < \mu(y_0)$ for some $x_0, y_0 \in H$. Let

$$t = \frac{1}{2}(\mu(x_0 y_0) + \mu(y_0)).$$

Then $t \in [0, 1]$ and $\mu(x_0 y_0) < t < \mu(y_0)$, which proves that $y_0 \in \mu_t$. In this case too, $\mu(x_0 y_0) \in \mu_t$, because μ_t is an ideal. Hence $\mu(x_0 y_0) \geq t$, a contradiction. Thus (F2) must be satisfied.

Finally, if (F3) is not true, then there are $u_0, v_0, x_0 \in H$ such that

$$\mu((u_0 \cdot v_0 x_0) x_0) < \min\{\mu(u_0),\, \mu(v_0)\}.$$

But in this case, for

$$p = \frac{1}{2}(\mu((u_0 \cdot v_0 x_0) x_0) + \min\{\mu(u_0),\, \mu(v_0)\})$$

we have $\mu((u_0 \cdot v_0 x_0) x_0) < p < \min\{\mu(u_0),\, \mu(v_0)\}$, which implies $u_0, v_0 \in \mu_p$, and, in the consequence, $(u_0 \cdot v_0 x_0) x_0 \in \mu_p$. This contradiction proves that (F3) is true and μ is a fuzzy ideal. □

Since every (fuzzy) ideal of a Hilbert algebra is a (fuzzy) deductive system, then from the result proved in [6] we obtain the following corollaries.

Corollary 2.8. *Two level ideals μ_s and μ_t $(s < t)$ of a Hilbert algebra $\mathcal{H}$ are equal if and only if there is no $x \in H$ such that $s \leq \mu(x) < t$.*

Corollary 2.9. *Let μ be a fuzzy ideal with finite image. If $\mu_s = \mu_t$ for some $s, t \in Im(\mu)$, then $s = t$.*

Corollary 2.10. *Let μ be a fuzzy ideal of a Hilbert algebra $\mathcal{H}$ and let $x \in G$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.*

Corollary 2.11. *Let μ and ρ be two fuzzy ideals of $\mathcal{H}$ with identical family of level subalgebras. Then $\mu = \rho$ if and only if $Im(\mu) = Im(\rho)$.*

For an endomorphism f of a Hilbert algebra $\mathcal{H}$ and a fuzzy set μ in $\mathcal{H}$, we define a new fuzzy set μ^f in $\mathcal{H}$ by $\mu^f(x) = \mu(f(x))$ for all $x \in H$.

Proposition 2.12. *Let f be an endomorphism of a Hilbert algebra $\mathcal{H}$. If μ is a fuzzy ideal of $\mathcal{H}$, then so is μ^f.*

Proposition 2.13. *Let f be an automorphism of a Hilbert algebra $\mathcal{H}$ and μ a fuzzy ideal of $\mathcal{H}$. Then $\mu^f = \mu$ if and only if $f(\mu_t) = \mu_t$ for all $t \in Im(\mu)$.*

Proof. Assume that $\mu^f = \mu$, $t \in Im(\mu)$ and $x \in \mu_t$. Then $\mu^f(x) = \mu(x) \geq t$, i.e. $\mu(f(x)) \geq t$, and so $f(x) \in \mu_t$, i.e. $f(\mu_t) \subseteq \mu_t$. Now, let $x \in \mu_t$ and let $y \in H$ be such that $f(y) = x$. Then $\mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq t$, whence $y \in \mu_t$, so that $x = f(y) \in f(\mu_t)$. Consequently, $\mu_t \subseteq f(\mu_t)$. Hence $f(\mu_t) = \mu_t$ for every $t \in Im(\mu)$.

Conversely, suppose that $f(\mu_t) = \mu_t$ for every $t \in Im(\mu)$. If $\mu(x) = t$, then, by virtue of Corollary 2.10, $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$. It follows from the hypothesis that $f(x) \in f(\mu_t) = \mu_t$, so that $\mu^f(x) = \mu(f(x)) \geq t$. Let $s = \mu^f(x)$ and assume that $s > t$. Then $f(x) \in \mu_s = f(\mu_s)$, which implies from the injectivity of f that $x \in \mu_s$, a contradiction. Hence $\mu^f(x) = \mu(f(x)) = t = \mu(x)$, which completes the proof. □

3. Normal fuzzy ideals

Definition 3.1. A fuzzy ideal μ of a Hilbert algebra $\mathcal{H}$ is called *normal* if there exists $x \in H$ such that $\mu(x) = 1$.

For a normal fuzzy ideal μ we obviously have $\mu(\mathbf{1}) = 1$. Thus a fuzzy ideal μ is normal if and only if $\mu(\mathbf{1}) = 1$.

Proposition 3.2. *Given a fuzzy ideal μ of $\mathcal{H}$, let μ^+ be a fuzzy set in $\mathcal{H}$ defined by $\mu^+(x) = \mu(x) + 1 - \mu(\mathbf{1})$ for all $x \in H$. Then μ^+ is a normal fuzzy ideal of $\mathcal{H}$ which contains μ.*

Proof. For all $x, y \in H$ we have $\mu^+(\mathbf{1}) = \mu(\mathbf{1}) + 1 - \mu(\mathbf{1}) = 1 \geq \mu^+(x)$ and

$$\mu^+(xy) = \mu(xy) + 1 - \mu(\mathbf{1}) \geq \mu(y) + 1 - \mu(\mathbf{1}) = \mu^+(y),$$

which proves (F1) and (F2) for μ^+. To prove (F3) note that

$$\begin{aligned}
\mu^+((y_1 \cdot y_2x)x) &= \mu((y_1 \cdot y_2x)x) + 1 - \mu(\mathbf{1}) \\
&\geq \min\{\mu(y_1),\ \mu(y_2)\} + 1 - \mu(\mathbf{1}) \\
&= \min\{\mu(y_1) + 1 - \mu(\mathbf{1}),\ \mu(y_2) + 1 - \mu(\mathbf{1})\} \\
&= \min\{\mu^+(y_1),\ \mu^+(y_2)\}
\end{aligned}$$

for all $y_1, y_2, x \in H$. This shows that (F3) holds and μ^+ is a fuzzy ideal of $\mathcal{H}$. Clearly, $\mu \subseteq \mu^+$, which completes the proof. □

Corollary 3.3. *If $\mu^+(x_0) = 0$ for some $x_0 \in H$, then also $\mu(x_0) = 0$.*

Using Proposition 2.6, we see that for any ideal A of $\mathcal{H}$ the characteristic function χ_A of A is a normal fuzzy ideal of $\mathcal{H}$. It is clear that μ is normal if and only if $\mu^+ = \mu$.

Proposition 3.4. *If μ is a fuzzy ideal of $\mathcal{H}$, then $(\mu^+)^+ = \mu^+$. Moreover, if μ is normal, then $(\mu^+)^+ = \mu$.*

Proposition 3.5. *If μ and ν are fuzzy ideals of $\mathcal{H}$ such that $\mu \subseteq \nu$ and $\mu(\mathbf{1}) = \nu(\mathbf{1})$, then $H_\mu \subseteq H_\nu$.*

Proof. Let $x \in H_\mu$. Then $\nu(x) \geq \mu(x) = \mu(\mathbf{1}) = \nu(\mathbf{1})$ and so $\nu(x) = \nu(\mathbf{1})$, i.e. $x \in H_\nu$. Therefore $H_\mu \subseteq H_\nu$. □

Corollary 3.6. *If μ and ν are normal fuzzy ideals of $\mathcal{H}$ such that $\mu \subseteq \nu$, then $H_\mu \subseteq H_\nu$.*

Proposition 3.7. *Let μ be a fuzzy ideal of $\mathcal{H}$. If there exists a fuzzy ideal ν of $\mathcal{H}$ such that $\nu^+ \subseteq \mu$, then μ is normal.*

Proof. Assume that there exists a fuzzy ideal ν of $\mathcal{H}$ such that $\nu^+ \subseteq \mu$. Then $1 = \nu^+(\mathbf{1}) \leq \mu(\mathbf{1})$, and so $\mu(\mathbf{1}) = 1$ and we are done. □

Proposition 3.8. *Let μ be a fuzzy ideal of $\mathcal{H}$ and let $f : [0, \mu(\mathbf{1})] \to [0, 1]$ be an increasing function. Then a fuzzy set $\mu_f : H \to [0, 1]$, defined by $\mu_f(x) := f(\mu(x))$ for all $x \in H$, is a fuzzy ideal of $\mathcal{H}$. In particular, if $f(\mu(\mathbf{1})) = 1$, then μ_f is normal. Moreover, μ is contained in μ_f if $f(t) \geq t$ for all $t \in [0, \mu(\mathbf{1})]$.*

Proof. It is not difficult to verify that μ_f is a normal fuzzy ideal of $\mathcal{H}$. If $f(t) \geq t$ for all $t \in [0, \mu(\mathbf{1})]$, then $\mu_f(x) = f(\mu(x)) \geq \mu(x)$ for all $x \in H$, which proves that μ is contained in μ_f. □

Denote by $\mathcal{N}(\mathcal{H})$ the set of all normal fuzzy ideals of $\mathcal{H}$. Note that $\mathcal{N}(\mathcal{H})$ is a poset under the set inclusion.

Theorem 3.9. *Let $\mu \in \mathcal{N}(\mathcal{H})$ be a non-constant such that it is a maximal element of $(\mathcal{N}(\mathcal{H}), \subseteq)$. Then μ takes only the values 0 and 1.*

Proof. Note that $\mu(\mathbf{1}) = 1$ since μ is normal. Let $x \in H$ be such that $\mu(x) \neq 1$. We claim that $\mu(x) = 0$. If not, then there exists $x_0 \in H$ such that $0 < \mu(x_0) < 1$. Let ν be a fuzzy set in $\mathcal{H}$ defined by $\nu(x) = \frac{1}{2}(\mu(x) + \mu(x_0))$ for all $x \in H$. Then clearly ν is well-defined. Moreover, as it is easy to see, ν is a fuzzy ideal of $\mathcal{H}$. From Proposition 3.2 follows that $\nu^+ \in \mathcal{N}(\mathcal{H})$, where ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(\mathbf{1})$ for all $x \in H$. Clearly, $\nu^+(x) \geq \mu(x)$ for all $x \in H$. Note that

$$\begin{aligned} \nu^+(x_0) &= \nu(x_0) + 1 - \nu(\mathbf{1}) \\ &= \frac{1}{2}(\mu(x_0) + \mu(x_0)) + 1 - \frac{1}{2}(\mu(\mathbf{1}) + \mu(x_0)) \\ &= \frac{1}{2}(\mu(x_0) + 1) > \mu(x_0) \end{aligned}$$

and $\nu^+(x_0) < 1 = \nu^+(\mathbf{1})$. Hence ν^+ is non-constant, and μ is not a maximal element of $\mathcal{N}(\mathcal{H})$. This is a contradiction. □

Definition 3.10. A non-constant fuzzy ideal μ of $\mathcal{H}$ is called *maximal* if μ^+ is a maximal element of $(\mathcal{N}(\mathcal{H}), \subseteq)$.

Theorem 3.11 *If μ is a maximal fuzzy ideal of $\mathcal{H}$, then*

(i) μ is normal,

(ii) μ takes only the values 0 and 1,

(iii) $\mu_{H_\mu} = \mu$,

(iv) H_μ is a maximal ideal of $\mathcal{H}$.

Proof. Let μ be a maximal fuzzy ideal of $\mathcal{H}$. Then μ^+ is a non-constant maximal element of the poset $(\mathcal{N}(\mathcal{H}), \subseteq)$. It follows from Theorem 3.9 that μ^+ takes only the values 0 and 1. Note that $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(\mathbf{1})$, and $\mu^+(x) = 0$ if and only if $\mu(x) = \mu(\mathbf{1}) - 1$. By Corollary 3.3, we have $\mu(x) = 0$, that is, $\mu(\mathbf{1}) = 1$. Hence μ is normal, and clearly $\mu^+ = \mu$. This proves *(i)* and *(ii)*.

(iii) Clearly, $\mu_{H_\mu} \subseteq \mu$ and μ_{H_μ} takes only the values 0 and 1. Let $x \in H$. If $\mu(x) = 0$, then obviously $\mu \subseteq \mu_{H_\mu}$. If $\mu(x) = 1$, then $x \in H_\mu$, and so $\mu_{H_\mu}(x) = 1$. This shows that $\mu \subseteq \mu_{H_\mu}$.

(iv) H_μ is a proper ideal of $\mathcal{H}$ because μ is non-constant. Let A be an ideal of $\mathcal{H}$ such that $H_\mu \subseteq A$. Noticing that, for any ideals A and B of $\mathcal{H}$, $A \subseteq B$ if and only if $\mu_A \subseteq \mu_B$, then we obtain $\mu = \mu_{H_\mu} \subseteq \mu_A$. Since μ and μ_A are normal and since $\mu = \mu^+$ is a maximal element of $\mathcal{N}(\mathcal{H})$, we have that either $\mu = \mu_A$ or $\mu_A = \omega$, where $\omega : H \to [0,1]$ is a fuzzy set defined by $\omega(x) = 1$ for all $x \in H$. The last case implies that $A = H$. If $\mu = \mu_A$, then $H_\mu = H_{\mu_A} = A$ by Proposition 2.6. This proves that H_μ is a maximal ideal of $\mathcal{H}$. □

4. Cartesian product of fuzzy ideals

Definition 4.1. ([1]) By a *fuzzy relation* defined on a set S we mean a fuzzy set

$$\mu : S \times S \to [0,1].$$

Definition 4.2. ([1]) If μ is a fuzzy relation on a set S and ν is a fuzzy set in S, then μ is called *a fuzzy relation* on ν if

$$\mu(x,y) \leq \min\{\nu(x),\, \nu(y)\},\ \forall x,y \in S.$$

Definition 4.3. ([1]) The *Cartesian product* of two fuzzy sets μ and ν in S is defined by

$$(\mu \times \nu)(x,y) = \min\{\mu(x),\, \nu(y)\},\ \forall x,y \in S.$$

Lemma 4.4. ([1]) *Let μ and ν be fuzzy sets in a set S. Then*

(i) $\mu \times \nu$ is a fuzzy relation on S,

(ii) $(\mu \times \nu)_t = \mu_t \times \nu_t$ for all $t \in [0,1]$.

Definition 4.5. ([1]) Let ν be a fuzzy set in a set S. *The strongest fuzzy relation* on S is a fuzzy relation μ_ν defined by

$$\mu_\nu(x,y) = \min\{\nu(x),\, \nu(y)\},\ \forall x,y \in S.$$

Lemma 4.6. ([1]) *For a given fuzzy set ν in a set S, let μ_ν be the strongest fuzzy relation on S. Then for $t \in [0,1]$, we have that $(\mu_\nu)_t = \nu_t \times \nu_t$.*

Proposition 4.7. *For a given fuzzy set ν in a Hilbert algebra $\mathcal{H}$, let μ_ν be the strongest fuzzy relation on $\mathcal{H}$. If μ_ν is a fuzzy ideal of $\mathcal{H} \times \mathcal{H}$, then $\nu(x) \leq \nu(\mathbf{1})$ for all $x \in H$.*

Proof. Since $\mathcal{H} \times \mathcal{H}$ is a Hilbert algebra, then from the fact that μ_ν is a fuzzy ideal of $\mathcal{H} \times \mathcal{H}$ follows $\mu_\nu(x,y) \leq \mu_\nu(\mathbf{1},\mathbf{1})$ for all $x, y \in H$. Hence $\min\{\nu(x), \nu(y)\} \leq \min\{\nu(\mathbf{1}), \nu(\mathbf{1})\}$, which gives $\nu(x) \leq \nu(\mathbf{1})$ for $x \in H$. □

The following corollary is an immediate consequence of Lemma 4.6, and we omit the proof.

Corollary 4.8. If ν is a fuzzy ideal of a Hilbert algebra $\mathcal{H}$, then the level ideals of μ_ν are given by $(\mu_\nu)_t = \nu_t \times \nu_t$ for all $t \in [0,1]$.

Proposition 4.9. *Let μ and ν be fuzzy ideals of a Hilbert algebra $\mathcal{H}$. Then $\mu \times \nu$ is a fuzzy ideal of $\mathcal{H} \times \mathcal{H}$. Moreover, if μ and ν are normal, then $\mu \times \nu$ is also normal.*

Proof. Straightforward. □

Theorem 4.10. *Let μ and ν be fuzzy sets in a Hilbert algebra $\mathcal{H}$ such that $\mu \times \nu$ is a fuzzy ideal of $\mathcal{H} \times \mathcal{H}$. Then*

(i) either $\mu(x) \leq \mu(\mathbf{1})$ or $\nu(x) \leq \nu(\mathbf{1})$ for all $x \in H$,

(ii) if $\mu(x) \leq \mu(\mathbf{1})$ for all $x \in H$, then either $\mu(x) \leq \nu(\mathbf{1})$ or $\nu(x) \leq \nu(\mathbf{1})$,

(iii) if $\nu(x) \leq \nu(\mathbf{1})$ for all $x \in H$, then either $\mu(x) \leq \mu(\mathbf{1})$ or $\nu(x) \leq \mu(\mathbf{1})$,

(iv) either μ or ν is a fuzzy ideal of $\mathcal{H}$.

Proof. (i) Suppose that $\mu(x) > \mu(\mathbf{1})$ and $\nu(y) > \nu(\mathbf{1})$ for some $x, y \in H$. Then $(\mu \times \nu)(x,y) = \min\{\mu(x), \nu(y)\} > \min\{\mu(\mathbf{1}), \nu(\mathbf{1})\} = (\mu \times \nu)(\mathbf{1},\mathbf{1})$, which is a contradiction. Thus either $\mu(x) \leq \mu(\mathbf{1})$ or $\nu(x) \leq \nu(\mathbf{1})$ for all $x \in H$.

(ii) Assume that $\mu(x) > \nu(\mathbf{1})$ and $\nu(y) > \nu(\mathbf{1})$ for some $x, y \in H$. Then $(\mu \times \nu)(\mathbf{1},\mathbf{1}) = \min\{\mu(\mathbf{1}), \nu(\mathbf{1})\} = \nu(\mathbf{1})$ and hence

$$(\mu \times \nu)(x,y) = \min\{\mu(x), \nu(y)\} > \nu(\mathbf{1}) = (\mu \times \nu)(\mathbf{1},\mathbf{1}).$$

This is a contradiction. Hence *(ii)* holds.

(iii) Similarly as (ii).

(iv) Since, by (i), either $\mu(x) \leq \mu(\mathbf{1})$ or $\nu(x) \leq \nu(\mathbf{1})$ for all $x \in H$, without loss of generality we may assume that $\nu(x) \leq \nu(\mathbf{1})$ for all $x \in H$. It follows from (iii) that either $\mu(x) \leq \mu(\mathbf{1})$ or $\nu(x) \leq \mu(\mathbf{1})$.

As first, we consider the case when $\nu(x) \leq \mu(\mathbf{1})$ for all $x \in H$. We prove that in this case ν is a fuzzy ideal. Indeed, by the assumption (F1) is satisfied. Moreover, if $\cdot$ denotes the operation in $\mathcal{H} \times \mathcal{H}$, then

$$\begin{aligned} \nu(xy) &= \min\{\mu(\mathbf{1}), \nu(xy)\} = (\mu \times \nu)(\mathbf{1}, xy) \\ &= (\mu \times \nu)(x\mathbf{1}, xy) = (\mu \times \nu)((x, x) \cdot (\mathbf{1}, y)) \\ &\geq (\mu \times \nu)(\mathbf{1}, y) = \min\{\mu(\mathbf{1}), \nu(y)\} = \nu(y) \end{aligned}$$

for all $x, y \in H$, which proves that ν satisfies the condition (F2).

Similarly, for all $x, y_1, y_2 \in H$, we have

$$\begin{aligned} \nu((y_1 \cdot y_2x)x) &= \min\{\mu(\mathbf{1}),\ \nu((y_1 \cdot y_2x)x)\} \\ &= (\mu \times \nu)(\mathbf{1},\ (y_1 \cdot y_2x)x) = (\mu \times \nu)((\mathbf{1} \cdot \mathbf{1}x)x,\ (y_1 \cdot y_2x)x) \\ &= (\mu \times \nu)([(\mathbf{1}, y_1) \cdot [(\mathbf{1}, y_2) \cdot (x, x)]] \cdot (x, x)) \\ &\geq \min\{(\mu \times \nu)(\mathbf{1}, y_2),\ (\mu \times \nu)(\mathbf{1}, y_1)\} \\ &= \min\{\min\{\mu(\mathbf{1}), \nu(y_2)\}, \min\{\mu(\mathbf{1}), \nu(y_1)\}\} \\ &= \min\{\nu(y_1),\ \nu(y_2)\}, \end{aligned}$$

which proves (F3) for ν. Thus ν is a fuzzy ideal of $\mathcal{H}$.

Now we consider the case $\mu(x) \leq \mu(\mathbf{1})$ for all $x \in H$. In this case μ is a fuzzy ideal. (F1) obviously holds. To prove (F2) suppose that $\nu(y) > \mu(\mathbf{1})$ for some $y \in H$. Then $\nu(\mathbf{1}) \geq \nu(y) > \mu(\mathbf{1})$. Since $\mu(\mathbf{1}) \geq \mu(x)$ for all $x \in H$, it follows that $\nu(\mathbf{1}) > \mu(x)$ for any $x \in H$. Hence $(\mu \times \nu)(x, \mathbf{1}) = \min\{\mu(x),\ \nu(\mathbf{1})\} = \mu(x)$ for all $x \in H$. Thus

$$\begin{aligned} \mu(xy) &= (\mu \times \nu)(xy,\ \mathbf{1}) = (\mu \times \nu)(xy,\ x\mathbf{1}) \\ &= (\mu \times \nu)((x, x) \cdot (y, \mathbf{1})) \geq (\mu \times \nu)((y,\ \mathbf{1})) = \mu(y) \end{aligned}$$

for all $x, y \in H$, which gives (F2). Moreover

$$\begin{aligned} \mu((y_1 \cdot y_2x)x) &= (\mu \times \nu)((y_1 \cdot y_2x)x,\ \mathbf{1}) \\ &= (\mu \times \nu)((y_1 \cdot y_2x)x,\ (\mathbf{1} \cdot \mathbf{1}x)x) \\ &= (\mu \times \nu)([(y_1, \mathbf{1}) \cdot [(y_2, \mathbf{1}) \cdot (x, x)]] \cdot (x, x)) \\ &\geq \min\{(\mu \times \nu)(y_1, \mathbf{1}), (\mu \times \nu)(y_2, \mathbf{1})\} = \min\{\mu(y_1), \mu(y_2)\} \end{aligned}$$

for all $x, y_1, y_2 \in H$, which gives (F3). Thus μ is a fuzzy ideal of $\mathcal{H}$. This completes the proof. □

Now we give an example to show that if $\mu \times \nu$ is a fuzzy ideal of $\mathcal{H} \times \mathcal{H}$, then μ and ν both need not be fuzzy ideals of $\mathcal{H}$.

Example 4.11. Let $\mathcal{H}$ be a Hilbert algebra with $|G| \geq 2$ and let $s, t \in [0, 1)$ be such that $s \leq t$. Define fuzzy sets μ and ν in $\mathcal{H}$ by $\mu(x) = s$ and

$$\nu(x) = \begin{cases} t & \text{if } x = \mathbf{1}, \\ 1 & \text{otherwise}, \end{cases}$$

for all $x \in H$, respectively. Then $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} = s$ for all $(x, y) \in H \times H$, that is, $\mu \times \nu$ is a constant function and so $\mu \times \nu$ is a fuzzy ideal of $H \times H$. Now μ is a fuzzy ideal of $\mathcal{H}$, but ν is not a fuzzy ideal of $\mathcal{H}$ since for $x \neq \mathbf{1}$ we have $\nu(\mathbf{1}) = t < 1 = \nu(x)$.

Theorem 4.12. *Let ν be a fuzzy set in a Hilbert algebra $\mathcal{H}$ and let μ_ν be the strongest fuzzy relation on $\mathcal{H}$. Then ν is a fuzzy ideal of $\mathcal{H}$ if and only if μ_ν is a fuzzy ideal of $\mathcal{H} \times \mathcal{H}$.*

Proof. Assume that ν is a fuzzy ideal of $\mathcal{H}$. Clearly, $\mu_\nu(\mathbf{1}, \mathbf{1}) \geq \mu_\nu(x, y)$ for all $(x, y) \in H \times H$. Now

$$\begin{aligned} \mu_\nu((x_1, x_2) \cdot (y_1, y_2)) &= \mu_\nu(x_1 y_1,\ x_2 y_2) = \min\{\nu(x_1 y_1),\ \nu(x_2 y_2)\} \\ &\geq \min\{\nu(y_1),\ \nu(y_2)\} = \mu_\nu(y_1, y_2) \end{aligned}$$

for all $(x_1, x_2), (y_1, y_2) \in H \times H$, and

$$\mu_\nu([(x_1, y_1) \cdot [(x_2, y_2) \cdot (x, y)]] \cdot (x, y)) = \mu_\nu((x_1 \cdot x_2 x)x,\ (y_1 \cdot y_2 y)y) \geq$$

$$\begin{aligned} &\geq \min\{\nu((x_1 \cdot x_2 x)x),\ \nu((y_1 \cdot y_2 y)y)\} \\ &= \min\{\min\{\nu(x_1),\ \nu(x_2)\},\ \min\{\nu(y_1),\ \nu(y_2)\}\} \\ &= \min\{\min\{\nu(x_1),\ \nu(y_1)\},\ \min\{\nu(x_2),\ \nu(y_2)\}\} \\ &= \min\{\mu_\nu(x_1, y_1),\ \mu_\nu(x_2, y_2)\} \end{aligned}$$

for all $(x, y), (x_1, x_2), (y_1, y_2) \in H \times H$. Hence μ_ν is a fuzzy ideal of $\mathcal{H} \times \mathcal{H}$.

Conversely, suppose that μ_ν is a fuzzy ideal of $\mathcal{H} \times \mathcal{H}$. Then

$$\min\{\nu(\mathbf{1}), \nu(\mathbf{1})\} = \mu_\nu(\mathbf{1}, \mathbf{1}) \geq \mu_\nu(x, y) = \min\{\nu(x), \nu(y)\}$$

for all $(x, y) \in H \times H$. It follows that $\nu(1) \geq \nu(x)$ for all $x \in H$. Now we have

$$\begin{aligned}\nu(xy) &= \min\{\nu(xy),\ \nu(1)\} = \mu_\nu(xy, 1) = \mu_\nu(xy, x1) \\ &= \mu_\nu((x, x) \cdot (y, 1)) \geq \mu_\nu(y, 1) = \min\{\nu(y),\ \nu(1)\} = \nu(y)\end{aligned}$$

for all $x, y \in H$, and

$$\begin{aligned}\nu((y_1 \cdot y_2 x)x) &= \min\{\nu((y_1 \cdot y_2 x)x),\ \nu((y_1 \cdot y_2 x)x)\} \\ &= \mu_\nu((y_1 \cdot y_2 x)x,\ (y_1 \cdot y_2 x)x) \\ &= \mu_\nu([(y_1, y_1) \cdot [(y_2, y_2) \cdot (x, x)]] \cdot (x, x)) \\ &\geq \min\{\mu_\nu(y_1, y_1),\ \mu_\nu(y_2, y_2)\} = \min\{\mu(y_1),\ \mu(y_2)\}\end{aligned}$$

for all $x, y_1, y_2 \in H$. Hence ν is a fuzzy ideal of $\mathcal{H}$. □

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