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ON SUBALGEBRAS IN HILBERT ALGEBRAS

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Abstract

The number of subalgebras of a finite Hilbert algebra is determined and ideals introduced by I. Chajda and R. Halaš are described.

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1. Introduction

The concept of Hilbert algebra was introduced in the early 50s by L. Henkin and T. Skolem for some investigations of implication in intuitionistic and other non-classical logics. In the 60s, these algebras were studied especially by A. Horn and A. Diego from the algebraic point of view. A. Diego proved (cf. [4]) that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by D. Busneag (cf. [1, 2]) and Y. B. Jun (cf. [6]) and their filters forming deductive systems have been recognized recently. I. Chajda and R. Halaš introduced in [3] the concept of ideals in Hilbert algebras and described connections between such ideals and congruences.

In this note we determine the number of subalgebras of a finite Hilbert algebra and prove that every ideal is a union of some deductive systems.

Since there exist various modifications of the definition of Hilbert algebra, we use the one from [1].

Definition 1.1. A *Hilbert algebra* is a triplet $\mathcal{H} = (H;*,\mathbf{1})$, where H is a nonempty set, $*$ is a binary operation and $\mathbf{1}$ is a fixed element of H such that the following axioms hold for each $x, y, z \in H$:

(I) $x * (y * x) = \mathbf{1}$,

(II) $(x * (y * z)) * ((x * y) * (x * z)) = \mathbf{1}$,

(III) $x * y = \mathbf{1}$ and $y * x = \mathbf{1}$ imply $x = y$.

The following result was proved (cf. for example [4]).

Lemma 1.2. *Let $\mathcal{H} = (H;*,\mathbf{1})$ be a Hilbert algebra and $x, y, z \in H$. Then*

(1) $x * x = \mathbf{1}$,

(2) $\mathbf{1} * x = x$,

(3) $x * \mathbf{1} = \mathbf{1}$,

(4) $x * (y * z) = y * (x * z)$,

(5) $x * (y * z) = (x * y) * (x * z)$.

It is easily checked that in a Hilbert algebra $\mathcal{H}$ the relation $\leq$ defined by

$$x \leq y \iff x * y = \mathbf{1}$$

is a partial order on H with $\mathbf{1}$ as the largest element.

Lemma 1.3. *Let $\mathcal{H} = (H;*,\mathbf{1})$ be a Hilbert algebra and $x, y, z \in H$. Then*

(6) $(x * y) * y \geq x$,

(7) $(y * z) * (x * z) \geq x * y$,

(8) $(x * y) * (x * z) \geq y * z$.

Proof. The property (6) is a consequence of (4) and (1). (7) is proved in [4]; (8) follows from (7) and (4). □

Lemma 1.4. *Let $\mathcal{H} = (H;*,\mathbf{1})$ be a Hilbert algebra and $x, y, z \in H$. Then*

(9) $x * y = \mathbf{1}$ *and* $x \neq y$ *implies* $y * x \neq \mathbf{1}$,

(10) $x * y = \mathbf{1}$ *and* $y * z = \mathbf{1}$ *implies* $x * z = \mathbf{1}$,

(11) $x * y = z$ *implies* $y * z = \mathbf{1}$.

Proof. The condition (9) follows from (III); (10) is a consequence of (7) and (2); (11) follows from (I). □

2. Subalgebras

For a given Hilbert algebra $\mathcal{H} = (H; *, \mathbf{1})$ we define the matrix H_n as follows

$$H_n = \begin{pmatrix} a_1 * a_1 & a_1 * a_2 & \dots & a_1 * a_n \\ a_2 * a_1 & a_2 * a_2 & \dots & a_2 * a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n * a_1 & a_n * a_2 & \dots & a_n * a_n \end{pmatrix},$$

where $a_1, a_2, \dots, a_n$ are arbitrary nonunit elements of H such that $a_i \neq a_j$ for $i \neq j$. If in the i-th column of this matrix $a_j * a_i = \mathbf{1}$ only for $j = i$, then this column is called *proper.*

Lemma 2.1. *If all elements* $a_1, a_2, \dots, a_n$ $(n \geq 2)$ *are different, then the matrix* H_n *has at least one proper column.*

Proof. The proof is by induction on n. For $n = 2$ and $a_1 \neq a_2$ we have

$$H_2 = \begin{pmatrix} \mathbf{1} & a_1 * a_2 \\ a_2 * a_1 & \mathbf{1} \end{pmatrix},$$

which by (9) proves that for $n = 2$ the lemma is true.

Now assume that our lemma is true for $n - 1 \geq 2$, i.e. the matrix

$$H_{n-1} = \begin{pmatrix} a_1 * a_1 & a_1 * a_2 & \dots & a_1 * a_{n-1} \\ a_2 * a_1 & a_2 * a_2 & \dots & a_2 * a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} * a_1 & a_{n-1} * a_2 & \dots & a_{n-1} * a_{n-1} \end{pmatrix}$$

has at least one proper column. Without loss of generality, we can assume that the first column is proper, i.e. $a_j * a_1 \neq \mathbf{1}$ for all $j = 2, 3, \dots, n-1$. If

$a_n * a_1 = 1$, then $a_1 * a_n \neq 1$ (by (9)) and $a_j * a_n \neq 1$ for all $j = 2, \ldots, n-1$, because $a_j * a_n = 1$ (for some $j = 2, 3, \ldots, n-1$) implies (by (10)) $a_j * a_1 = 1$, which is impossible. This proves that $a_n * a_1 \neq 1$. Hence the first column of H_n is proper. □

Theorem 2.2. *Every Hilbert algebra of the order $n \geq 2$ has at least one subalgebra of the order $n-1$.*

Proof. For $n = 2$ the theorem is obvious. Let $H = \{a_1, a_2, \ldots, a_{n-1}, 1\}$ be a fixed Hilbert algebra of the order $n \geq 3$. Then, by Lemma 2.1, the matrix H_{n-1} has at least one proper column. Without loss of generality, we can assume that the $(n-1)$–th column is proper, i.e. $a_j * a_{n-1} \neq 1$ for all $j = 1, 2, \ldots, n-2$. Then $S = \{a_1, a_2, \ldots, a_{n-2}, 1\}$ is a subalgebra of H. Indeed, if S is not a subalgebra, then it is not closed under the operation $*$, i.e. there exist $a_i, a_j \in S$ such that $a_i * a_j = a_{n-1}$. Hence by (11) $a_j * a_{n-1} = 1$, which is impossible because the $(n-1)$–th column is proper. □

Corollary 2.3. *Every Hilbert algebra of the order $n \geq 2$ has at least one subalgebra of the order $2 \leq i \leq n-1$.*

Let $N(i)$ denote the number of subalgebras of the order i. Since for $1 \leq i \leq n$ a subalgebra of the order i (if it exists) contains 1 and $i-1$ nonunit elements, then $1 \leq N(i) \leq \binom{n-1}{i-1}$. It is clear that every nonunit element together with 1 forms a subalgebra. Hence for every Hilbert algebra of the order $n \geq 2$ we have $N(2) = \binom{n-1}{2-1} = n-1$.

Theorem 2.4. *If in a Hilbert algebra $\mathcal{H}$ for some fixed $i \geq 2$ all subsets containing 1 and i nonunit elements are subalgebras, then every its subset containing 1 is a subalgebra.*

Proof. Let $M = \{1, a_1, a_2, \ldots, a_{i+1}\}$ be an arbitrary subset of H, where i is as in the assumption of the theorem. Then M is a subalgebra. Indeed, since $S_1 = \{1, a_2, a_3, \ldots, a_{i+1}\}$, $S_2 = \{1, a_1, a_3, \ldots, a_{i+1}\}$ and $S_3 = \{1, a_1, a_2, a_4, \ldots, a_{i+1}\}$ are subalgebras, then $x * y \in M = S_1 \cup S_2 \cup S_3$ for all $x, y \in M$. Hence, by induction, every subset containing 1 and at least $j \geq i$ nonunit elements is a subalgebra.

Also, all subsets containing 1 and at most $j < i$ nonunit elements are subalgebras. If not, then there exists $S_j = \{1, a_1, \ldots, a_j\}$ such that $x * y =$

$a_k \notin S_j$ for some $x, y \in S_j$. Thus $S_j \cup \{a_{j+1}, \ldots, a_i\}$ containing **1** and i nonunit elements is not a subalgebra, which is a contradiction. Hence all subsets with **1** must be subalgebras. □

Corollary 2.5. *If in a Hilbert algebra every subset containing* **1** *and two nonunit elements is a subalgebra, then all subsets containing* **1** *are subalgebras.*

Corollary 2.6. *In a Hilbert algebra of the order $n > 3$ for every $i = 3, \ldots, n$ we have either $N(i) = \binom{n-1}{i-1}$ or $N(i) < \binom{n-1}{i-1}$.*

Example 2.7. The set $H = \{a, b, c, d, \mathbf{1}\}$ with the operation defined by the following Cayley table:

$*$	a	b	c	d	$\mathbf{1}$
a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
b	a	$\mathbf{1}$	c	$\mathbf{1}$	$\mathbf{1}$
c	a	b	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
d	a	b	c	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	a	b	c	d	$\mathbf{1}$

Table 1

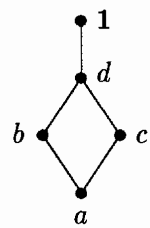

Diagram 1

is an example of a Hilbert algebra (cf. [5]) in which $N(i) = \binom{n-1}{i-1}$ for every $i = 2, 3, \ldots, n$.

Example 2.8. The set $H = \{a, b, c, d, \mathbf{1}\}$ with the multiplication defined by the table:

$\star$	a	b	c	d	$\mathbf{1}$
a	$\mathbf{1}$	b	b	d	$\mathbf{1}$
b	a	$\mathbf{1}$	a	d	$\mathbf{1}$
c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	d	$\mathbf{1}$
d	$\mathbf{1}$	b	b	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	a	b	c	d	$\mathbf{1}$

Table 2

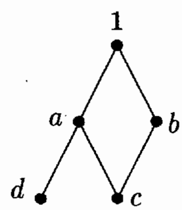

Diagram 2

is an example of a Hilbert algebra in which $1 < N(i) < \binom{n-1}{i-1}$ for every $i = 3, 4, \ldots, n$. In this algebra $N(3) = 3$, $N(4) = 2$ since it has only the following subalgebras $\{1\}$, $\{a, 1\}$, $\{b, 1\}$, $\{c, 1\}$, $\{d, 1\}$, $\{a, b, 1\}$, $\{a, d, 1\}$, $\{b, d, 1\}$, $\{a, b, c, 1\}$, $\{a, b, d, 1\}$.

Example 2.9. The set $H = \{a, b, c, 1\}$ with the operation defined by the following Cayley table:

$\cdot$	a	b	c	1
a	1	1	1	1
b	c	1	c	1
c	b	b	1	1
1	a	b	c	1

Table 3

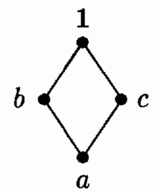

Diagram 3

is an example of a Hilbert algebra which has only one subalgebra of the order $n - 1$.

Problem 1. *Describe Hilbert algebras in which* $N(i) = \binom{n-1}{i-1}$ *for all* $i \geq 2$.

A Hilbert algebra $\mathcal{H}$ containing an element θ such that

$$\theta * x = \mathbf{1}$$

for all $x \in H$ is called *bounded*. In the algebras defined by Tables 1 and 3 the role of θ plays an element a. The algebra defined by Table 2 is not bounded. An element θ (if it exists) is obviously the smallest element with respect to the partial oder $\leq$.

A subalgebra S containing θ is called *extremal*. Such subalgebra contains at least two elements: θ and $\mathbf{1}$. Extremal subalgebras are considered in [7], but some of the results obtained in that paper are not true.

Example 2.10. The set $H = \{a, b, c, d, 1\}$ with the operation defined by the following Cayley table:

$*$	a	b	c	d	$\mathbf{1}$
a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
b	b	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
c	b	d	$\mathbf{1}$	d	$\mathbf{1}$
d	b	c	c	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	a	b	c	d	$\mathbf{1}$

Table 4

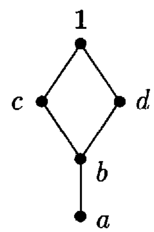

Diagram 4

is an example of a bounded Hilbert algebra which has only two extremal subalgebras: $\{a, \mathbf{1}\}$ and $\{a, b, \mathbf{1}\}$. This proves that Theorems 6 and 7 from [7] (the analogue of our Theorem 2.2) are not true.

Theorem 2.11. *If in a bounded Hilbert algebra $\mathcal{H}$ for some fixed $i \geq 2$ all subsets of the form $\{\theta, a_1, a_2, \ldots, a_i, \mathbf{1}\}$ are subalgebras, then every subset of H containing θ, $\mathbf{1}$ and at least two elements is a subalgebra.*

Proof. A modification of the proof of Theorem 2.4. □

Let $N_e(i)$ denote the number of extremal subalgebras of the order $i \geq 2$. Since every such subalgebra contains θ and $\mathbf{1}$, then

$$N_e(i) \leq \binom{n-2}{i-2}$$

for all bounded Hilbert algebras of the order $n \geq 2$.

Corollary 2.12. *In a Hilbert algebra of the order $n \geq 3$ for every $2 \leq i \leq n$ we have either $N_e(i) = \binom{n-2}{i-2}$ or $N_e(i) < \binom{n-2}{i-2}$.*

In the Hilbert algebra defined in Example 2.7 we have $N_e(i) = \binom{n-2}{i-2}$ for all $i = 2, 3, \ldots, n$, but there are Hilbert algebras in which $N_e(i) = 1$ for all $i = 2, 3, \ldots, n$.

Example 2.13. The set $H = \{a, b, c, d, \mathbf{1}\}$ with the operation defined by the following Cayley table:

*	a	b	c	d	$\mathbf{1}$
a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
b	c	$\mathbf{1}$	c	$\mathbf{1}$	$\mathbf{1}$
c	b	b	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
d	a	b	c	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	a	b	c	d	$\mathbf{1}$

Table 5

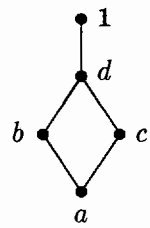

Diagram 5

is an example of a bounded Hilbert algebra which for every $i \geq 2$ has only one extremal subalgebra of the order i.

3. Extensions

From the above results follows that every finite Hilbert algebra may be treated as one-element extension of some its subalgebra. But in general, such extension is not unique. For example, the algebra defined in Example 2.8 may be treated as one-element extension of a subalgebra $\{a, b, c, \mathbf{1}\}$ or $\{a, b, d, \mathbf{1}\}$.

We give three methods of an extension of a given Hilbert algebra. Since the proofs are a verification of the axioms, we omit it.

Construction A. *Let $(X; \cdot, \mathbf{1})$ be a Hilbert algebra and let $d \notin X$. Then the set $X' = X \cup \{d\}$ with the operation $*$ defined as follows:*

$$x * y = \begin{cases} xy & if \quad x, y \in X, \\ d & if \quad x = \mathbf{1},\ y = d\,, \\ \mathbf{1} & if \quad x \neq \mathbf{1},\ y = d\,, \\ y & if \quad x = d,\ y \in X, \end{cases}$$

is a Hilbert algebra.

Construction B. *Let $(X; \cdot, \mathbf{1})$ be a Hilbert algebra and let $e \notin X$. Then the set $X' = X \cup \{e\}$ with the operation*

$$x * y = \begin{cases} xy & if \quad x, y \in X, \\ \mathbf{1} & if \quad x = e,\ y \in X', \\ e & if \quad x \in X,\ y = e\,, \end{cases}$$

is a bounded Hilbert algebra with e as the smallest element.

Construction C. *Let* $(X;\cdot,\mathbf{1})$ *be a Hilbert algebra with an element* x_0 *such that* $x \le x_0 < \mathbf{1}$ *for all* $x \in X$. *If* $m \notin X$, *then the set* $X' = X \cup \{m\}$ *with the operation*

$$x * y = \begin{cases} xy & if \quad x, y \in X\,, \\ \mathbf{1} & if \quad x = m,\ y = \mathbf{1}\,, \\ x_0 & if \quad x = m,\ y \in X \setminus \{\mathbf{1}\}\,, \\ \mathbf{1} & if \quad x = m,\ y = m\,, \\ m & if \quad x \in X,\ y = m\,, \end{cases}$$

is a Hilbert algebra.

4. Ideals

The following concept of ideals of a Hilbert algebra was introduced in [3] by I. Chajda and R. Halaš.

Definition 4.1. A nonempty subset I of a Hilbert algebra $\mathcal{H} = (H;*,\mathbf{1})$ is called *an ideal* of $\mathcal{H}$ if

(i) $\mathbf{1} \in I$,

(ii) $x * y \in I$ for all $x \in H,\ \ y \in I$,

(iii) $(y_2 * (y_1 * x)) * x \in I$ for all $x \in H,\ \ y_1, y_2 \in I$.

Observe that (i) follows from (ii) and (1). From (ii) follows also that any ideal is a subalgebra. Moreover, putting in the above definition $y_1 = a$ and $y_2 = \mathbf{1}$, we obtain

Proposition 4.2. *If* I *is an ideal of a Hilbert algebra* $\mathcal{H}$, *then* $(a*x)*x \in I$ *for all* $a \in I$ *and* $x \in H$.

Corollary 4.3 *If* $a \le x$ *and* $a \in I$, *where* I *is an ideal of a Hilbert algebra* $\mathcal{H}$, *then also* $x \in I$.

Proof. Since $x * x = \mathbf{1}$ by the assumption, then by (2) and Proposition 4.2 we obtain

$$x = 1 * x = (a * x) * x \in I,$$

which completes the proof. □

Definition 4.4. A nonempty subset D of a Hilbert algebra $\mathcal{H}$ is called *a deductive system* of $\mathcal{H}$ if

(d_1) $\mathbf{1} \in D$,

(d_2) $x \in D$ and $x * y \in D$ imply $y \in D$.

Theorem 4.5. *An ideal is a deductive system.*

Proof. Let A be an ideal. Since $\mathbf{1} \in A$, then (d_1) is satisfied. To prove (d_2) assume $a \in A$ and $a * x = a_1 \in A$ for some $x \in H$. Then, by Proposition 4.2, $a_2 = (a * x) * x \in A$, which together with (1) and (2) gives

$$x = \mathbf{1} * x = [((a * x) * x) * ((a * x) * x)] * x = [a_2 * (a_1 * x)] * x \in A.$$

Thus $a \in A$ and $a * x \in A$ imply $x \in A$, which proves (d_2). Hence A is a deductive system. □

In [6] is proved that for every fixed $a, b \in H$ the set

$$A(a, b) = \{x \in H : a \leq b * x\} = \{x \in H : a * (b * x) = \mathbf{1}\}$$

is a deductive system. Moreover, any deductive system may be presented as a union of some $A(a, b)$. We prove that the similar results hold for ideals.

Theorem 4.6. *$A(a, b)$ is an ideal for every fixed $a, b \in H$.*

Proof. Let $a, b \in H$ be fixed. Then $\mathbf{1} \in A(a, b)$. If $x \in H$ and $y \in A(a, b)$, then $a * (b * y) = \mathbf{1}$ and, in the consequence,

$$a * (b * (x * y)) = a * (x * (b * y)) = x * (a * (b * y)) = x * \mathbf{1} = \mathbf{1},$$

by (4). Hence $x * y \in A(a, b)$, which proves (ii). To prove (iii) observe that for all $y_1, y_2 \in A(a, b)$ and $x \in H$ we have

$$\begin{aligned} &a * (b * ((y_2 * (y_1 * x)) * x)) = \\ &= a * (((b * y_2) * ((b * y_1) * (b * x))) * (b * x)) = \\ &= ((a * (b * y_2)) * ((a * (b * y_1)) * (a * (b * x)))) * (a * (b * x))) = \\ &= (\mathbf{1} * (\mathbf{1} * (a * (b * x)))) * (a * (b * x)) = \mathbf{1}, \end{aligned}$$

by (5) and (2). Hence $(y_2 * (y_1 * x)) * x \in A(a, b)$. □

Corollary 4.7. $A(a) = A(a, \mathbf{1}) = \{x \in H : a * x = \mathbf{1}\} = \{x \in H : a \leq x\}$ *is the smallest ideal containing* a.

It is clear that $a, b \in A(a) \cup A(b) \subseteq A(a, b)$, but, as shows Example 2.8, in generally $A(a, b) \neq A(a) \cup A(b)$.

Theorem 4.8. *A nonempty subset* A *of a Hilbert algebra* $\mathcal{H}$ *is an ideal if and only if* $A(a, b) \subseteq A$ *for all* $a, b \in A$.

Proof. If $a, b \in A$, where A is an ideal, then for every $y \in A(a, b)$ we have $a * (b * y) = \mathbf{1} \in A$, which, by Corollary 4.3, implies $b * y \in A$, and, in the consequence, $y \in A$. Thus $A(a, b) \subseteq A$ for all $a, b \in A$.

Conversely, if $A(a, b) \subseteq A$ for all $a, b \in A$, then $\mathbf{1} \in A$ and A is an ideal. Indeed, for every $y \in A$ and $x \in H$ we have $\mathbf{1} \leq y * (x * y) = \mathbf{1}$. Thus $x * y \in A(\mathbf{1}, y) \subseteq A$, i.e. $x * y \in A$. For every $y_1, y_2 \in A$ we have also

$$\begin{aligned} & y_1 * (y_2 * ((y_2 * (y_1 * x)) * x)) = \\ &= y_1 * (((y_2 * y_2) * ((y_2 * y_1) * (y_2 * x))) * (y_2 * x)) = \\ &= ((y_1 * (y_2 * y_2)) * ((y_1 * (y_2 * y_1)) * (y_1 * (y_2 * x)))) * (y_1 * (y_2 * x))) = \\ &= (\mathbf{1} * (\mathbf{1} * (y_1 * (y_2 * x)))) * (y_1 * (y_2 * x)) = \mathbf{1}, \end{aligned}$$

by (5) and (2). Hence $(y_2 * (y_1 * x)) * x \in A(y_1, y_2) \subseteq A$. This proves that A is an ideal. □

Corollary 4.9 *Every ideal* A *of a Hilbert algebra* $\mathcal{H}$ *may be represented in the form*

$$A = \bigcup_{a,b \in A} A(a, b) = \bigcup_{a \in A} A(a).$$

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