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WEIGHTED BLOCK DESIGNS AND STEINER SYSTEMS

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Abstract

We consider weighted block designs and complete Steiner systems, and compare them with totally symmetric (n, m)-quasigroups. We show that a complete Steiner system $S'(2, k, v)$ is equivalent to a totally symmetric $(2, k-2)$-quasigroup, and that any complete Steiner quadruple system $S'(3, 4, v)$ is equivalent to a totally symmetric $(3, 1)$-quasigroup.

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An incidence structure is a triple $\mathbf{D} = (V, \mathbf{B}, I)$, where V and $\mathbf{B}$ are disjoint sets and $I \subseteq V \times \mathbf{B}$. The elements of V are called *points*, and the elements of $\mathbf{B}$ are called *blocks*. If A is a point of V, the set of all blocks incident with A is denoted by (A). Thus, $(A) = \{b |\ b \in \mathbf{B}, AIb\}$. Moreover, for $A_1, A_2, \dots, A_n$, the set of all the blocks incident with all the points A_i is denoted by $(A_1, A_2, \dots, A_n)$. Thus,

$$(A_1, A_2, \dots, A_n) = \{b |\ b \in \mathbf{B}, A_i I b \text{ for all } i \in \mathbf{N}_n\},$$

where $\mathbf{N}$ is the set of all positive integers and $\mathbf{N}_n = \{1, 2, \ldots, n\}$. Dually, for $b, b_1, b_2, \ldots, b_n \in \mathbf{B}$, $(b) = \{A|\ A \in V, AIb\}$, and

$$(b_1, b_2, \ldots, b_n) = \{A|\ A \in V, AIb_i \text{ for all } i \in \mathbf{N}_n\}.$$

We consider only the incidence structures where distinct blocks have distinct sets of points. We identify each block b with the set (b) and identify the incidence relation with the membership relation $\in$.

Definition 1. A finite incidence structure $\mathbf{D} = (V, \mathbf{B}, \in)$ is called *t-design* with parameters $v, k, \lambda \in \mathbf{N}$, if:

(T.1) $|V| = v$,

(T.2) $|(A_1, A_2, \ldots, A_t)| = \lambda$, for any t distinct points $A_1, A_2, \ldots, A_t \in V$,

(T.3) $|(b)| = k$, for any $b \in \mathbf{B}$.

A 2-design with parameters v, k, λ, is usually called *block* design with parameters v, k, λ. A t-design with parameters $v, k, 1$ is called a *Steiner t-system*, and is denoted by $S(t, k, v)$. A Steiner 2-system $S(2, k, v)$ is called only Steiner system with parameters $v, k, 1$.

The following definition generalizes the notion of t-design.

Definition 2. A finite incidence structure $\mathbf{D} = (V, \mathbf{B}, \in)$ is called *weighted t-design* with parameters v, k, λ, if for any $b \in \mathbf{B}$ there is a map $f_b : (b) \to \mathbf{N}$, such that:

(WT.1) $|V| = v$,

(WT.2) $|(A_1, A_2, \ldots, A_t)| \doteq \lambda$, for any t distinct points $A_1, A_2, \ldots, A_t \in V$,

(WT.3) $k_b = k$, for any $b \in \mathbf{B}$, where:

(a) the image $f_b(A)$ is denoted by t_{Ab}, and is called the *weight* of the point A in the block b,

(b) for $A \in V$, its *weight* is $t_A = \sum_{A \in b} t_{Ab}$, and

(c) for $b \in \mathbf{B}$, the number $k_b = \sum_{A \in b} t_{Ab}$ is called the *size* of b.

Every block design, i.e. 2-design with parameters v, k, λ is a weighted block design, where for all $A \in b$, $t_{A,b} = 1$, for all $b \in \mathbf{B}$, $k_b = k$, and for all $A \in V, t_A = r$, where $r = |(A)|$ is the number of blocks containing A.

Definition 3. A weighted t-design $\mathbf{D}' = (V', \mathbf{B}, \in)$ is an *extension* of a weighted t-design $\mathbf{D} = (V, \mathbf{B}, \in)$, if $V \subseteq V'$ and for each $b \in \mathbf{B}$ there is $b' \in \mathbf{B}'$ such that (b)$\subseteq (b')$, and for each $A \in (b)$, $t_{Ab'} = t_{Ab}$.

Definition 4. An extension $(V', \mathbf{B}', \in)$ of a Steiner system $(V, \mathbf{B}, \in)$ with parameters $v, k, 1$, defined by

(a) $V' = V$,

(b) $\mathbf{B}' = \mathbf{B} \cup \mathbf{B}''$ where $\mathbf{B}'' = \{\{A\}|\ A \in V\}$, and

(c) for each $A \in V$, $t_A = r + k$, where r is the number of blocks in $\mathbf{B}$ containing A,

is called a *complete Steiner system* with parameters $v, k, 1$, and is denoted by $S'(2, k, v)$.

A Steiner 3-system with parameters $v, 4, 1$, i.e. $S(3, 4, v)$, is called a *Steiner quadruple system.*

Definition 5. Let $(V, \mathbf{B}, \in)$ be a Steiner quadruple system. An extension $(V', \mathbf{B}', \in)$ of $(V, \mathbf{B}, \in)$ with parameters $v, 4, 1$, defined by:

(a) $V' = V$,

(b) $\mathbf{B}' = \mathbf{B} \cup \mathbf{C} \cup \mathbf{P}$, where $\mathbf{C} = \{\{A\}|\ A \in V\}$ and $\mathbf{P} = \{\{A, B\}| A \neq B \in V\}$, and

(c) for each $A \in V$, $t_A = r + 4 + 2(v - 1)$, where r is the number of blocks in $\mathbf{B}$ containing A,

is called a *complete Steiner quadruple system* with parameters $v, 4, 1$, and is denoted by $S'(3, 4, v)$.

Next we compare Steiner systems and quadruple systems with the notion of totally symmetric (n, m)-quasigroups given below.

Definition 6. Let $Q \neq \emptyset$, $n, m \in \mathbf{N}$. A map $f : Q^n \to Q^m$ is called an (n,m)-*operation* of Q, and the pair (Q, f) is called a (n,m)-*groupoid*. An (n,m)-groupoid is called *(n,m)-quasigroup*, if

(A) for each $(a_1, a_2, \ldots, a_n) \in Q^n$, and each injection $\varphi : \mathbf{N}_n \to \mathbf{N}_{n+m}$, there exists a unique $(b_1, b_2, \ldots, b_{n+m}) \in Q^{n+m}$, such that for each $i \in \mathbf{N}_n$, $a_i = b_{\varphi(i)}$ and

$$f(b_1, b_2, \ldots, b_n) = (b_{n+1}, b_{n+2}, \ldots, b_{n+m}).$$

In the paper [3], an (n,m)-quasigroup is interpreted as an $(n+m)$-ary relation, as follows:

Definition 7. An $(n+m)$-ary relation $\rho \subseteq Q^{n+m}$ is called (n,m)-quasigroup relation, if

(A') for each $(a_1, a_2, \ldots, a_n) \in Q^n$, and each injection $\varphi : \mathbf{N}_n \to \mathbf{N}_{n+m}$, there exists a unique $(b_1, b_2, \ldots, b_{n+m}) \in Q^{n+m}$, such that for each $i \in \mathbf{N}_n$, $a_i = b_{\varphi(i)}$ and $(b_1, b_2, \ldots, b_{n+m}) \in \rho$.

The following theorem is proved in [3].

Theorem 1. *An (n,m)-groupoid (Q, f) is an (n,m)-quasigroup if and only if the $(n+m)$-ary relation defined by*

$$(x_1, x_2, \ldots, x_{n+m}) \in \rho \Leftrightarrow f(x_1, x_2, \ldots, x_n) = (x_{n+1}, x_{n+2}, \ldots, x_{n+m})$$

is an (n,m)-quasigroup relation.

Definition 8. An (n,m)-quasigroup is called *totally symmetric*, if

$$f(x_1, \ldots, x_n) = (x_{n+1}, \ldots, x_{n+m}) \Leftrightarrow f(y_1, \ldots, y_n) = (y_{n+1}, \ldots, y_{n+m})$$

for any $(x_1, x_2, \ldots, x_{n+m}) \in Q^{n+m}$ and any permutation $(y_1, y_2, \ldots, y_{n+m})$ of $(x_1, x_2, \ldots, x_{n+m})$. The $(n+m)$-ary relation ρ in this case is called *totally symmetric.*

Theorem 2. *Every complete Steiner system $(V, \mathbf{B}, \in)$ defines a totally symmetric $(2, k-2)$-quasigroup relation $\rho \subseteq V^k$, where*

$$(A_1, A_2, \ldots, A_k) \in \rho \Leftrightarrow \{A_1, A_2, \ldots, A_k\} \in \mathbf{B}.$$

Conversely, any totally symmetric $(2, k-2)$-quasigroup relation $\rho \subseteq V^k$ satisfying $(A, A, \ldots, A) = (A^k) \in \rho$ for any $A \in V$, defines a complete Steiner system $S'(2, k, v) = (V, \mathbf{B}, \in)$, where

$$\{A_1, A_2, \ldots, A_k\} \in \mathbf{B} \Leftrightarrow (A_1, A_2, \ldots, A_k) \in \rho.$$

Proof. Let $S'(2, k, v) = (V, \mathbf{B}, \in)$ be a complete Steiner system with parameters $v, k, 1$, and $\rho \subseteq V^k$ be defined as above. From the definition it follows that if $(A_1, A_2, \ldots, A_k) \in \rho$, then either $|\{A_1, A_2, \ldots, A_k\}| = k$ or $A_1 = A_2 = \ldots = A_k$, and moreover, $(A_1, A_2, \ldots, A_k) \in \rho$ if and only if $(B_1, B_2, \ldots, B_k) \in \rho$ for an arbitrary permutation $(B_1, B_2, \ldots, B_k)$ of $(A_1, A_2, \ldots, A_k)$. Hence ρ is totally symmetric k relation. For any two distinct points $A \neq B$, there is a unique block containing A, B, i.e. there is a unique $(A_1, A_2, \ldots, A_k) \in \rho$, such that $A, B \in \{A_1, A_2, \ldots, A_k\}$. And for any $A \in V$, the pair (A, A) is in the unique $(A, A, \ldots, A) \in \rho$. Hence, ρ is a totally symmetric $(2, k-2)$-quasigroup relation.

Conversely, let $\rho \subset V^k$ be a totally symmetric $(2, k-2)$-quasigroup relation satisfying $(A, A, \ldots, A) \in \rho$, and let $(V, \mathbf{B}, \in)$ be defined as above. If $(A_1, A_2, \ldots, A_k) \in \rho$ and $A_i = A_j = A$ for some $i \neq j$, then, since $(A, A, \ldots, A) \in \rho$, it follows that $A_1 = A_2 = \ldots = A_k = A$. Hence, if $(A_1, A_2, \ldots, A_k) \in \rho$, then $|\{A_1, A_2, \ldots, A_k\}| = k$ or $A_1 = A_2 = \ldots = A_k$. Let $\mathbf{B}' = \mathbf{B} \setminus \{\{A\}|\ A \in V\}$. Then it is easy to check that $(V, \mathbf{B}', \in)$ is a Steiner system with parameters $v, k, 1$, and $(V, \mathbf{B}, \in)$ is its extension. Hence, $(V, \mathbf{B}, \in)$ is a complete Steiner system with parameters $v, k, 1$. □

Example 1. A projective plane (V, B, I) of order 3 is a Steiner system $S(2, 4, 3^2 + 3 + 1)$. The weighted block design (V', B', I), where $V = V'$ $B' = B \cup B''$, $B'' = \{\{A\}|\ A \in V\}$, is a complete Steiner sistem with the same parameters as those of (V, B, I). The relation $\rho \subset V^4$ defined by $(A_1, A_2, A_3, A_4) \in \rho$ if and only if $(A_1, A_2, A_3, A_4) \in B$ or $A_1 = A_2 = A_3 = A_4$, is a totally simmetric $(2, 2)$-quasigroup relation satisfying the condition $(A, A, A, A) \in \rho$. The number of points is $|V| = 3^2 + 3 + 1 = 13$, the number of blocks is $|B'| = 13 + 13 = 26$, and $t_A = 4 + 4 = 8$, for all $A \in V$.

Theorem 3. *Every complete Steiner quadruple system $(V, \mathbf{B}, \in)$ defines a totally symmetric (3,1)-quasigroup relation $\rho \subseteq V^4$, where*

$$(A_1, A_2, A_3, A_4) \in \rho \Leftrightarrow \{A_1, A_2, A_3, A_4\} \in \mathbf{B}.$$

Conversely, any totally symmetric (3,1)-quasigroup relation $\rho \subseteq V^k$, *satisfying* $(A, A, B, B) \in \rho$, *for any* $A, B \in V$, *defines a complete Steiner quadruple system* $S'(3, 4, v) = (V, \mathbf{B}, \in)$, *where*

$$\{A_1, A_2, A_3, A_4) \in \mathbf{B} \Leftrightarrow (A_1, A_2, A_3, A_4) \in \rho.$$

Proof. Let $(V, \mathbf{B}, \in)$ be a complete quadruple Steiner system $S'(3,4,v)$ and $\rho \subseteq V^4$ be defined as above. From the definition it follows that if we have $(A_1, A_2, A_3, A_4) \in \rho$, then either $|\{A_1, A_2, A_3, A_4\}| = 4$ or $A_1 = A_2 \neq A_3 = A_4$ or $A_1 = A_3 \neq A_2 = A_4$ or $A_1 = A_4 \neq A_2 = A_3$ or $A_1 = A_2 = A_3 = A_4$, and moreover, $(A_1, A_2, A_3, A_4) \in \rho$ if and only if $(B_1, B_2, B_3, B_4) \in \rho$ for any permutation (B_1, B_2, B_3, B_4) of (A_1, A_2, A_3, A_4). Hence, ρ is a totally symmetric 4-relation. For any three distinct points $A \neq B \neq C \neq A$, there is a unique block containing A, B, C, i.e. there is a unique $(A_1, A_2, A_3, A_4) \in \rho$ such that $A, B, C \in \{A_1, A_2, A_3, A_4\}$. For any two distinct points $A \neq B \in V$, there is a unique block containing A, B, i.e. there is a unique $(A_1, A_2, A_3, A_4) \in \rho$, such that $\{A_1, A_2, A_3, A_4\} = \{A, B\}$. For any $A \in V$, there is unique $(A, A, A, A) \in \rho$. Hence, ρ is a totally symmetric (3,1)-quasigroup relation.

Conversely, let $\rho \subset V^4$ be a totally symmetric (3,1)-quasigroup relation satisfying $(A, A, B, B) \in \rho$, for any $A, B \in V$, and let $(V, \mathbf{B}, \in)$ be defined as above. If $(A_1, A_2, A_3, A_4) \in \rho$ and $A_i = A_j = A_s$ for some $i \neq j \neq s \neq i$, then, since $(A, A, A, A) \in \rho$, it follows that $A_1 = A_2 = A_3 = A_4$. If $(A_1, A_2, A_3, A_4) \in \rho$ and $A_i = A_j$, while $A_s \neq A_i$ and $A_s \neq A_j$ for some $i \neq j \neq s$, then, since $(A_i, A_i, A_s, A_s) \in \rho$, it follows that $\{A_1, A_2, A_3, A_4\} = \{A, B\}$. Hence, if $(A_1, A_2, A_3, A_4) \in \rho$, then either $|\{A_1, A_2, A_3, A_k\}| = 4$ or $A_1 = A_2 \neq A_3 = A_4$ or $A_1 = A_3 \neq A_2 = A_4$ or $A_1 = A_4 \neq A_2 = A_3$ or $A_1 = A_2 = A_3 = A_4$. Let $\mathbf{B}' = \mathbf{B} \setminus (\{\{A\}|\ A \in V\} \cup \{\{A, B\}|\ A \neq B \in V\})$. Then it is easy to check that $(V, \mathbf{B}', \in)$ is a Steiner quadruple system $S(3, 4, v)$, and $(V, \mathbf{B}, \in)$ is its extension. Hence $(V, \mathbf{B}, \in)$ is a complete Steiner system $S'(3, 4, v)$. □

Example 2. Let (V, B, I) be a Steiner quadruple systems $S(3, 4, 8)$. $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $B = \{(1234), (5678), (1256), (3478), (1278), (3456), (1357), (2468), (1368), (2457), (1458), (2367), (1467), (2358)\}$. The weighted block design (V', B', I), where $V = V'$, $B' = B \cup B'' \cup B'''$, $B'' = \{\{A\}|A \in V\}$, $B''' = \{\{A, B\}|A \neq B \in V\}$ is a complete Steiner quadruple systems with the same parameters as those of (V, B, I).

The relation $\rho \subset V^4$ defined by $(A_1, A_2, A_3, A_4) \in \rho$ if and only if $(A_1, A_2, A_3, A_4) \in B$ or $A_1 = A_2 = A_3 = A_4$, or $A_1 = A_2 \neq A_3 = A_4$

or $A_1 = A_3 \neq A_2 = A_4$ or $A_1 = A_4 \neq A_2 = A_3$ is a totally symmetric $(3,1)$-quasigroup. The number of points is $|V| = 8$, the number of blocks is $|B'| = |B| + |B''| + |B'''| = 14 + 8 + 7 + \cdots + 2 + 1 = 50$. For every point $A \in V$ its weight is $t_A = 8 + 4 + 7 \cdot 2 = 26$.

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