NOVI SAD J. MATH.
VOL. 29, NO. 2 (1999), 71–84
PROC. VIII INT. CONF.
"ALGEBRA & LOGIC" (NOVI SAD, 1998)

GOOD QUOTIENT RELATIONS AND POWER ALGEBRAS

Ivica Bošnjak and Rozália Sz. Madarász
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia
e-mail: *{ivb,rozi}@unsim.ns.ac.yu*

Abstract

In this paper we study the sets of good, very good, Hoare good and Smyth good relations on an algebra. In particular, we give a necessary and sufficient condition for the $\subseteq$-ordered set of good relations on an algebra to be a lattice. Also, we investigate the sets of very good, Hoare good and Smyth good relations in respect to various ways of powering.

AMS Math. Subject Classification (1991): 08A30, 08A99
Key words and phrases: good relations, congruences, power algebras

1. Introduction

The notion of a *generalized quotient algebra* and the corresponding notion of a *good (quotient) relation* has been introduced in [4] and [5] as an attempt to generalize the notion of a quotient algebra to relations on an algebra which are not necessarily congruences. It follows from the definition of a good quotient relation (Definition 2) that, for example, any partial order on A is a good relation on any algebra $\mathcal{A}$ with the universe A. Hence, every non-trivial algebra has good relations which are not congruences. Some other examples of good relations are: compatible quasi-orders, structure preserving relations (Definition 3(b)) and quasi-congruences (Definition 3(d)). On the other hand, there are good relations on $\mathcal{A}$ which are neither reflexive, nor symmetric, nor transitive, which are not compatible and which are not argument preserving (see [5]).

As a justification for the name "generalized quotient algebra" we can quote the following "extended" homomorphism theorem (which can be proved easily by the definition of a good quotient relation):

Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type. Then $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$ if and only if there is a good relation R on $\mathcal{A}$ such that $\mathcal{B} \cong \mathcal{A}/R$.* □

Although the well-known isomorphism theorems cannot be proved for the whole class of good relations, there are some special classes of good relations for which they hold.

The most of the results on good relations has been obtained in the context of *power algebras* (Definition 4). The concept of a power algebra $\mathcal{P}(\mathcal{A})$ originates with Frobenius in group theory. Since any subset of a group was referred to as a *complex*, power algebras are also sometimes known as *complex algebras.* We refer to the paper [2] for an overview on power structures.

Since for any algebra $\mathcal{A}$ and any relation $R \subseteq A^2$, the generalized quotient set A/R (see Definition 1a) is a subset of the power set $\mathcal{P}(A)$, one of the most natural questions is whether the generalized quotient algebra $\mathcal{A}/R$ is a subalgebra of the power algebra $\mathcal{P}(\mathcal{A})$.

Theorem. ([3]) *Let R be a good quotient relation on an algebra $\mathcal{A}$. Then $\mathcal{A}/R$ is a subalgebra of the power algebra $\mathcal{P}(\mathcal{A})$ if and only if R preserves the structure of $\mathcal{A}$.* □

The concept of a *power relation* has been introduced for some special relations (such as $\leq$ or $\subseteq$), independently, in one form or another, by various authors. The general definition of n-ary power relation R^+ (see Definition 5) was introduced in [7]. The motivation for the definition of relations $R^{\rightarrow}$ and $R^{\leftarrow}$ (Definition 5) comes from theoretical computer science. These two power relations, in general form, were introduced in [2]. In that paper the following "power" version of the homomorphism theorem was proved.

Theorem. *If θ is a congruence on $\mathcal{A}$, then θ^+ is a congruence on $\mathcal{P}(\mathcal{A})$ and*

$$\mathcal{P}(\mathcal{A}/\theta) \cong \mathcal{P}(\mathcal{A})/\theta^+ . \quad \square$$

It is natural to ask whether this theorem can be extended to the generalized quotient algebras and/or for "weak" power relations $R^{\rightarrow}$ and $R^{\leftarrow}$. The answer is not simple, because the "lifted" power relations R^{+}, $R^{\rightarrow}$ and $R^{\leftarrow}$ are not necessarily good on the power algebra. This fact was the motivation for introducing the corresponding notions of a *very good relation* ([4], [5]) and *Hoare good* and *Smyth good* relations ([3]). In [3], the following versions of the "power" homomorphism theorems were proved.

Theorem. *Let $\mathcal{A}$ be an algebra and $R \subseteq A^2$.*

(a) If R is Hoare good on $\mathcal{A}$, then $\mathcal{P}(\mathcal{A})/R^{\rightarrow}$ is a homomorphic image of $\mathcal{P}(\mathcal{A}/R)$.
(b) If R is Smyth good on $\mathcal{A}$, then $\mathcal{P}(\mathcal{A})/R^{\leftarrow}$ is a homomorphic image of $\mathcal{P}(\mathcal{A}/R)$.
(c) If R is very good on $\mathcal{A}$, then $\mathcal{P}(\mathcal{A})/R^{+}$ is a homomorphic image of $\mathcal{P}(\mathcal{A}/R)$. □

In [1], the relationships between Hoare good, Smyth good and very good relations were described.

In the present paper we study the sets of good, very good, Hoare good and Smyth good relations in respect to the set-theoretical operations and various ways of powering. Also, we give a necessary and sufficient condition for $\subseteq$-ordered set of good relations to be a lattice.

2. Basic notions

It is well known that the notion of a quotient algebra $\mathcal{A}/R$ is defined for any universal algebra $\mathcal{A} = \langle A, F\rangle$ and any congruence $R \subseteq A^2$ on $\mathcal{A}$ in the following way:

$$\mathcal{A}/R = \langle A/R, \{\lceil f \rceil : f \in F\}\rangle,$$

where A/R is the corresponding *quotient set* (i.e. the set of all equivalence classes $a/R = \{b \in A : bRa\}, a \in A$), and for any $f \in F$ of arity n, operation $\lceil f \rceil : (A/R)^n \to A/R$ is defined by

$$\lceil f \rceil (a_1/R, \ldots, a_n/R) = f(a_1, \ldots, a_n)/R. \qquad (1)$$

Definition 1. Let R be an arbitrary binary relation on $\mathcal{A}$.

(a) For any $a \in A$ we define $a/R = \{b : bRa\}$. The corresponding *generalized quotient set* is $A/R = \{a/R : a \in A\}$.

(b) Relation $\varepsilon(R) \subseteq A^2$ is defined by:

$$(a,b) \in \varepsilon(R) \iff a/R = b/R.$$

Of course, if $R \subseteq A^2$ is not a congruence on $\mathcal{A} = \langle A, F\rangle$, then the operations $\ulcorner f \urcorner$ ($f \in F$) are not necessarily well defined by (1). It is easy to see that Definition 1 is "good" (for every $f \in F$) if and only if $\varepsilon(R)$ is a congruence on $\mathcal{A}$.

Definition 2. Let $\mathcal{A} = \langle A, F\rangle$ be an algebra and $R \subseteq A^2$.

(a) We call R a *good (quotient) relation* on $\mathcal{A}$ if $\varepsilon(R)$ is a congruence on $\mathcal{A}$. The set of all good relations on $\mathcal{A}$ we denote by $G(\mathcal{A})$.

(b) If R is a good quotient relation on $\mathcal{A}$, the corresponding *generalized quotient algebra* $\mathcal{A}/R$ is

$$\mathcal{A}/R = \langle A/R, \{\ulcorner f \urcorner : f \in F\}\rangle,$$

where the operations $\ulcorner f \urcorner$ ($f \in F$) are defined by (1).

Definition 3. ([2], [3]) Let $\mathcal{A} = \langle A, F\rangle$ be an algebra and $R \subseteq A^2$.

(a) R is *argument preserving* on $\mathcal{A}$ if for any n-ary operation $f \in F$, and any $x_1, \ldots, x_n, z \in A$ we have

$$zRf(x_1, \ldots, x_n) \Rightarrow$$

$$\Rightarrow (\exists z_1, \ldots, z_n \in A)(z_1 R x_1 \ \& \ \ldots \ \& \ z_n R x_n \ \& \ z = f(z_1, \ldots, z_n)).$$

(b) R is *structure preserving* on $\mathcal{A}$ if R is compatible and argument preserving on $\mathcal{A}$.

(c) R is a *quasi-equivalence* on A if for all $x, y \in A$ $x/R = y/R \iff (xRy \ \& \ yRx)$.

(d) A compatible quasi-equivalence on $\mathcal{A}$ is called a *quasi-congruence* (on $\mathcal{A}$).

It is not hard to verify ([5], [3]) that any structure preserving relation and any quasi-congruence on an algebra are good quotient relations.

Definition 4.

(a) Let $f : A^n \to A$. We define the *power operation* $f^+ : \mathcal{P}(A)^n \to \mathcal{P}(A)$ in the following way:

$$f^+(X_1, \ldots, X_n) = \{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$

(b) If $\mathcal{A} = \langle A, \{f : f \in F\}\rangle$ is an algebra, the *power algebra* $\mathcal{P}(\mathcal{A})$ is defined to be

$$\mathcal{P}(\mathcal{A}) = \langle \mathcal{P}(A), \{f^+ : f \in F\}\rangle.$$

Definition 5. For any set A and any binary relation R on A the *power relations* $R^{\rightarrow}$, $R^{\leftarrow}$ and R^+ on $\mathcal{P}(A)$ are defined in the following way: for any $X, Y \in \mathcal{P}(A)$

$$X R^{\rightarrow} Y \iff (\forall x \in X)(\exists y \in Y)\, xRy$$

$$X R^{\leftarrow} Y \iff (\forall y \in Y)(\exists x \in X)\, xRy$$

$$R^+ = R^{\rightarrow} \cap R^{\leftarrow}.$$

Note that there are examples of good relations over an algebra $\mathcal{A}$, such that the "lifted" power relations $R^{\rightarrow}$, $R^{\leftarrow}$ and R^+ are not good over the power algebra $\mathcal{P}(\mathcal{A})$.

Definition 6. Let $\mathcal{A}$ be an algebra and $R \subseteq A^2$.

(a) We call R a *very good relation* on $\mathcal{A}$ if R^+ is good on $\mathcal{P}(\mathcal{A})$.

(b) We say that R is *Hoare good* on $\mathcal{A}$ if $R^{\rightarrow}$ is a good relation on $\mathcal{P}(\mathcal{A})$; R is *Smyth good* on $\mathcal{A}$ if $R^{\leftarrow}$ is a good relation on $\mathcal{P}(\mathcal{A})$.

It is not hard to see that every very good (or Hoare good, or Smyth good) relation is good on $\mathcal{A}$. But the converse is not true. Note that the names "Hoare" and "Smyth" point at the origins of these relations in theoretical computer science.

3. The set of good relations

In this section we study the set of all good relations on an algebra. In particular, we give a necessary and sufficient condition for the partially ordered set $\mathcal{G}(\mathcal{A}) = \langle G(\mathcal{A}), \subseteq \rangle$ to be a lattice.

Definition 7. Let $R \subseteq A^2$ and $\varphi : A/R \to \mathcal{P}(A)$. The relation $R_\varphi \subseteq A^2$ we define as

$$aR_\varphi b \iff a \in \varphi(b/R).$$

Theorem 1. *Let $\mathcal{A}$ be an algebra and $R \subseteq A^2$. Then R is a good quotient relation if and only if there is a congruence θ on $\mathcal{A}$ and an injection $\varphi : A/\theta \to \mathcal{P}(A)$ such that $R = \theta_\varphi$.*

Proof. ($\Leftarrow$) Let $\theta \in Con\mathcal{A}$ and $\varphi : A/\theta \to \mathcal{P}(A)$ be an injection. Then it can be proved that $\varepsilon(\theta_\varphi) = \theta$. Hence $\varepsilon(\theta_\varphi)$ is a congruence on $\mathcal{A}$ and θ_φ is a good relation on $\mathcal{A}$.

($\Rightarrow$) Let R be a good relation on $\mathcal{A}$. If we put $\theta = \varepsilon(R)$ then θ is a congruence on $\mathcal{A}$. Define $\varphi : A/\theta : \mathcal{P}(A)$ by $\varphi(a/\theta) = a/R$, $a \in A$. Then it is easy to prove that φ is an injection and $R = \theta_\varphi$. □

Theorem 2. *Let $\mathcal{A}$ be an algebra. The following conditions are equivalent:*

(1) Any $R \subseteq A^2$ is good on $\mathcal{A}$.
(2) $EqvA = Con\mathcal{A}$.

Proof. ($\Leftarrow$) Let $R \subseteq A^2$ and $\theta = \varepsilon(R)$. Define $\varphi : A/\theta \to \mathcal{P}(A)$ as $\varphi(a/\theta) = a/R$, $a \in A$. By the assumption, θ is a congruence. It is easy to prove that φ is an injection. Hence, according to Theorem 1, θ_φ is a good relation on $\mathcal{A}$. On the other hand, $\theta_\varphi = R$. Consequently, R is a good relation on $\mathcal{A}$.

($\Rightarrow$) Let $\theta \in EqvA$. Then $\theta = \varepsilon(\theta)$ and since θ is good, we conclude that $\varepsilon(\theta)$ is a congruence. So, $\theta \in Con\mathcal{A}$. □

Corollary 1. *Let $\mathcal{A} = \langle A, F \rangle$ be an algebra. The following conditions are equivalent:*

(1) $G(\mathcal{A}) = \mathcal{P}(A^2)$,
(2) All fundamental operations of $\mathcal{A}$ are projections or essentially nullary, or $|A| \le 2$.

Proof. It follows from Theorem 2 and from the well-known characterization of algebras which satisfy condition (2) of Theorem 2. □

Now we are able to find a necessary and sufficient condition for $\mathcal{G}(\mathcal{A})$ to be a lattice.

Lemma 1. *Let $\mathcal{A}$ be an algebra such that $E = Eqv A \setminus Con\mathcal{A} \neq \emptyset$. Then $\langle E, \subseteq \rangle$ has a minimal element.*

Proof. Let $\mathcal{A} = \langle A, F \rangle$ and $\theta \in E$. Then for some $f \in F$ of arity $n \geq 1$ and some $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ we have $(\forall i \leq n)\ a_i \theta b_i$, but not $f(a_1, \ldots, a_n) \theta f(b_1, \ldots, b_n)$. Let us define $\rho \subseteq A^2$ as the smallest equivalence relation on A which contains $\{(a_i, b_i) \mid i \leq n\}$. Of course, $\rho \subseteq \theta$, $\rho \in E$ and there are only finitely many relations $\sigma \in Eqv A$ such that $\sigma \subseteq \rho$. Hence, the set $\{\sigma \in E \mid \sigma \subseteq \rho\}$ has a minimal element. □

Theorem 3. *Let $\mathcal{A}$ be an algebra. The following conditions are equivalent:*

(1) $\mathcal{G}(\mathcal{A})$ is a lattice,
(2) $Eqv A = Con\mathcal{A}$.

Proof. ($\Rightarrow$) Suppose that $Eqv A \neq Con\mathcal{A}$ and let $E = Eqv A \setminus Con\mathcal{A}$. Since $E \neq \emptyset$, according to Lemma 1, $\langle E, \subseteq \rangle$ has a minimal element θ. As $\theta \neq \Delta_A$, there exist distinct elements $x_1, x_2 \in A$ such that $x_1 \theta x_2$. Since $\theta \neq A^2$, there exists an element $c \in A$ such that $\neg x_1 \theta c$. Let us define an equivalence relation $\rho \subseteq \theta$ in the following way:

$$x \rho y \iff (x \theta y \;\&\; x \neq x_1 \;\&\; y \neq x_1) \text{ or } x = y = x_1.$$

Since θ is minimal in E and $\rho \subseteq \theta$, we have $\rho \in Con\mathcal{A}$. According to Theorem 1, for any injection $\varphi : A/\rho \to \mathcal{P}(A)$ relation ρ_φ is a good relation on $\mathcal{A}$. Let us define injections $\varphi_1, \varphi_2 : A/\rho \to \mathcal{P}(A)$ such that $\rho_{\varphi_1} \cap \rho_{\varphi_2} = \theta$:

$$\varphi_1(x/\rho) = \begin{cases} x/\theta & \text{if } \neg x \theta x_2 \\ x_2/\theta \cup \{c\} & \text{if } x = x_1 \\ x_2/\theta & \text{if } x \theta x_2 \;\&\; x \neq x_1 \end{cases}$$

$$\varphi_2(x/\rho) = \begin{cases} x/\theta & \text{if } \neg x \theta x_2 \\ x_2/\theta & \text{if } x = x_1 \\ x_2/\theta \cup \{c\} & \text{if } x \theta x_2 \;\&\; x \neq x_1 \end{cases}$$

It is easy to see that φ_1 and φ_2 are injections, so ρ_{φ_1} and ρ_{φ_2} are good relations such that $\rho_{\varphi_1} \cap \rho_{\varphi_2} = \theta$. Similarly, we define injections $\varphi_3, \varphi_4 : A/\rho \to \mathcal{P}(A)$ such that for good relations ρ_{φ_3} and ρ_{φ_4} it holds $\rho_{\varphi_3} \cup \rho_{\varphi_4} = \theta$:

$$\varphi_3(x/\rho) = \begin{cases} x/\theta & \text{if } \neg x\theta x_2 \\ x_2/\theta \setminus \{x_2\} & \text{if } x = x_1 \\ x_2/\theta & \text{if } x\theta x_2 \,\&\, x \neq x_1 \end{cases}$$

$$\varphi_4(x/\rho) = \begin{cases} x/\theta & \text{if } \neg x\theta x_2 \\ x_2/\theta & \text{if } x = x_1 \\ x_2/\theta \setminus \{x_2\} & \text{if } x\theta x_2 \,\&\, x \neq x_1 \end{cases}$$

Then good relations ρ_{φ_1} and ρ_{φ_2} do not have an infimum in $\mathcal{G}(\mathcal{A})$ because

$$\inf(\rho_{\varphi_1}, \rho_{\varphi_2}) \subseteq \rho_{\varphi_1} \cap \rho_{\varphi_2} = \theta,$$

$$\rho_{\varphi_3} \cup \rho_{\varphi_4} = \theta \subseteq \inf(\rho_{\varphi_1}, \rho_{\varphi_2}).$$

Thus, we would have $\theta = \inf(\rho_{\varphi_1}, \rho_{\varphi_2})$, which is a contradiction with $\theta \notin G(\mathcal{A})$.

($\Leftarrow$) According to Theorem 2, if $EqvA = Con\mathcal{A}$, then $G(\mathcal{A}) = \mathcal{P}(A^2)$, so $\mathcal{G}(\mathcal{A})$ is a lattice. □

4. Very good, Hoare good and Smyth good relations

It is clear from the proof of Theorem 3 that the set of good relations of an algebra is not necessarily closed under union and intersection. The same is true for the sets of all very good, Hoare good and Smyth good relations.

Notation. In the sequel, we will use the notation $G^+(\mathcal{A})$, $G^\rightarrow(\mathcal{A})$ and $G^\leftarrow(\mathcal{A})$ for the sets of all very good, Hoare good and Smyth good relations on an algebra $\mathcal{A}$, respectively.

Example 1. The sets $G^+(\mathcal{A})$, $G^\rightarrow(\mathcal{A})$ and $G^\leftarrow(\mathcal{A})$ are not necessarily closed under $\cap$ and $\cup$. Namely, let $\mathcal{A} = \langle A, F\rangle$ be a simple algebra on $A = \{a, b, c\}$, $R = \{(a,b),(b,c),(c,a)\}$, $S = \{(a,b),(b,a),(c,c)\}$. Then R and S are very good, Hoare good and Smyth good relations, but $R \cap S$ and $R \cup S$ are neither very good nor Hoare good nor Smyth good.

On the other hand, using only the definition of a good relation, we can prove that $G(\mathcal{A})$ is closed under complementation (in respect to A^2). It is not a surprising fact that both $G^+(\mathcal{A})$ and $G^\rightarrow(\mathcal{A})$ are not closed under complementation.

Example 2. Let $\mathcal{A} = \langle\{a,b,c\}, f\rangle$, where $ar(f) = 1$ such that $f(a) = a$, $f(b) = b$, $f(c) = a$. If we define $R = \{(c,a),(b,b),(a,c)\}$, then R is Hoare good on $\mathcal{A}$, but $\overline{R}$ is not Hoare good on $\mathcal{A}$.

Example 3. Let $\mathcal{A} = \langle\{a,b\}, f\rangle$, $ar(f) = 1$, $f(a) = b$, $f(b) = a$ and $R = \{(a,a),(b,a),(a,b)\}$. Then R is very good on $\mathcal{A}$, but $\overline{R}$ is not very good on $\mathcal{A}$.

However, despite these facts, we can prove that $G^\leftarrow(\mathcal{A})$ is closed under complementation. To prove this, we need the following lemmas.

Lemma 2. ([1]) *Let R be a binary relation on A. Then for any $X, Y \subseteq A$ we have*

(a) $X/R^\leftarrow = \{Z \subseteq A : (\forall x \in X)\, Z \cap x/R \neq \emptyset\}$,
(b) $X \subseteq Y \Rightarrow Y/R^\leftarrow \subseteq X/R^\leftarrow$,
(c) $(X \cup Y)/R^\leftarrow = X/R^\leftarrow \cap Y/R^\leftarrow$,
(d) $(\forall x, y \in A)\,(x/R \subseteq y/R \iff \{x\}/R^\leftarrow \subseteq \{y\}/R^\leftarrow)$. □

Lemma 3. *Let $R \subseteq A^2$ be a Smyth good relation on an algebra $\mathcal{A} = \langle A, F\rangle$. Then for any $f \in F$, $ar(f) = n \geq 1$ and $X_1, \ldots, X_n, Y_1, \ldots, Y_n \subseteq A$ we have*

$$(\forall i \leq n)\; X_i/R^\leftarrow \subseteq Y_i/R^\leftarrow \Rightarrow f^+(X_1, \ldots, X_n)/R^\leftarrow \subseteq f^+(Y_1, \ldots, Y_n)/R^\leftarrow.$$

Proof. Let $X_i/R^\leftarrow \subseteq Y_i/R^\leftarrow$ for all $i \in \{1, \ldots, n\}$. According to Lemma 2(c) we have

$$(X_i \cup Y_i)/R^\leftarrow = X_i/R^\leftarrow \cap Y_i/R^\leftarrow = X_i/R^\leftarrow, \;\; i \in \{1, \ldots, n\}.$$

Since R is Smyth good, using Lemma 2(b) we conclude

$$f^+(X_1, \ldots, X_n)/R^\leftarrow =$$

$$= f^+(X_1 \cup Y_1, \ldots, X_n \cup Y_n)/R^\leftarrow \subseteq f^+(Y_1, \ldots, Y_n)/R^\leftarrow. \quad \square$$

Lemma 4. *Let $\mathcal{A}$ be an algebra and $R \subseteq A^2$. If $X, Y \in \mathcal{P}(A)$ and $X/R^{\leftarrow} \subseteq Y/R^{\leftarrow}$ then*

$$(\forall y \in Y)(\exists x \in X)\ x/R \subseteq y/R.$$

Proof. Suppose that for every $x \in X$, $x/R \not\subseteq y/R$. Then we would have

$$(\cup\{x/R \mid x \in X\}) \setminus y/R \in X/R^{\leftarrow},$$

$$(\cup\{x/R \mid x \in X\}) \setminus y/R \notin Y/R^{\leftarrow},$$

which is a contradiction. □

Theorem 4.

(a) *For any algebra $\mathcal{A}$, the set $G(\mathcal{A})$ is closed under complementation.*
(b) *The sets $G^{+}(\mathcal{A})$ and $G^{\rightarrow}(\mathcal{A})$ are not necessarily closed under complementation.*
(c) *For any algebra $\mathcal{A}$, the set $G^{\leftarrow}(\mathcal{A})$ is closed under complementation.*

Proof. *(a)* It follows from the definition of a good relation because $\varepsilon(R) = \varepsilon(\overline{R})$.

(b) See examples 2 and 3.

(c) Let $\mathcal{A} = \langle A, F\rangle$, $R \in G^{\leftarrow}(\mathcal{A})$ and $f \in F$, $ar(f) = n \geq 1$. We have to prove that for any $X_1, \ldots, X_n, Y_1, \ldots, Y_n \subseteq A$, if $X_i/\overline{R}^{\leftarrow} = Y_i/\overline{R}^{\leftarrow}$, $i \in \{1, \ldots, n\}$ then

$$f^{+}(X_1, \ldots, X_n)/\overline{R}^{\leftarrow} = f^{+}(Y_1, \ldots, Y_n)/\overline{R}^{\leftarrow}.$$

Let $Z \notin f^{+}(X_1, \ldots, X_n)/\overline{R}^{\leftarrow}$. Using Lemma 2(a), we conclude that there are $x_1 \in X_1, \ldots, x_n \in X_n$ such that $Z \cap f(x_1, \ldots, x_n)/\overline{R} = \emptyset$, or, equivalently,

$$Z \subseteq f(x_1, \ldots, x_n)/R. \tag{2}$$

On the other hand, because of Lemma 4, there are $y_1 \in Y_1, \ldots, y_n \in Y_n$ such that $y_i/\overline{R} \subseteq x_i/\overline{R}$, $i \in \{1, \ldots, n\}$, or, equivalently, $x_i/R \subseteq y_i/R$, $i \in \{1, \ldots, n\}$. According to Lemma 2(d) this implies $\{x_i\}/R^{\leftarrow} \subseteq \{y_i\}/R^{\leftarrow}$, $i \in \{1, \ldots, n\}$. Using Lemma 3 we get

$$f^{+}(\{x_1\}, \ldots, \{x_n\})/R^{\leftarrow} \subseteq f^{+}(\{y_1\}, \ldots, \{y_n\})/R^{\leftarrow},$$

and again, because of Lemma 2(d) this implies

$$f(x_1, \ldots, x_n)/R \subseteq f(y_1, \ldots, y_n)/R.$$

According to (2), we now conclude that $Z \subseteq f(y_1, \ldots y_n)/R$ or, equivalently, $Z \cap f(y_1, \ldots, y_n)/\overline{R} = \emptyset$. According to Lemma 2(a) this means that $Z \notin f^+(Y_1, \ldots, Y_n)/\overline{R}^{\leftarrow}$. □

As we have already mentioned, if R is a good relation on $\mathcal{A}$, then R^+, $R^{\rightarrow}$ and $R^{\leftarrow}$ are not necessarily good on the corresponding power algebra. A similar statement can be proved for very good relations. To verify this, we need the following results from [1].

Theorem 5. ([1]) *Let $\mathcal{A}$ be an algebra. If $R \subseteq A^2$ is a Hoare good relation on $\mathcal{A}$, then R is a Smyth good relation on $\mathcal{A}$.* □

Theorem 6. ([1]) *For any non-trivial type $\mathcal{F}$ of algebras and any cardinal $\lambda \geq 2$, there is an algebra $\mathcal{A}$ of type $\mathcal{F}$ with λ elements and a relation $R \subseteq A^2$ such that R is very good on $\mathcal{A}$, but R is not Smyth good on $\mathcal{A}$.* □

Corollary 2. *Let R be a very good relation on $\mathcal{A}$. Then*

(a) $R^{\rightarrow}$ does not have to be very good on $\mathcal{P}(\mathcal{A})$;
(b) $R^{\leftarrow}$ does not have to be very good on $\mathcal{P}(\mathcal{A})$.

Proof. (a) Suppose that for any algebra $\mathcal{A}$ and any very good relation R on $\mathcal{A}$, $R^{\rightarrow}$ is again very good. Then $R^{\rightarrow}$ must be good on $\mathcal{P}(\mathcal{A})$. But this is a contradiction with Theorem 5 and Theorem 6.

(b) Suppose that for any algebra $\mathcal{A}$ and any very good relation R on $\mathcal{A}$, $R^{\leftarrow}$ is again very good. Then $R^{\leftarrow}$ must be good on $\mathcal{P}(\mathcal{A})$. But this is a contradiction with Theorem 6. □

Example 4. Let $A = \{a, b, c\}$, $R = \{(a,a),(a,b),(b,b),(a,c),(c,c)\}$ and $f(a) = c$, $f(b) = a$, $f(c) = b$. Then R is very good on $\mathcal{A} = \langle A, f \rangle$ (because all R^+-classes are different), but R^+ is not very good on $\mathcal{P}(\mathcal{A})$.

Corollary 3. *Let R be a very good relation on $\mathcal{A}$. Then R^+ is not necessarily very good on $\mathcal{P}(\mathcal{A})$.*

Proof. See Example 4. □

In the sequel, we try to answer the same question for the sets of Smyth good and Hoare good relations.

Theorem 7. ([1]) *Let $R \subseteq A^2$ be a Hoare good relation on algebra $\mathcal{A}$. Then R is very good on $\mathcal{A}$.* □

Theorem 8. ([1]) *For any non-trivial type $\mathcal{F}$ of algebras and any cardinal $\lambda \geq 3$, there is an algebra $\mathcal{A}$ of type $\mathcal{F}$ with λ elements and a relation $R \subseteq A^2$ such that R is Smyth good on $\mathcal{A}$, but R is not very good on $\mathcal{A}$.* □

Corollary 4. *Let R be a Smyth good relation on $\mathcal{A}$. Then*

(a) R^+ is not necessarily Smyth good on $\mathcal{P}(\mathcal{A})$;
(b) $R^{\rightarrow}$ is not necessarily Smyth good on $\mathcal{P}(\mathcal{A})$.

Proof. (a) Suppose that for any algebra $\mathcal{A}$ and any Smyth good relation R on $\mathcal{A}$, R^+ is again Smyth good. Then R^+ must be good on $\mathcal{P}(\mathcal{A})$. But this is a contradiction with Theorem 8.

(b) Suppose that for any algebra $\mathcal{A}$ and any Smyth good relation R on $\mathcal{A}$, $R^{\rightarrow}$ is again Smyth good. Then $R^{\leftarrow}$ must be good on $\mathcal{P}(\mathcal{A})$. But this is a contradiction with Theorem 8 and Theorem 7. □

Not all the answers are negative for Smyth good and Hoare good relations.

Theorem 9. *Let R be a Smyth good relation on $\mathcal{A}$. Then $R^{\leftarrow}$ is a Smyth good relation on $\mathcal{P}(\mathcal{A})$.*

Proof. Let $\mathcal{A} = \langle A, F \rangle$ be an algebra, $f \in F$, $ar(f) = n \geq 1$ and $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ be subsets of $\mathcal{P}(A)$, such that $\alpha_i/(R^{\leftarrow})^{\leftarrow} = \beta_i/(R^{\leftarrow})^{\leftarrow}$, $i \in \{1, \ldots, n\}$. We have to prove that

$$(f^+)^+(\alpha_1, \ldots, \alpha_n)/(R^{\leftarrow})^{\leftarrow} = (f^+)^+(\beta_1, \ldots, \beta_n)/(R^{\leftarrow})^{\leftarrow}.$$

Let $\gamma \in (f^+)^+(\alpha_1, \ldots, \alpha_n)/(R^{\leftarrow})^{\leftarrow}$ and $Y \in (f^+)^+(\beta_1, \ldots, \beta_n)$. Then $Y = f^+(Y_1, \ldots, Y_n)$ for some $Y_1 \in \beta_1, \ldots, Y_n \in \beta_n$. According to Lemma 4, there exists $X_1 \in \alpha_1, \dots, X_n \in \alpha_n$ such that $(\forall i \leq n)\ X_i/R^{\leftarrow} \subseteq Y_i/R^{\leftarrow}$. Using Lemma 3 we conclude

$$f^+(X_1, \dots, X_n)/R^{\leftarrow} \subseteq f^+(Y_1, \dots, Y_n)/R^{\leftarrow}. \tag{3}$$

As $\gamma \in (f^+)^+(\alpha_1, \dots, \alpha_n)/(R^{\leftarrow})^{\leftarrow}$, then $(\exists X \in \gamma)\ X\ R^{\leftarrow}\ f^+(X_1, \dots, X_n)$ According to (3), this implies $(\exists X \in \gamma)\ X\ R^{\leftarrow}\ f^+(Y_1, \dots, Y_n) = Y$, and therefore

$$\gamma \in (f^+)^+(\beta_1, \dots, \beta_n)/(R^{\leftarrow})^{\leftarrow}. \quad \square$$

Lemma 5. ([1]) *Let $R \subseteq A^2$ be a Hoare good relation on an algebra $\mathcal{A} = \langle A, F\rangle$ and $f \in F$, $ar(f) = n \geq 1$. Then for any $X_1, \dots, X_n, Y_1, \dots, Y_n \subseteq A$ we have*

$$(\forall i \leq n)\ X_i/R^{\rightarrow} \subseteq Y_i/R^{\rightarrow} \Rightarrow$$
$$\Rightarrow f^+(X_1, \dots, X_n)/R^{\rightarrow} \subseteq f^+(Y_1, \dots, Y_n)/R^{\rightarrow}. \quad \square$$

Theorem 10. *Let R be a Hoare good relation on $\mathcal{A}$. Then $R^{\rightarrow}$ is a Hoare good relation on $\mathcal{P}(\mathcal{A})$.*

Proof. Let $\mathcal{A} = \langle A, F\rangle$ be an algebra, $f \in F$, $ar(f) = n \geq 1$, and $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n$ be subsets of $\mathcal{P}(A)$, such that $\alpha_i/(R^{\rightarrow})^{\rightarrow} = \beta_i/(R^{\rightarrow})^{\rightarrow}$, $i \in \{1, \dots, n\}$. We have to prove that

$$(f^+)^+(\alpha_1, \dots, \alpha_n)/(R^{\rightarrow})^{\rightarrow} = (f^+)^+(\beta_1, \dots, \beta_n)/(R^{\rightarrow})^{\rightarrow}.$$

Let $\gamma \in (f^+)^+(\alpha_1, \dots, \alpha_n)/(R^{\rightarrow})^{\rightarrow}$ and $X \in \gamma$. Then there exists $X_1 \in \alpha_1, \dots, X_n \in \alpha_n$ such that $X\ R^{\rightarrow}\ f^+(X_1, \dots, X_n)$. For $X_i \in \alpha_i$ we define X_i' as

$$X_i' = \{y \in A \mid (\exists x \in X_i \mid y\ R\ x\}$$

Now we have

$$X_i'\ R^{\rightarrow}\ X_i \Rightarrow \{X_i'\} \in \alpha_i/(R^{\rightarrow})^{\rightarrow} \Rightarrow$$
$$\Rightarrow \{X_i'\} \in \beta_i/(R^{\rightarrow})^{\rightarrow} \Rightarrow (\exists Y_i \in \beta_i)\ X_i'\ R^{\rightarrow}\ Y_i \Rightarrow$$
$$\Rightarrow (\exists Y_i \in \beta_i)\ X_i' \subseteq Y_i' \Rightarrow (\exists Y_i \in \beta_i)\ X_i/R^{\rightarrow} \subseteq Y_i/R^{\rightarrow}.$$

Since this holds for an arbitrary $i \in \{1, \dots, n\}$, using Lemma 5 we get the following

$$X_i/R^{\rightarrow} \subseteq Y_i/R^{\rightarrow} \Rightarrow f^+(X_1, \dots, X_n)/R^{\rightarrow} \subseteq f^+(Y_1, \dots, Y_n)/R^{\rightarrow} \Rightarrow$$
$$\Rightarrow X\ R^{\rightarrow}\ f^+(Y_1, \dots, Y_n) \Rightarrow \gamma \in (f^+)^+(\beta_1, \dots, \beta_n)/(R^{\rightarrow})^{\rightarrow}. \quad \square$$

References

[1] Bošnjak, I., Madarász, R. Sz., Power algebras and generalized quotient algebras, Algebra Universalis, to appear.

[2] Brink, C., Power structures, Algebra Universalis 30 (1993), 177–216.

[3] Brink, C., Jacobs, D. F., Nelte, K. and Sekaran, R., Generalized quotient algebras and power algebras, manuscript, 11pp., 1996.

[4] Goranko, V., Notes on "Power Structures", Section 7 (Generalized Quotient Algebras), private communication, 1994.

[5] Madarász, R., Generalized factor algebras, Mat. Bilten, 18 (XLIV) (1994), 29–38.

[6] Madarász, R., Remarks on Power Structures, Algebra Universalis 34 (1995), 179–184.

[7] Whitney, S., Théories linéaries, Ph.D. thesis, Université Laval, Québec, 1977.

Received November 11, 1998.