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DIRECTABLE AUTOMATA AND THEIR GENERALIZATIONS: A SURVEY

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1. Introduction and preliminaries

1.1 Introduction

Directable automata, known also as synchronizable, cofinal and reset automata, are a significant type of automata that has been a subject of interest of many eminent authors since 1964, when these automata were introduced in a paper by Černý. Some of their special types were investigated even several years earlier. As automata that correspond to so-called definite languages, definite automata were studied in 1956 by Kleene and in 1963 by Perles, Rabin and Shamir (see also Steinby's paper [79]), whereas nilpotent automata were investigated in 1962 by Shevrin (see also the book by Gécseg and Peák).

Several other important types of automata were also studied in the 60's. Reverse definite automata and languages, that are antipodes to definite automata and languages, were studied in 1963 by Brzozowski and in 1966 by Ginzburg. As a common generalization of definite and reverse definite automata, generalized definite automata were introduced in 1966 by Ginzburg, and they were also studied in 1969 by Steinby.

Various other specializations and generalizations of directable automata have appeared recently. In the papers by Petković, Ćirić and Bogdanović, and Bogdanović, Ćirić, Imreh, Petković and Steinby, and in the Ph.D. thesis by Petković, trap-directable, trapped, generalized directable, locally directable, uniformly locally directable and other related kinds of automata have been introduced and studied. Certain generalizations of directability and definiteness have also appeared in theories of nondeterministic automata and tree automata and languages (*cf.* [19], [33, 34], [40], [51] and [80], for example).

The main purpose of this survey is to describe certain properties of directable automata, their specializations and generalizations. These are mainly algebraic and structural properties, as well as the properties of their transition semigroups.

The paper is divided into three chapters and eleven sections. In the first chapter we introduce the notions and notations concerning automata, semigroups and ordered sets, which are needed for further work. In the second one we define the class of directable automata, certain classes that are contained in it and various classes that contain this class. Treating automata as unary algebras, we show that it is possible to study these classes as varieties and generalized varieties of automata, which is one of the main ideas of the paper.

The third chapter is devoted to structural and other properties of the automata introduced in the previous chapter. When an automaton starts work from some of its states, then that work is localized inside its monogenic subautomaton generated by that state. Therefore, if we want to know what are the possibilities of the given automaton, we have to know what are the properties possesed by all its monogenic subautomata. This fact is a motivation for defining two "localization" operators on classes of automata, which were introduced and studied by Steinby in [81], Petković, Ćirić and Bogdanović in [56], and Bogdanović, Ćirić, Imreh, Petković and Steinby in [4]. One of these operators is used in Section 2.2 to define various generalizations of directable automata such as locally directable and uniformly locally directable automata and many other related kinds of automata. The properties of both of these two operators, in the cases when they are applied to varieties and generalized varieties of automata, are described in Section 3.1. It is shown that these operators are closely related to the regularization operator on varieties and generalized varieties. This operator has been studied by numerous universal algebraists for many years, mainly by Plonka, Graczyńska and others.

Using various decompositions and composition methods, such as direct sum and subdirect decompositions, parallel compositions, extensions and retractive extensions of automata, in Section 3.2 we describe the structure of many automata introduced in Chapter 2. We also characterize the properties of these automata in terms of the properties of their transition semigroups.

In Section 3.3 we present a new kind of semigroups which are associated to directable automata and we show that these semigroups are more suitable for studying directable automata than the "classical" transition semigroups. In Section 3.4 we give variuos characterizations of subdirectly irreducible nilpotent, definite, reverse definite and generalized definite automata, and finally, in Section 3.5 we present various properties of directing words and languages consisting of them, directing congruences, etc.

1.2 Basic notions and notations

In this section we introduce the notions and notations that will be used in what follows. First we give some definitions concerning semigroups, quasi-ordered sets and partially ordered sets.

An element z of a semigroup S is called a *bi-zero* of S if $zsz = z$, for each $s \in S$, a *left zero* of S if $zs = z$, for each $s \in S$, a *right zero* of S if $sz = z$, for each $s \in S$, and a *zero* of S, if it is both a left and a right zero of S. A semigroup whose every element is a bi-zero (resp. left zero, right zero) is called a *rectangular band* (resp. *left zero band*, *right zero band*).

One may associate with any ideal T of a semigroup S a congruence relation ϱ_T on S, called the *Rees congruence* on A determined by T, which is defined in the following way: for $s, t \in S$ we say that $(s,t) \in \varrho_T$ if and only if either $s = t$ or $s, t \in T$. The factor semigroup S/ϱ_T, usually denoted by S/T, is called the *Rees factor* of S determined by T. A semigroup S is said to be an *ideal extension* of a semigroup T by a semigroup Q with zero if T is an ideal of S and the Rees factor S/T is isomorphic to Q. If, in addition, Q is a nilpotent semigroup, i.e. if $Q^k = \{0\}$, for some $k \in \mathbf{N}$, where 0 is the zero of Q, then we say that S is a *nilpotent extension* of T. Note that throughout the paper, $\mathbf{N}$ denotes the set of positive natural numbers and $\mathbf{N}^0 = \mathbf{N} \cup \{0\}$.

Let Q be a non-empty set. A binary relation $\preceq$ on Q is said to be a *quasi-order* on Q if it is reflexive and transitive, and in this case the system $(Q, \preceq)$ is called a *quasi-ordered set*. We say that a quasi-ordered set $(Q, \preceq)$ is *directed* if for any $a, b \in Q$ there exists $c \in Q$ such that $a \preceq c$ and $b \preceq c$, i.e. if each finite subset of Q has an upper bound. A subset Q' of Q is *cofinal* in Q if for any $a \in Q$ there exists $b \in Q'$ such that $a \preceq b$.

An anti-symmetric quasi-order is called a *partial order*. Partial orders are usually denoted by $\leq$. The system $(P, \leq)$, where P is a non-empty set and $\leq$ is a partial order on P, is called a *partially ordered set*. Let $(P, \leq)$ be a partially ordered set. A mapping $\varphi : P \to P$ is called *extensive*, if $a \leq a\varphi$, for each $a \in P$, *isotone*, if for any $a, b \in P$, $a \leq b$ implies $a\varphi \leq b\varphi$, and *idempotent*, if $(a\varphi)\varphi = a\varphi$, for each $a \in P$, and if φ is extensive, isotone and idempotent, then it is called a *closure operator* on the partially ordered set $(P, \leq)$. An element $a \in P$ is called *closed with respect to* φ, or φ*-closed*, if $a\varphi = a$.

Next we present some notions and notations from automata theory. The automata considered throughout the paper are automata without outputs, in the sense of the definition given by F. Gécseg and I. Peák in [22], and we call them simply automata. Therefore, an *automaton* is defined as a triple (A, X, δ), where A and X are non-empty sets, not necessarily finite, A is called the *set of states* and X is called the *input alphabet*, and $\delta : A \times X \to A$ is the *transition function* of this automaton. All automata that will be considered in the paper will have the same input alphabet X with $|X| \geq 2$. The *free monoid* and the *free semigroup* over X are denoted by X^* and X^+, respectively. The length of a word $u \in X^*$ is denoted by $|u|$. For any $k \in \mathbf{N}^0$, the subsets X^k, $X^{\leq k}$ and $X^{\geq k}$ of X^* are defined by $X^k = \{u \in X^* \mid |u| = k\}$, $X^{\leq k} = \{u \in X^* \mid |u| \leq k\}$ and $X^{\geq k} = \{u \in X^* \mid |u| \geq k\}$. There will be no danger of confusion if we denote the automaton and its set of states by the same letter, usually by A. This will be done throughout the paper, in order to simplify our notations.

Any input symbol $x \in X$ can be interpreted as a unary functional symbol which determines a unary operation x^A on A defined by $x^A : a \mapsto \delta(a, x)$. Thus, the alphabet X can be treated as a type of algebras consisting only of unary functional symbols and any automaton with X as its input alphabet can be treated as a unary algebra of type X, so the notions such as *congruence* on an automaton, *subautomaton*, *generating set*, *variety of automata*, etc. will have their usual algebraic meanings (*cf.* [7]). For $x \in X$ and $a \in A$, the image of a under the mapping x^A is denoted by ax^A. For a word $u = x_1x_2 \cdots x_n$, where $x_1, x_2, \ldots, x_n \in X$, the mapping $u^A : A \to A$ is defined by $au^A = ax_1^Ax_2^A \cdots x_n^A$, $a \in A$, and for the empty word e, we define e^A to be the identity mapping on A. When an automaton A is known from the context, we shall write simply au for au^A.

Let A be an automaton and let H be a non-empty subset of its states. The smallest subautomaton of A containing H is denoted by $\langle H \rangle$ and called the *subautomaton of A generated by H*. As it is known, it is represented by $\langle H \rangle = \{au \mid a \in H, u \in X^*\}$. If H is finite, i.e. if $H = \{a_1, a_2, \ldots, a_n\}$, then we write $\langle a_1, a_2, \ldots, a_n \rangle$ for $\langle H \rangle$. We say that a subautomaton B of A

is *finitely generated* if it is generated by a finite subset of A, and that it is a *monogenic subautomaton* of A if it is generated by a single state of A.

For a subset H of the states of an automaton A and a word $u \in X^*$, we write $Hu = \{au \mid a \in H\}$. The set of all subsets of an automaton A, with the transitions given by $H \mapsto Hx$, $x \in X$, is an automaton called the *power-set automaton* of A. An automaton A is said to be the *direct sum* of its subautomata A_α, $\alpha \in Y$, if $A = \bigcup_{\alpha \in Y} A_\alpha$ and $A_\alpha \cap A_\beta = \emptyset$, for all $\alpha, \beta \in Y$ with $\alpha \neq \beta$. By a *parallel composition* of automata A and B we mean any subautomaton of their direct product.

The class of all finite automata is denoted by **Fin**. For a class K of automata, we denote by $\underline{K}$ the class consisting of all finite automata from K, i.e. $\underline{K} = K \cap \mathbf{Fin}$. An automaton A is *connected* if for all $a, b \in A$ there exist $u, v \in X^*$ such that $au = bv$. The class of all connected automata is denoted by **Conn**. We say that A is *strongly connected* if it has no proper subautomata, i.e. if for all $a, b \in A$ there exists $u \in X^*$ such that $au = b$. A state a of an automaton A is called a *trap* if $au = a$, for all $u \in X^*$, i.e. if $\{a\}$ is a subautomaton of A. The set of all traps of A is denoted by $Tr(A)$. An automaton whose any state is a trap is called *discrete*. The class of all discrete automata is denoted by **D**. In particular, we denote the two-element discrete automaton by D_2. By **O** we denote the class of all automata having only one state.

If K_1 and K_2 are two classes of automata, then the *Mal'cev's product* of K_1 and K_2, denoted by $K_1 \circ K_2$, is the class of all automata A having the property that there exists a congruence relation θ on A such that the factor A/θ belongs to K_2 and every θ-class which is a subautomaton of A belongs to K_1. In particular, for a class K of automata, $K \circ \mathbf{D}$ is the class of all direct sums of automata from K.

The *transition semigroup* $S(A)$ of an automaton A, in some origins called the *characteristic semigroup* of A, can be defined in two equivalent ways. The first one is to define $S(A)$ as the subsemigroup of the full transformation semigroup on A consisting of all *transition mappings* on A, i.e. $S(A) = \{u^A \mid u \in X^+\}$. Another way is to define $S(A)$ to be the factor semigroup of the input semigroup X^+ with respect to the *Myhill congruence* μ_A on X^+ defined by: $(u, v) \in \mu_A$ if and only if $au = av$, for every $a \in A$. Note that $(u, v) \in \mu_A$ if and only if $u^A = v^A$.

One may associate with any subautomaton B of an automaton A a congruence relation ϱ_B on A, called the *Rees congruence* on A determined by

B, which is defined in the following way: for $a, b \in A$ we say that $(a, b) \in \varrho_B$ if and only if either $a = b$ or $a, b \in B$. The factor automaton A/ϱ_B is usually denoted by A/B, and it is called the *Rees factor automaton* of A with respect to B. We say that an automaton A is an *extension* of an automaton B by an automaton C if B is a subautomaton of A and the Rees factor automaton A/B is isomorphic to C. In other words, the automaton C can be viewed as an automaton originated from A by contraction of B into a single state, the trap of C.

Let an automaton A be an extension of an automaton B. If A/B is a nilpotent automaton (the definition of a nilpotent automaton can be found in Section 2.1), then we say that A is a *nilpotent extension* of B. On the other hand, if there exists a homomorphism φ of A onto B such that $a\varphi = a$, for every $a \in B$, then we say that A is a *retractive extension* of B and φ is called a *retraction* of A onto B. A congruence θ on A is said to be a *B-congruence* if the restriction of θ to B is the equality relation on B, that is $\theta \cap \nabla_B = \Delta_B$, and if the equality relation Δ_A is the only B-congruence on A, then A is said to be a *dense extension* of B.

An automaton having only one state is called *trivial*. The smallest non-trivial subautomaton of an automaton A, if it exists, is called the *core* of A. Let H be a subset of the states of an automaton A and let θ be an equivalence relation on A. If H is the union of some family of θ-classes, then we say that it is *saturated by θ*, or that *θ saturates H*. On the other hand, for H the relation π_H on A defined by:

$$(a, b) \in \pi_H \;\Leftrightarrow\; (\forall u \in X^*)(au \in H \;\Leftrightarrow\; bu \in H),$$

is a congruence on A. It is the greatest congruence on A which saturates H and it is called the *principal congruence* on A determined by H. A subset H of the states of an automaton A is called *disjunctive* in A if $\pi_H = \Delta_A$, whereas an element $a \in A$ is called a *disjunctive element* of A if the singleton $\{a\}$ is a disjunctive subset of A.

For undefined notions and notations we refer to the books by Bogdanović and Ćirić [2], Burris and Sankappanavar [7], Gécseg and Peák [22], Grätzer [32], Howie [35], Lallement [46], Madarász and Crvenković[47], and Saliĭ[73].

1.3 Varieties and generalized varieties of automata

Let G be a non-empty set whose elements are called *variables*. By a *term of type X over G* we mean any word over the set $G \cup X$ of the form gu, with

$g \in G$ and $u \in X^*$. The set of all terms of type X over G, denoted by $T(G)$, is an automaton with the input alphabet X and the transitions defined by $(gu)x = g(ux)$, for $gu \in T(G)$ and $x \in X$. This automaton is called the *term automaton* over G. Any pair (s,t) of terms from $T(G)$, usually written as the formal equality $s = t$, is called an (*automaton*) *identity* over G or over $T(G)$. If the terms s and t contain the same variable, i.e. if the identity $s = t$ has the form $gu = gv$, with $g \in G$ and $u, v \in X^*$, then it is called a *regular identity*. Otherwise, if s and t contain distinct variables, i.e. if the identity $s = t$ has the form $gu = hv$, with $g, h \in G$, $g \neq h$, and $u, v \in X^*$, then it is called a *nonregular* or *irregular identity*.

We say that an automaton A *satisfies the identity* $s = t$ over $T(G)$, and we write $A \models s = t$, if the pair (s,t) of terms belong to the kernel of any homomorphism of $T(G)$ into A, i.e. if $s\varphi = t\varphi$, for any homomorphism φ of $T(G)$ into A. In other words, $A \models gu = gv$, with $g \in G$ and $u, v \in X^*$, if and only if $au = av$, for any $a \in A$, and $A \models gu = hv$, with $g, h \in G$, $g \neq h$, and $u, v \in X^*$, if and only if $au = bv$, for all $a, b \in A$. Evidently, when we work with identities satisfied on any automaton, it is enough to work with the two-element set $G = \{g, h\}$ of variables, and this will be done throughout the paper.

If Σ is a set of identities over $T(G)$, we say that an automaton A *satisfies* Σ, and we write $A \models \Sigma$, if it satisfies every identity from Σ. The class of all automata that satisfy Σ is denoted by $[\Sigma]$. If Σ is represented as $\Sigma = \{s_i = t_i \mid i \in I\}$, then we write $[s_i = t_i \mid i \in I]$ for $[\Sigma]$, and if $I = \{1, 2, \dots, n\}$, then we write $[s_1 = t_1, \dots, s_n = t_n]$. A class K of automata is called a *variety* if $K = [\Sigma]$, for some set of identities Σ. As it is known, varieties can be characterized as classes closed under subautomata, homomorphic images and direct products, or equivalently, as classes closed under homomorphisms and subdirect products. If $K = [\Sigma]$, then we say that K is the *variety defined* (or *represented*) *by the set of identities* Σ. Varieties that can be represented by sets of regular identities are called *regular varieties*, and varieties in which some nonregular identity is satisfied are called *nonregular* or *irregular varieties*.

Let Σ be a set of identities over $T(G)$, where $G = \{g, h\}$. If Σ can be written as $\Sigma = \{s_i = t_i\}_{i \in I}$, where $(I, \preceq)$ is a directed quasi-ordered set, then we say that Σ is a *directed set of identities*. In this case we also say that an automaton A of type X *ultimately satisfies* Σ, if there exists $k \in I$ such that $A \models s_i = t_i$, for each $i \succeq k$, and we write $A \models_u \Sigma$. The class of all automata ultimately satisfying Σ is denoted by $[\Sigma]_u$ or $[s_i = t_i \mid i \in I]_u$.

We say that a class K of automata is *ultimately defined* by a directed set of identities Σ if $K = [\Sigma]_u$. In the case when $I = \mathbf{N}$ with the usual ordering of natural numbers, that is $\Sigma = \{s_n = t_n\}_{n\in\mathbf{N}}$ is a sequence of identities, we write $[s_n = t_n \mid n \in \mathbf{N}]_\mathbf{u}$ or simply $[s_n = t_n]_u$, and for $K = [s_n = t_n]_u$ we say that it is *ultimately defined* by a sequence $\{s_n = t_n\}_{n\in\mathbf{N}}$ of identities. A family of varieties is called a *directed family of varieties* if it is a directed partially ordered set with respect to the inclusion of sets.

Classes that are ultimately defined by directed sets of identities are characterized by the following well-known result.

Theorem 1.3.1. ([1]) *The following conditions for a class K of automata are equivalent:*

- (i) *K is closed under homomorphic images, subautomata, finite direct products and arbitrary direct powers;*
- (ii) *K is the union of a directed family of varieties;*
- (iii) *K is ultimately defined by some directed set of identities.*

A class of automata satisfying anyone of the equivalent conditions of the previous theorem is called a *generalized variety* of automata. On the other hand, a class of finite automata satisfying anyone of the equivalent conditions of the next well-known theorem is called a *pseudovariety* of automata.

Theorem 1.3.2. ([1], [18]) *The following conditions for a class K of finite automata are equivalent:*

- (i) *K is closed under homomorphic images, subautomata and finite direct products;*
- (ii) *K is the class of all finite automata from some generalized variety;*

If the input alphabet X of the considered automata is finite, then the above two conditions are equivalent to the following one:

- (iii) *K is ultimately defined by some sequence of identities.*

The sets of all varieties of automata and all generalized varieties of automata, partially ordered by the set inclusion, are complete lattices called the *lattice of varieties* and the *lattice of generalized varieties* of automata, respectively. The meets in both of these lattices coincide with the set intersection, and the smallest element in both of them is the variety $\mathbf{O}$ of all automata having only one state.

2. Directability, its specializations and generalizations

2.1 Directable automata and their specializations

In this section we give the definitions of directable automata and its important special types and introduce the notations for the classes consisting of these automata and for certain languages associated to them. We also give some historical comments concerning the considered classes of automata.

For a given word $u \in X^*$, an automaton A is called *u-directable* if $au = bu$, for all $a, b \in A$, and in that case u is called a *directing word* of A. Furthermore, A is said to be *directable* if there exists a word $u \in X^*$ such that A is u-directable. In other words, the directing word u directs the states of the automaton A into a single state that will be denoted by d_u and called a *u-neck* of A. A state $d \in A$ is called a *neck* of A if there exists $u \in X^*$ such that d is a u-neck of A.

There are two main ways to specialize the notion of a directable automaton. First, for a given number $k \in \mathbf{N^0}$, an automaton A is called *k-definite* if each word from $X^{\geq k}$ is a directing word of A, and A is said to be *definite* if there exists $k \in \mathbf{N^0}$ such that A is k-definite. The smallest number $k \in \mathbf{N^0}$ for which A is k-definite is called the *degree of definiteness* of A. In particular, the 1-definite automata are called *reset automata*.

On the other hand, if $u \in X^*$ such that A is u-directable and has a trap a_0, then a_0 is both the unique trap and the unique neck of A. In this case u directs the states of A into the trap a_0, thus, A is called *trap-u-directable* and u is called a *trap-directing word* of A. An automaton A is said to be *trap-directable* if there exists a word $u \in X^*$ such that A is trap-u-directable.

Now we can give a common specialization of the notions of definite and trap-directable automata. Namely, for a given number $k \in \mathbf{N^0}$, an automaton A is called *k-nilpotent* if each word from $X^{\geq k}$ is a trap-directing word of A, or equivalently, if A is k-definite and has a trap. Furhermore, A is said to be *nilpotent* if there exists a number $k \in \mathbf{N^0}$ such that A is k-nilpotent. The smallest number $k \in \mathbf{N^0}$ for which A is k-nilpotent is called the *degree of nilpotency* of A.

To denote the classes of automata consisting of the above defined automata and the sets of words associated to an automaton A we use the notations given by Table 2.1.1.

notation	class of automata	notation	class of automata
$\mathbf{Dir}_u$	u-directable	$\mathbf{Dir}$	directable
$\mathbf{TDir}_u$	trap-u-directable	$\mathbf{TDir}$	trap-directable
$\mathbf{Def}_k$	k-definite	$\mathbf{Def}$	definite
$\mathbf{Nilp}_k$	k-nilpotent	$\mathbf{Nilp}$	nilpotent

notation	set of words	notation	set of words
$DW(A)$	directing words of A	$TDW(A)$	trap-directing words of A

Table 2.1.1.

It is obvious that

$$(1)\qquad \mathbf{Dir} = \bigcup_{u\in X^*} \mathbf{Dir}_u \qquad \mathbf{Def} = \bigcup_{k\in \mathbf{N}^0} \mathbf{Def}_k$$
$$\mathbf{TDir} = \bigcup_{u\in X^*} \mathbf{TDir}_u \qquad \mathbf{Nilp} = \bigcup_{k\in \mathbf{N}^0} \mathbf{Nilp}_k$$

Directable automata were defined and studied first by J. Černý in [9], 1964, and by P. H. Starke in [78], 1969, whereas the definite automata appeared even several years earlier. They were studied first as automata corresponding to the so-called definite languages, by S. C. Kleene in [44], 1956, and M. Perles, M. O. Rabin and E. Shamir in [54], 1963. Nilpotent automata were first studied by L. N. Shevrin in [77], 1962, and afterwards in the book of F. Gécseg and I. Peák [22], 1972. Only the trap-directable automata were introduced recently, by T. Petković, M. Ćirić and S. Bogdanović in [56], where they were studied under the name one-trapped automata. All those automata were a subject of interest of many other authors, and we shall mention the papers by S. Bogdanović, M. Ćirić, B. Imreh, T. Petković and M. Steinby [4], J. Černý, A. Piricka and B. Rosenauerová [10], M. Ćirić, B. Imreh and M. Steinby [15], B. Imreh [36, 37], B. Imreh and M. Steinby [39], M. Ito and J. Duske [41], J. E. Pin [58, 59], T. Petković, M. Ćirić and S. Bogdanović [56], T. Petković [55], I. Rystsov [66]–[72], M. Steinby [79], and others.

In some sources, various other names for directable automata and directing words were used. For example, J. E. Pin used in [58, 59] the names "synchronizable automata" and "synchronizing words", M. Ito and J. Duske in [41] used the name "cofinal automata", whereas the names "reset automata" and "reset words" were used by I. Rystsov. In [48] definite automata were studied under the name "local automata". Note again that the name "reset automaton" is used here as a synonym for "1-definite automaton". We also repeat that the name "one-trapped automaton", used in [56], is changed here into "trap-directable automaton".

2.2 Generalizations of directability

For a given word $u \in X^*$, an automaton A is called *generalized u-directable* if $auvu = au$ for every $a \in A$ and $v \in X^*$, and in this case u is called a *generalized directing word* of A. Furthermore, A is said to be *generalized directable* if there exists a word $u \in X^*$ such that A is generalized u-directable. If, for a given number $k \in \mathbf{N^0}$, every word from $X^{\geq k}$ is a generalized directing word of A, then A is called *generalized k-definite*, and A is said to be *generalized definite* if there exists $k \in \mathbf{N^0}$ such that A is generalized k-definite.

In a similar way the notions of a trap-directable automaton and a nilpotent automaton can be generalized. Namely, for a given word $u \in X^*$, an automaton is called *u-trapped* if $au \in Tr(A)$, for all $a \in A$. In this case u is called a *trapping word* of A, and A is said to be *trapped* if there exists a word $u \in X^*$ such that A is u-trapped. If, for a given number $k \in \mathbf{N^0}$, every word from $X^{\geq k}$ is a trapping word of A, then A is called *reverse k-definite*, and A is said to be *reverse definite* if there exists $k \in \mathbf{N^0}$ such that A is reverse k-definite.

It is evident that, for some $u \in X^*$, an automaton A is trap-u-directable if and only if it is u-directable and u-trapped, and for $k \in \mathbf{N^0}$, we have that A is k-nilpotent if and only if it is k-definite and reverse k-definite.

To denote the just defined classes of automata and sets of words associated to an automaton A, we use the notations given in Tables 2.2.1 and 2.2.2.

It is evident that

$$(2)\qquad \mathbf{GDir} = \bigcup_{u \in X^*} \mathbf{GDir}_u \qquad \mathbf{GDef} = \bigcup_{k \in \mathbf{N^0}} \mathbf{GDef}_k$$
$$\mathbf{Trap} = \bigcup_{u \in X^*} \mathbf{Trap}_u \qquad \mathbf{RDef} = \bigcup_{k \in \mathbf{N^0}} \mathbf{RDef}_k$$

Reverse definite automata appeared first in the paper by J. A. Brzo-

notation	class of automata	notation	class of automata
$\mathbf{GDir}_u$	generalized u-directable	**GDir**	generalized directable
$\mathbf{GDef}_k$	generalized k-definite	**GDef**	generalized definite
$\mathbf{Trap}_u$	u-trapped	**Trap**	trapped
$\mathbf{RDef}_k$	reverse k-definite	**RDef**	reverse definite

Table 2.2.1.

notation	set of words
$GDW(A)$	generalized directing words of A
$TW(A)$	trapping words of A

Table 2.2.2.

zowski [5], 1963, and in the book by A. Ginzburg [23], 1966, while generalized definite automata were first defined also by A. Ginzburg in the mentioned book. They were also studied by M. Steinby in [79], M. Ćirić, B. Imreh and M. Steinby in [15] and others. The remaining types of automata, generalized directable and trapped automata, were introduced and studied first in a recent paper by T. Petković, M. Ćirić and S. Bogdanović [56]. The notion "trapped automaton" was used by A. Nagy in [49] for automata which are 1-nilpotent in our terminology.

Another way to generalize the notions of directability, trap-directability, definiteness and nilpotency is to use two operators $L : K \mapsto L(K)$ and $CL : K \mapsto CL(K)$ on classes of automata defined as follows: If K is an arbitrary class of automata, then $L(K)$ denotes the class of all automata whose every monogenic subautomaton belongs to K, and $CL(K)$ denotes the class of all automata whose every finitely generated subautomaton belongs to K. The automata from $L(K)$ are said to *belong locally* to K, or that they are *locally K-automata*, whereas the automata from $CL(K)$ are said to *belong completely locally* to K, or that they are *completely locally K-automata*. The first of these two operators was originally used by M. Steinby in [81], 1994, while the second one was introduced by S. Bogdanović, M. Ćirić, B. Imreh, T. Petković and M. Steinby in [4]. The definitions that will be given in the further text are taken from the second of these papers and the paper by T.

Petković, M. Ćirić and S. Bogdanović [56], but they are slightly modified.

For a given word $u \in X^*$ and a number $k \in \mathbf{N}^0$, in the first column of Table 2.2.3 we give definitions of several new classes of automata, whereas in the second one we give the names for automata belonging to the corresponding classes.

definition	name
$\mathbf{LDir}_u = L(\mathbf{Dir}_u)$	locally u-directable
$\mathbf{LTDir}_u = L(\mathbf{TDir}_u)$	locally trap-u-directable
$\mathbf{LDef}_k = L(\mathbf{Def}_k)$	locally k-definite
$\mathbf{LNilp}_k = L(\mathbf{Nilp}_k)$	locally k-nilpotent
$\mathbf{LDir} = L(\mathbf{Dir})$	locally directable
$\mathbf{LTDir} = L(\mathbf{TDir})$	locally trap-directable
$\mathbf{LDef} = L(\mathbf{Def})$	locally definite
$\mathbf{LNilp} = L(\mathbf{Nilp})$	locally nilpotent
$\mathbf{ULDir} = \bigcup_{u \in X^*} \mathbf{LDir}_u$	uniformly locally directable
$\mathbf{ULTDir} = \bigcup_{u \in X^*} \mathbf{LTDir}_u$	uniformly locally trap-directable
$\mathbf{ULDef} = \bigcup_{k \in \mathbf{N}^0} \mathbf{LDef}_k$	uniformly locally definite
$\mathbf{ULNilp} = \bigcup_{k \in \mathbf{N}^0} \mathbf{LNilp}_k$	uniformly locally nilpotent

Table 2.2.3.

Note again that the terminology used by T. Petković, M. Ćirić and S. Bogdanović in [56] differs slightly from the one used in this paper. Namely, the automata whose name in [56] had the prefix "locally", in this paper have the prefix "uniformly locally", while the prefix "locally" is reserved here for another classes of automata.

If, for a given word $u \in X^*$, an automaton A is locally u-directable, then u is called a *locally directing word* of A, and if it is locally trap-u-directed, then u is called a *locally trap-directing word* of A. The notations for the sets consisting of such words are given in Table 2.2.4.

notation	set of words
$LDW(A)$	locally directing words of A
$LTDW(A)$	locally trap-directing words of A

Table 2.2.4.

2.3 Algebraic properties of the classes

The subject under discussion in this section is the classes defined in the preceeding two sections. It will be shown that these classes have some very interesting algebraic properties. For the classes having subscripts in their notations we prove that they are varieties of automata, we give their representations through automaton identities and we explain the mutual relationships between the classes with various subscripts. For the remaining classes we show that they are pairwise different generalized varieties of automata, and we give their inclusion diagram and the inclusion diagram for the corresponding pseudovarieties of automata.

The proofs of the presented results can be found in [56], [55] and [4].

The first theorem treats the classes that are varieties.

Theorem 2.3.1. *For an arbitrary word $u \in X^*$ and an arbitrary number $k \in \mathbf{N}^0$, the classes listed below are varieties of automata with the given representations:*

$\mathbf{Dir}_u = [gu = hu]$;
$\mathbf{TDir}_u = [gux = hu \mid x \in X]$;
$\mathbf{Trap}_u = [gux = gu \mid x \in X]$;
$\mathbf{GDir}_u = [guwu = gu \mid w \in X^*]$;
$\mathbf{LDir}_u = [gwu = gu \mid w \in X^*]$;
$\mathbf{LTDir}_u = [gwux = gu \mid w \in X^*, x \in X]$;
$\mathbf{Def}_k = [gu = hu \mid u \in X^{\geq k}]$;
$\mathbf{RDef}_k = [gux = gu \mid u \in X^{\geq k}, x \in X]$;
$\mathbf{Nilp}_k = [gux = hu \mid u \in X^{\geq k}, x \in X]$;
$\mathbf{GDef}_k = [guwu = gu \mid u \in X^{\geq k}, w \in X^*]$;
$\mathbf{LDef}_k = [gxu = gu \mid u \in X^{\geq k}, x \in X]$;

$$\mathbf{LNilp}_k = \big[gxu = gu,\, gux = gu \;\big|\; u \in X^{\geq k},\, x \in X\big].$$

We note that the varieties $\mathbf{Dir}_u$, $\mathbf{TDir}_u$, $\mathbf{Def}_k$ and $\mathbf{Nilp}_k$ are nonregular, and all other varieties from Theorem 2.3.1 are regular. The next two theorems explain the mutual relationships between some of these varieties.

Theorem 2.3.2. *Let $u, v \in X^*$ be arbitrary words and let $\mathbf{E}$ be an arbitrary element of the set $\{\mathbf{Dir}, \mathbf{Trap}, \mathbf{TDir}, \mathbf{GDir}, \mathbf{LDir}, \mathbf{LTDir}\}$. Then the following conditions hold:*

(a) $\mathbf{E}_u \subseteq \mathbf{E}_v$ *if and only if u is a subword of v;*
(b) $\mathbf{E}_u = \mathbf{E}_v$ *if and only if $u = v$;*
(c) $\mathbf{E}_u \cup \mathbf{E}_v \subseteq \mathbf{E}_{uv}$;

Moreover, for the empty word e we have that

$$\mathbf{E}_e = \begin{cases} \mathbf{O}, & \textit{for } \mathbf{E} \in \{\mathbf{Dir}, \mathbf{TDir}\} \\ \mathbf{D}, & \textit{for } \mathbf{E} \in \{\mathbf{Trap}, \mathbf{GDir}, \mathbf{LDir}, \mathbf{LTDir}\} \end{cases}$$

Theorem 2.3.3. *Let $\mathbf{E}$ be any element of the set $\{\mathbf{Def}, \mathbf{RDef}, \mathbf{Nilp}, \mathbf{GDef}, \mathbf{LDef}, \mathbf{LNilp}\}$. Then*

$$\mathbf{E}_0 \subset \mathbf{E}_1 \subset \ldots \subset \mathbf{E}_k \subset \mathbf{E}_{k+1} \subset \ldots \mathbf{E}.$$

Moreover,

$$\mathbf{E}_0 = \begin{cases} \mathbf{O}, & \textit{for } \mathbf{E} \in \{\mathbf{Def}, \mathbf{Nilp}\} \\ \mathbf{D}, & \textit{for } \mathbf{E} \in \{\mathbf{RDef}, \mathbf{GDef}, \mathbf{LDef}, \mathbf{LNilp}\} \end{cases}$$

By Theorems 2.3.2 and 2.3.3 it follows that the classes having no subscripts in their notations are directed unions of the corresponding varieties, that is they are generalized varieties of automata. The following theorem gives some more properties of these classes.

Theorem 2.3.4. *The classes listed in Figure 2.3.1 are pairwise different generalized varieties of automata and the figure represents their inclusion diagram. Furthermore, they form a semilattice under the set intersection.*

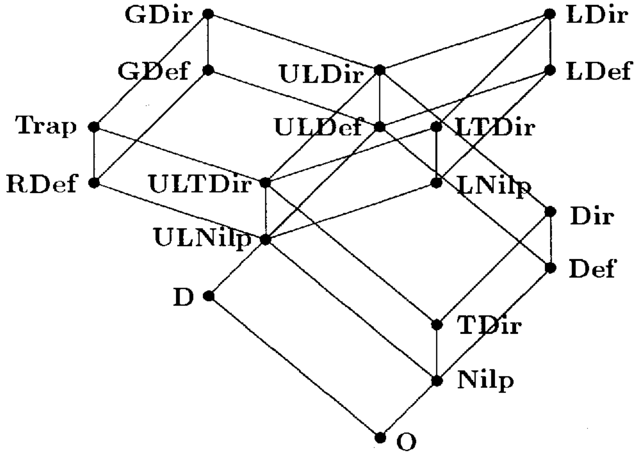

Figure 2.3.1.

As far as the corresponding pseudovarieties of automata are concerned, the situation slightly differs, as the next theorem shows.

Theorem 2.3.5. *The classes listed in Figure 2.3.2 are pairwise different pseudovarieties of automata and the figure represents their inclusion diagram. Furthermore, they form a semilattice under the set intersection.*

In other words, there are no finite automata in the classes

LDir \ **ULDir**, **LTDir** \ **ULTDir**, **LDef** \ **ULDef** and **LNilp** \ **ULNilp**.

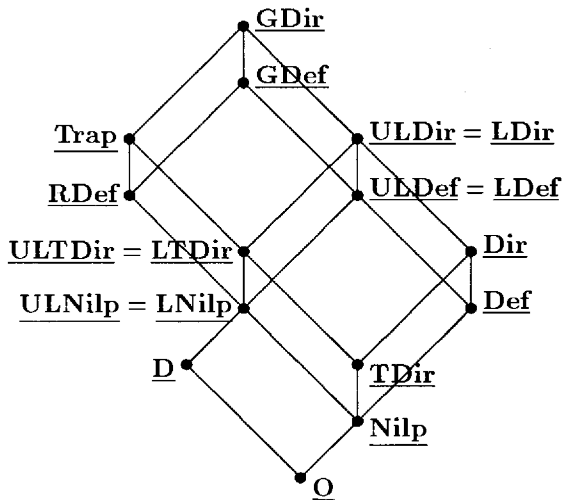

Figure 2.3.2.

3. Structural and other properties

3.1 Local closure operators

In Section 2.2 two operators $L : K \mapsto L(K)$ and $CL : K \mapsto CL(K)$ on classes of automata were used to define some new classes of automata. In the general case, these operators are isotone and meet-preserving operators on the partially ordered set of all classes of automata, but they are not closure operators. However, their properties are much better when they are applied on varieties and generalized varieties of automata, and the main goal of this section is to present the results that describe these properties.

First we give the following result.

Theorem 3.1.1. ([4]) *The operators L and CL are closure operators both on the lattice of varieties and on the lattice of generalized varieties of automata.*

Furthermore, $CL(\boldsymbol{V}) = \boldsymbol{V}$, for every variety of automata $\boldsymbol{V}$.

Hence, L and CL are closure operators on the lattices of varieties and generalized varieties of automata, and it is interesting to study the varieties and generalized varieties closed under the operators L and CL. On the other hand, it is also interesting to investigate the properties of the variety $L(\boldsymbol{V})$ associated to a variety $\boldsymbol{V}$ in terms of the properties of $\boldsymbol{V}$, and the properties of generalized varieties $L(\boldsymbol{K})$ and $CL(\boldsymbol{K})$, assigned to a generalized variety $\boldsymbol{K}$, in terms of the properties of $\boldsymbol{K}$.

In the case of both varieties and the case of generalized varieties it will be shown that the properties of $L(\boldsymbol{V})$, i.e. of $L(\boldsymbol{K})$ and $CL(\boldsymbol{K})$, depend on the possible regularity or nonregularity of $\boldsymbol{V}$ and $\boldsymbol{K}$. That is why we give first some information about regular and nonregular varieties of automata, and later we introduce the notions of regular and nonregular generalized varieties.

Recall that a variety of automata is called *regular* if it is defined by a set of regular identities. Otherwise, i.e. if some nonregular identity is satisfied on it, it is called *nonregular* or *irregular*. As it is known, the notion of a regular variety is also defined for arbitrary algebras. It is first introduced by J. Płonka in [60, 61], and after that it was a subject of interest of many universal algebraists. For more information about regular and nonregular varieties we refer to the cited papers by J. Płonka and E. Graczyńska, especially to the survey paper [28] and the papers [62, 64] and [27], concerning unary algebras.

Let us first describe the main properties of regular varieties of automata.

Theorem 3.1.2. ([60, 61, 43, 42, 65]) *Let $\boldsymbol{V}$ be a variety of automata. Then the following conditions are equivalent:*

(i) $\boldsymbol{V}$ *is a regular variety;*
(ii) $\boldsymbol{V} \circ \mathbf{D} = \boldsymbol{V}$*;*
(iii) $\mathbf{D} \subseteq \boldsymbol{V}$*;*
(iv) $D_2 \in \boldsymbol{V}$*.*

On the other hand, nonregular varieties are characterized as follows.

Theorem 3.1.3. *Let $\boldsymbol{V}$ be a variety of automata. Then the following conditions are equivalent:*

(i) $\boldsymbol{V}$ *is an irregular variety;*
(ii) $\boldsymbol{V} \subseteq \mathbf{Dir}_u$*, for some word $u \in X^*$;*
(iii) $\mathbf{D} \cap \boldsymbol{V} = \mathbf{O}$*;*
(iv) $\mathbf{D} \not\subseteq \boldsymbol{V}$*;*
(v) $D_2 \notin \boldsymbol{V}$*.*

Note that the condition (ii) of Theorem 3.1.3 is especially interesting, because it emphasizes the role of directable automata in studying regular and nonregular varieties of automata.

For an arbitrary variety of automata $\boldsymbol{V}$, there exists the smallest regular variety $R(\boldsymbol{V})$ containing $\boldsymbol{V}$, called the *regularization* of $\boldsymbol{V}$. Evidently, one obtains $R(\boldsymbol{V})$ as the intersection of all regular varieties containing $\boldsymbol{V}$, or as the variety determined by the set of all regular identities satisfied in $\boldsymbol{V}$. The regularization operator $R : \boldsymbol{V} \mapsto R(\boldsymbol{V})$ is a closure operator on the lattice of varieties of automata and R-closed elements in it are exactly the regular varieties. The next theorem shows that the local closure operator L coincides with the regularization operator R on the lattice of varieties of automata, and gives several other characterizations of the varietity $L(\boldsymbol{V})$ assigned to a variety $\boldsymbol{V}$.

Theorem 3.1.4. ([62, 64, 4]) *Let $\boldsymbol{V}$ be an arbitrary variety of automata. Then*

$$L(\boldsymbol{V}) = \boldsymbol{V} \circ \mathbf{D} = R(\boldsymbol{V}) = \mathbf{D} \vee \boldsymbol{V}.$$

Recall that $V \circ \mathbf{D}$ is the class consisting of all direct sums of automata from V, and $\mathbf{D} \vee V$ denotes the join of the varieties $\mathbf{D}$ and V in the lattice of varieties of automata. Therefore, the previous theorem also describes the structure of automata from $L(V)$ in terms of direct sums and the structure of automata from the variety V. It also shows that a variety of automata V is L-closed if and only if it is regular.

An interesting question is the following: If a variety of automata V determined by a set of identities Σ is given, how cean one find a set of identities (with a number of elements as small as possible) determining the variety $L(V)$, that is, $R(V)$? An algorithm for finding such set of identities was given by J. Płonka in [62] (see also E. Graczyńska [27]), whereas another one was given by S. Bogdanović, M. Ćirić, B. Imreh, T. Petković and M. Steinby in [4].

When we redirect one's attention from the action of the operators L and CL on varieties to their action on generalized varieties, many differences appear, but the regularity and the nonregularity of generalized varieties play still a crucial role. The definitions of regular and nonregular generalized varieties, introduced in [4], are given in the further text.

Theorem 3.1.5. ([4]) *Let K be a generalized variety of automata. Then the following conditions are equivalent:*

(i) *K is ulitimately defined by a directed set of regular identities;*
(ii) *K is the union of a directed family of regular varieties of automata;*
(iii) $\mathbf{D} \subseteq K$;
(iv) $D_2 \in K$.

A generalized variety of automata satisfying anyone of the equivalent conditions of Theorem 3.1.5 is called *regular*. Otherwise it is called *nonregular* or *irregular*. Nonregular generalized varieties are characterized by the following theorem.

Theorem 3.1.6. ([4]) *Let K be a generalized variety of automata. Then the following conditions are equivalent:*

(i) *K is a nonregular generalized variety of automata;*
(ii) *Any dircted set of identities which ultimately defines K contains a nonregular identity;*

(iii) *Any directed set of identities which ultimately defines* $\boldsymbol{K}$ *contains a cofinal subset of nonregular identities;*
(iv) $\boldsymbol{K} \subseteq \mathbf{Dir}$;
(v) $\mathbf{D} \cap \boldsymbol{K} = \mathbf{O}$;
(vi) $\mathbf{D} \nsubseteq \boldsymbol{K}$;
(vii) $D_2 \notin \boldsymbol{K}$.

The equivalence of the conditions (i) and (iv) is especially interesting and it can be also stated in the following way.

Theorem 3.1.7. ([4]) *Nonregular generalized varieties of automata form the principal ideal of the lattice of generalized varieties of automata generated by the generalized variety* **Dir** *of directable automata.*

Now we are ready to describe the properties of the operators L and CL on generalized varieties of automata.

Theorem 3.1.8. ([4]) *Let* $\boldsymbol{K}$ *be a generalized variety of automata. Then*

(a) $\boldsymbol{K}$ *is regular if and only if* $CL(\boldsymbol{K}) = L(\boldsymbol{K})$.
(b) $\boldsymbol{K}$ *is nonregular if and only if* $CL(\boldsymbol{K}) = L(\boldsymbol{K}) \cap \mathbf{Conn}$. *In this case*

$$L(\boldsymbol{K}) = CL(\boldsymbol{K}) \circ \mathbf{D} = CL(\boldsymbol{K} \circ \mathbf{D}).$$

Recall that the class denoted by **Conn** that appears in the previous theorem is the class of all connected automata.

In contrast to varieties of automata, which are regular if and only if they are L-closed, regular generalized varieties of automata are not necessarily L-closed. Before we give an example that confirms this claim, we state the following theorem that characterizes all L-closed generalized varieties of automata.

Theorem 3.1.9. ([4]) *Let* $\boldsymbol{K}$ *be a generalized variety of automata. Then*

$$L(\boldsymbol{K}) = \boldsymbol{K} \;\Leftrightarrow\; \boldsymbol{K} \circ \mathbf{D} = \boldsymbol{K}.$$

Therefore, a generalized variety of automata is L-closed if and only if it is closed under direct sums. Since the generalized variety **Trap** of trapped automata is not closed under infinite direct sums, it is an example of a regular generalized variety of automata which is not L-closed.

Note also that when a generalized variety $\boldsymbol{K}$ is represented as the union of a directed family $\{\boldsymbol{V}_i \mid i \in I\}$ of varieties, the generalized variety $L(\boldsymbol{K})$ is not necessarily the union of varieties $L(\boldsymbol{V}_i)$. For example, the generalized variety **Dir** of directable automata is a directed union of a family $\{\mathbf{Dir}_u \mid u \in X^*\}$, but the union of a directed family of varieties $\{L(\mathbf{Dir}_u) \mid u \in X^*\}$ is exactly the generalized variety **ULDir**, which is a proper subclass of $\mathbf{LDir} = L(\mathbf{Dir})$, as was shown by Theorem 2.3.4.

We shall finish the section giving several remarks concerning the action of the operators L and CL on pseudovarieties of automata. If K is a class of finite automata, the classes $L(K)$ and $CL(K)$ do not necessarily consist of finite automata, and when we work with classes of finite automata we have to combine the operators L and CL and the operator $K \mapsto \underline{K}$, which assigns to any class of automata the class $\underline{K}$ of all finite members of K. In other words, we have to modify the operator L by introducing an operator $\underline{L} : K \mapsto \underline{L}(K)$ on classes of finite automata by: $\underline{L}(K) = \underline{L(K)}$, for $K \subseteq \mathbf{Fin}$. The operator $\underline{L}$ thus defined preserves the finiteness of the members of classes, so it is a closure operator on the lattice of pseudovarieties of automata, and its properties are very similar to the properties of the operator L on varieties of automata. Evidently, $\underline{CL(\boldsymbol{P})} = \boldsymbol{P}$, for each pseudovariety of automata $\boldsymbol{P}$, and thus, it does not make sense to do a similar modification of the operator CL.

3.2 The structure and transition semigroups

From the general results presented in the previous section we can deduce the theorems that describe the structure of many automata considered in Section 2.2, such as the automata belonging to the varieties of locally u-directable, locally trap-u-directable, locally k-definite and locally k-nilpotent automata, and the automata belonging to the generalized varieties of locally directable, locally trap-directable, locally definite and locally nilpotent automata. For example, for any $u \in X^*$, locally u-directable automata can be described as direct sums of u-directable automata, whereas locally directable automata are characterized as direct sums of completely locally directable automata.

The main aim of this section is to describe the structure of other automata treated in Sections 2.1 and 2.2. This will be done using various decomposition and composition techniques, such as direct sum decompositions, extensions and retractive extensions, subdirect decompositions and parallel compositions of automata. On the other hand, we also characterize these automata through certain properties of their transition semigroups.

First, we give a theorem that describes generalized directable automata.

Theorem 3.2.1. ([56]) *The following conditions on an automaton A are equivalent:*

(i) *$S(A)$ has a bi-zero;*
(ii) *A is an extension of a locally directable automaton by a trap-directable automaton;*
(iii) *A is a generalized directable automaton.*

As we have seen before, locally directable automata are characterized as direct sums of completely locally directable automata, and they are not necessarily direct sums of directable automata. On the other hand, the automata which are direct sums of directable automata are not necessarily uniformly locally directable, as the following theorem shows.

Theorem 3.2.2. ([56]) *The following conditions on an automaton A are equivalent:*

(i) *$S(A)$ has a right zero;*
(ii) *A is a direct sum of directable automata with the same directing words;*
(iii) *A is a uniformly locally directable automaton.*

If A is a finite automaton, then the condition (ii) *can be replaced by*

(ii') *A is a direct sum of directable automata.*

Therefore, we can note that the class of automata which are direct sums of directable automata lies between the classes of locally directable and uniformly locally directable automata.

The next theorem that we present describes the structure of trapped automata.

Theorem 3.2.3. ([56]) *The following conditions on an automaton A are equivalent:*

(i) *$S(A)$ has a left zero;*

(ii) *A is an extension of a discrete automaton by a trap-directable automaton;*
(iii) *A is a trapped automaton.*

An especially interesting theorem is the following one which shows that the structure of uniformly locally trap-directable automata is very rich.

Theorem 3.2.4. ([56]) *The following conditions on an automaton A are equivalent:*

(i) *$S(A)$ has a zero;*
(ii) *A is a retractive extension of a discrete automaton by a trap-directable automaton;*
(iii) *A is a direct sum of trap-directable automata with the same trapping word;*
(iv) *A is a subdirect product of a discrete and a trap-directable automaton;*
(v) *A is a parallel composition of a discrete and a trap-directable automaton;*
(vi) *A is a uniformly locally trap-directable automaton;*

If A is a finite automaton, then the condition (iii) *can be replaced by*

(iii') *A is a direct sum of trap-directable automata.*

The next type of automata whose structure will be described are generalized definite automata.

Theorem 3.2.5. ([56]) *The following conditions on an automaton A are equivalent:*

(i) *$S(A)$ is a nilpotent extension of a rectangular band;*
(ii) *A is a nilpotent extension of a locally definite automaton;*
(iii) $(\exists m, n \in \mathbf{N})(\forall u \in X^{\geq m})(\forall v \in X^{\geq n})(\forall a \in A)(\forall p, q \in X^*)\ aupv = auqv$;
(iv) *A is a generalized definite automaton.*

Note that the condition (iii) of the previous theorem is the original definition of a generalized definite automaton given by A. Ginzburg in [23].

In a similar way as for uniformly locally directable and uniformly locally trap-directable automata, we give the characterization of the structure of uniformly locally definite automata.

Theorem 3.2.6. ([56]) *The following conditions on an automaton A are equivalent:*

(i) *$S(A)$ is a nilpotent extension of a right zero band;*
(ii) *A is a direct sum of definite automata with bounded degrees of definiteness;*
(iii) *A is a uniformly locally definite automaton.*

If A is a finite automaton, then the condition (ii) *can be replaced by the following one:*

(ii') *A is a direct sum of definite automata.*

The automata whose every monogenic subautomaton is a reset automaton are called *locally reset automata*. Therefore, as a special case of Theorem 3.2.6 we obtain the following theorem.

Theorem 3.2.7. ([56]) *The following conditions on an automaton A are equivalent:*

(i) *$S(A)$ is a right zero band;*
(ii) *A is a direct sum of reset automata;*
(iii) *A is a locally reset automaton.*

The following theorem, that can be deduced from the theorem characterizing trapped automata, describes the structure and transition semigroups of reverse definite automata.

Theorem 3.2.8. ([56]) *The following conditions on an automaton A are equivalent:*

(i) *$S(A)$ is a nilpotent extension of a left zero band;*
(ii) *A is a nilpotent extension of a discrete automaton;*
(iii) *A is a reverse definite automaton.*

The last theorem of the section describes the structure of uniformly locally nilpotent automata. The equivalence of the conditions (i) and (iii) was proved by L. N. Shevrin in [77] (see also the book by F. Gécseg and I. Peák [22]), whereas the equivalence of all other conditions was proved by T. Petković, M. Ćirić and S. Bogdanović in [56].

Theorem 3.2.9. ([56], [77], [22]) *The following conditions on an automaton A are equivalent:*

(i) *$S(A)$ is a nilpotent semigroup;*
(ii) *A is a retractive nilpotent extension of a discrete automaton;*
(iii) *A is a direct sum of nilpotent automata with bounded degrees of nilpotency;*
(iv) *A is a subdirect product of a discrete and a nilpotent automaton;*
(v) *A is a parallel composition of a discrete and a nilpotent automaton;*
(vi) *A is a uniformly locally nilpotent automaton.*

If A is a finite automaton, then the condition (iii) *can be replaced by*

(iii') *A is a direct sum of nilpotent automata.*

3.3 Characteristic semigroups of directable automata

In the previous section we saw that much information about an automaton can be obtained from information concerning its transition semigroup. Recall that the transition semigroup $S(A)$ of an arbitrary automaton A is defined as the subsemigroup of the full transformation semigroup on the set of states of A consisting of the mappings of the form u^A, where $u \in X^+$. On the other hand, we mentioned that it can be also defined as the factor semigroup of the free semigroup X^+ with respect to the Myhill congruence μ_A on X^+ that corresponds to the automaton A. But, we can observe that the Myhill congruence μ_A of A is defined by regular identities satisfied on A, that is to say, for $u, v \in X^+$ we have that

$$(u, v) \in \mu_A \;\Leftrightarrow\; A \models gu = gv.$$

Therefore, if an automaton A satisfies some nonregular identity, i.e. if it is directable, then the transition semigroup $S(A)$ of A does not give enough information about A. Motivated by this fact, T. Petković, M. Ćirić and S. Bogdanović introduced in [57] the notion of a characteristic semigroup of a directable automaton, defined in terms of nonregular identities satisfied on it. Namely, if A is a directable automaton, then a binary relation θ_A on X^+ defined by

$$(u, v) \in \theta_A \;\Leftrightarrow\; u = v \text{ or } A \models gu = hv$$

is a congruence relation on X^+, and the factor semigroup X^+/θ_A, denoted by $C(A)$, is called the *characteristic semigroup* of A. An equivalent definition of the congruence θ_A can be given using the following theorem that determines some conditions under which an automaton satisfies a nonregular identity.

Theorem 3.3.1. ([57]) *A nonregular identity $gu = hv$ is satisfied on an automaton A if and only if $u, v \in DW(A)$ and $d_u = d_v$ in A.*

For a directable automaton A let ϱ_A denote the Rees congruence on X^+ that corresponds to the ideal $DW(A)$ of X^+. Then the congruence θ_A can be expressed through ϱ_A and the Myhill congruence μ_A as follows.

Theorem 3.3.2. ([57]) *Let A be any directable automaton. Then*

$$\theta_A = \varrho_A \cap \mu_A.$$

By the previous theorem it follows immediately that the characteristic semigroup $C(A)$ is a subdirect product of the semigroup X^+/ϱ_A and the transition semigroup $S(A)$ of A.

As it is known, every semigroup is isomorphic to the transition semigroup of some automaton, and it is interesting to state and consider the following question: Is every semigroup the characteristic semigroup of some directable automaton? We shall show that the answer is negative, but we shall determine some necessary and sufficient conditions for a semigroup to be the characteristic semigroup of some directable automaton.

In order to describe such semigroups, T. Petković, M. Ćirić and S. Bogdanović introduced in [57] the notion of a 0-free semigroup as follows. Let S be a semigroup with the zero 0 and let it be generated by a set Y. We say that S is a *0-free semigroup* over Y if every nonzero element of S can be uniquely represented as the product of some elements from Y.

In the introductory section we mentioned that the input alphabet of the considered automata is fixed throughout the whole paper and it is denoted by X. The cardinality of this alphabet will be denoted by κ. A semigroup S is said to be *κ-generated* if it has a generating set of the cardinality not greater than κ.

Now we can state the theorem that describes characteristic semigroups of directable automata.

Theorem 3.3.3. ([57]) *Let S be a κ-generated semigroup. Then the following conditions on S are equivalent:*

(i) *S is a characteristic semigroup of some directable automaton;*
(ii) *S is a factor semigroup of the free semigroup X^+ with respect to a congruence relation θ on X^+ such that each nontrivial θ-class is a left ideal of X^+;*
(iii) *S is an ideal extension of a right zero band by a 0-free semigroup.*

Similarly, we characterize semigroups that are characteristic semigroups of trap-directable, definite and nilpotent automata. This is done by the next three theorems.

Theorem 3.3.4. ([57]) *Let S be a κ-generated semigroup. Then the following conditions on S are equivalent:*

(i) *S is a characteristic semigroup of some trap-directable automaton;*
(ii) *S is a Rees factor semigroup of the free semigroup X^+;*
(iii) *S is a 0-free semigroup.*

Theorem 3.3.5. ([57]) *Let S be a κ-generated semigroup. Then S is a characteristic semigroup of some definite automaton if and only if S is an ideal extension of a right zero band by a nilpotent 0-free semigroup.*

Theorem 3.3.6. ([57]) *Let S be a κ-generated semigroup. Then S is a characteristic semigroup of some nilpotent automaton if and only if S is a nilpotent 0-free semigroup.*

3.4 Subdirect irreducibility

The *direct products* and *subdirect decompositions* of automata are defined as usual. This kind of compositions can be interpreted as parallel connection of automata. The first investigation in this line was done by Yoeli [84] who characterized the finite subdirectly irreducible autonomous automata (automata with one input sign). Later this result was generalized for the infinite case by Wenzel [83]. A wider class, the class of commutative automata was studied in [20] and [21], where the subdirectly irreducible automata were characterized. For further special classes of finite automata such as nilpotent, definite, reverse definite, generalized definite, and transitive, the subdirectly irreducible automata were described in the papers [15], [36], [37], [76]. In [15], the description of subdirectly irreducible automata is achieved by considering certain characteristic congruences. These results are valid for the infinite case, too, and we recall them here.

A congruence θ on an automaton A is called *elementary* if there are two distinct states $a, b \in A$ such that $\theta = \Delta_A \cup \{(a,b),(b,a)\}$. The set of all elementary congruences on A is denoted by $\mathrm{Con}_e(A)$.

Let A be an automaton and $k \in \mathbf{N}^0$. Let us define the relation ρ_k on A as follows. For any $a, b \in A$,

$$a \, \rho_k \, b \quad \Leftrightarrow \quad (\forall u \in X^k) \; au = bu.$$

The relation ρ_1 was used already in [54], and the family of these relations was introduced in [79]. Using the notion of an elementary congruence and the relation ρ_1, we have the following assertion.

Theorem 3.4.1. ([15]) *A nontrivial definite automaton A is subdirectly irreducible if and only if* $\mathrm{Con}_e(A) = \{\rho_1\}$.

For any automaton A, the subsets A_k, $k \in \mathbf{N}^0$, of the state set are defined inductively as follows:

$$A_0 = \{a \in A \mid ax = a \;\text{ for all } x \in X\},$$
$$A_{k+1} = \{a \in A \mid ax \in A_k \;\text{ for all } x \in X\} \qquad (k \geq 0).$$

Moreover, if $A_0 \neq \emptyset$, then for each $k \in \mathbf{N}^0$, A_k is a subautomaton of A, and we denote by σ_k the Rees congruence that corresponds to A_k. Investigating these congruences, we have the following description of the subdirectly irreducible reverse definite automata.

Theorem 3.4.2. ([15]) *A nontrivial reverse definite automaton A is subdirectly irreducible if and only if either*

(1) $\mathrm{Con}_e(A) = \{\sigma_0\}$, *or*
(2) $\mathrm{Con}_e(A) = \{\sigma_1\}$.

Moreover, if (1) *holds, then* $|A_0| = 2$ *and A is not nilpotent, and if* (2) *holds, then* $|A_0| = 1$ *and A is nilpotent.*

By Theorem 3.4.2 we obtain the following description of the subdirectly irreducible nilpotent automata.

Theorem 3.4.3. ([15]) *A nilpotent automaton A with* $|A| > 2$ *is subdirectly irreducible if and only if* $\mathrm{Con}_e(A) = \{\sigma_1\}$.

Regarding the subdirectly irreducible generalized definite automata, the following characterization is valid.

Theorem 3.4.4. ([15]) *A generalized definite automaton A with at least three states is subdirectly irreducible if and only if either*

(1) $\mathrm{Con}_e(A) = \{\rho_1\}$, *or*
(2) $\mathrm{Con}_e(A) = \{\sigma_0\}$.

Extensions, kernels and cores of automata and their influence on the subdirect irreducibility are studied in [3]. Combining some earlier characterizations with the results of [3], we can conclude the following statements in terms of these notions.

Theorem 3.4.5. *Let A be a nilpotent automaton with the unique trap* a_0. *Then the following conditions are equivalent:*

(i) *A is subdirectly irreducible;*
(ii) *A satisfies the following two conditions:*
  (a) *there exists the greatest element* a_1 *in the partially ordered set* $(A \setminus \{a_0\}, \leq)$;
  (b) *for arbitrary* $a, b \in A \setminus \{a_0, a_1\}$ *there exists* $u \in X^*$ *such that* $au \neq bu$;
(iii) *A has a two-element core and* a_0 *is a disjunctive element;*
(iv) *A has a disjunctive element different than* a_0;
(v) *A is a dense extension of a two-element nilpotent automaton.*

Theorem 3.4.6. *A definite automaton A is subdirectly irreducible if and only if it satisfies one of the following two conditions:*

(1) *A is a subdirectly irreducible nilpotent automaton;*
(2) *A is a dense nilpotent extension of a subdirectly irreducible strongly connected automaton.*

Theorem 3.4.7. *A reverse definite automaton A is subdirectly irreducible if and only if it satisfies one of the following two conditions:*

(1) *A is a subdirectly irreducible nilpotent automaton;*
(2) *A is a dense nilpotent extension of a two-element discrete automaton.*

Theorem 3.4.8. *A generalized definite automaton A is subdirectly irreducible if and only if it satisfies one of the following conditions:*

(1) *A is a subdirectly irreducible nilpotent automaton;*
(2) *A is a dense nilpotent extension of a subdirectly irreducible strongly connected automaton;*
(3) *A is a dense nilpotent extension of a two-element discrete automaton;*
(4) *A is a dense nilpotent extension of a trap-extension of a subdirectly irreducible strongly connected automaton.*

3.5 Directing words

Throughout this section, by an automaton we always mean a finite automaton.

Regarding the directable automata, the main challenge from the very beginning has been Černý's Conjecture [9] which states that any n-state ($n \geq 1$) directable automaton has a directing word of length $(n-1)^2$ or less. The bound suggested by the conjecture is the lowest possible, but the best known upper bounds are of order $\mathcal{O}(n^3)$, and the conjecture remained one of the open problems of the automata theory. On the other hand, for some special classes of automata even better and accurate bounds have been presented (*cf.* [39],[58],[59], [66]–[72]).

From the practical point of view, it is important to know whether an automaton A having n states with $|X| = m$ is directable or not. It can be decided by constructing the power-set automaton. In this case, one should consider almost 2^n sets and form Awx, $x \in X$, for all sets Aw. Ito and Duske [41] suggested that the directability of an n-state automaton A can be tested by applying an input word t which contains all words over X of length k as subwords, where k denotes the maximum of the lengths of the shortest directing words of n-state directable automata. It is easy to see that A is directable if and only if $|At| = 1$. They show how one can construct such a word t, but the mere length $m^k + k - 1$ of the word renders the test unpractical even under small values of n and m. Supposing that Černý's Conjecture holds, which is the best we can hope for, the length of the test word will be of the order $\mathcal{O}(m^k)$. In [39], a more effective procedure is suggested for solving this decidability problem which is presented here. The time complexity of this algorithm is $\mathcal{O}(m \cdot n^2)$.

Let A be an arbitrary finite automaton. For any $k \in \mathbf{N^0}$, the relation $\xi_A(k)$ of *k-mergeability* on A is defined so that for $a, b \in A$,

$$(a,b) \in \xi_A(k) \quad \Leftrightarrow \quad (\exists w \in X^{\leq k})\ aw = bw.$$

Two states a and b are *mergeable* if they are k-mergeable for some $k \in \mathbf{N^0}$. We denote $\xi_A = \bigcup_{k \in \mathbf{N^0}} \xi_A(k)$. It is well-known (*cf.* [78]) that an automaton is directable if and only if all pairs of its states are mergeable.

As far as the computation of ξ_A is concerned, the following observations are important:

(1) $\xi_A(0) = \Delta_A$,

(2) $\xi_A(k) = \xi_A(k-1) \cup \{(a,b) \,|\, (\exists x \in X)\,(ax, bx) \in \xi_A(k-1)\}$, *for* $k > 0$,

(3) *If* $\xi_A(k) = \xi_A(k-1)$, *for some* $k > 0$, *then* $\xi_A(k) = \xi_A(k+1) = \ldots = \xi_A$,

(4) $\Delta_A = \xi_A(0) \subset \xi_A(1) \subset \ldots \subset \xi_A(k) = \xi_A(k+1) = \xi_A$ *for some* k, *where* $0 \leq k \leq \binom{n}{2}$.

The observations above suggest that the directability of A can be tested by computing successively $\xi_A(0)$, $\xi_A(1)$, $\xi_A(2)$, $\ldots$ until $\xi_A(k) = \xi_A(k-1)$. To do this effectively, we use the inverse transition table of A instead of the transition table itself. Also, we do not form explicitly each $\xi_A(k)$ although they appear in the sequence of computed relations.

The algorithm employs two data structures, a Boolean $n \times n$-matrix $\mathbf{M}$ and a list *NewPair* of pairs of states. For the sake of simplicity, we assume that $A = \{1, 2, \ldots, n\}$. Then $\mathbf{M}[i,j] = 1$ means that the pair i, j ($\in A$) is known to be mergeable. Since it suffices to consider just the pairs (i,j), where $1 \leq i < j \leq n$, we actually need just the upper part of $\mathbf{M}$. A pair appears in *NewPair* when i and j have found to be mergeable, but this fact has not yet been used for finding further mergeable pairs. The *inverted transition table*

$$\mathbf{I} = (\mathbf{I}[a,x])_{a \in A, x \in X}$$

is defined by $\mathbf{I}[a,x] = \{i \in A \,|\, ix = a\}$, for any $a \in A$, $x \in X$. Now, we have the following procedure.

Procedure 1.

Step 1. (Initialize **M** and *NewPair*) $\mathbf{M}[i,j] := 0$ for all $1 \leq i < j \leq n$, and $NewPair := \varepsilon$ (the empty list).

Step 2. Form the inverted transition table **I**.

Step 3. Find all pairs (a,x) $(\in A \times X)$ for which $|\mathbf{I}[a,x]| > 1$. For every such (a,x) consider each pair $i,j \in \mathbf{I}[a,x]$ with $i < j$. If $\mathbf{M}[i,j] = 0$, let $\mathbf{M}[i,j] := 1$ and append (i,j) to *NewPair*.

Step 4. Until $NewPair = \varepsilon$ do the following. Delete the first pair from *NewPair*; suppose it is (a,b). From **I** find all pairs (i,j), $i < j$, such that for some $x \in X$, $i \in \mathbf{I}[a,x]$ and $j \in \mathbf{I}[b,x]$, or $i \in \mathbf{I}[b,x]$ and $j \in \mathbf{I}[a,x]$. If $\mathbf{M}[i,j] = 0$, let $\mathbf{M}[i,j] := 1$ and append (i,j) to *NewPair*.

Step 5. If $\mathbf{M}[i,j] = 1$ whenever $1 \leq i < j \leq n$, then A is directable, otherwise not.

It can be seen (*cf.* [39]) that every automaton A has a unique minimal congruence denoted by ρ_A such that A/ρ_A is directable. It is an interesting question how can one determine it. For computing ρ_A, we consider the nonmergeable pairs of states.

For any automaton A, let $G_A = (V,E)$ be the directed graph defined as follows. The vertex set $V = \{\{a,b\} \mid a,b \in A, (a,b) \notin \xi_A\}$ consists of all unordered pairs of nonmergeable states of A. The edge set is $E = \{(\{a,b\},\{ax,bx\}) \mid \{a,b\} \in V, x \in X\}$. Note that $\{ax,bx\} \in V$ if $\{a,b\} \in V$ and $x \in X$. Let T denote the subset of V which is the union of (the vertex sets of) all strongly connected components of G_A from which no edges lead outside the component (*cf.* [16]). Then the following assertion is valid (see [39]).

Theorem 3.5.1. *The congruence ρ_A is equal to the transitive closure of the relation $\tau_A = \Delta_A \cup \{(a,b) \mid \{a,b\} \in T\}$.*

Using this observation, one can determine ρ_A in $O(m \cdot n^2 + n^3)$ time by the following algorithm.

Procedure 2.

Step 1. Compute ξ_A using Procedure 1.

Step 2. Form the graph $G_A = (V,E)$; the vertex set is obtained from ξ_A.

Step 3. Compute the strongly connected components forming the set T using the algorithm of [16].

Step 4. Form the relation τ_A and compute its transitive closure τ_A^∞. Then $\tau_A^\infty = \rho_A$.

A natural generalization of the directability is the directability of the nondeterministic automata. For nondeterministic automata, directability can be defined in several meaningful ways. In [40], the following three notions of directability are introduced.

An input word w of a nondeterministic automaton A is

(1) *D1-directing*, if the set of states aw in which A may be after reading w consists of the same single state c whatever the initial state a is;
(2) *D2-directing*, if the set aw is independent of the state a, for all $a \in A$,
(3) *D3-directing*, if there exists a state c which appears in all sets aw, $a \in A$.

The D1-directability of complete nondeterministic automata ($ax \neq \emptyset$, for all $a \in A$ and $x \in X$) was already studied by Burkhard [6]. He gave an exact exponential bound for the length of minimum-length D1-directing words of complete nondeterministic automata. In [26] it was shown that neither for D1- nor for D3-directing words the bound can be polynomial for general nondeterministic automata. On the other hand, Carpi [8] has found $\mathcal{O}(n^3)$ bounds for D1-directing words of unambiguous automata and for synchronizing pairs of maximal rational codes recognized by such automata. In [40], lower and upper bounds are derived for the lengths of the shortest D1-, D2- and D3-directing words.

In Chapter 2 we defined the languages $DW(A)$, $TDW(A)$, $LDW(A)$, $LTDW(A)$, $GDW(A)$ and $TW(A)$ associated to an automaton A. It is interesting to investigate the conditions under which a language in X^* can be represented as one of these languages. This problem was considered by Imreh and Ito in [38] and one of their results we can reformulate as follows.

Theorem 3.5.2. ([38]) *Let $L \subseteq X^*$ be a non-empty language. Then there exists a finite automaton A such that L is equal to one of the languages*

$$DW(A),\ TDW(A),\ LDW(A),\ LTDW(A),\ GDW(A) \text{ and } TW(A)$$

if and only if L is recognizable and $X^*LX^* = L$.

In other words, the condition $X^*LX^* = L$ means that L is an ideal of X^*. The automaton A associated with the given language L can be constructed from the recognizer accepting L. Note also that if we omit the requirements that A is finite and L is recognizable, then Theorem 3.5.2 is still valid.

In a similar way, one can define languages consisting of Di-directing words of nondeterministic automata ($i = 1, 2, 3$). These languages and the languages that correspond to deterministic automata were compared also in [38].

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