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AN INEQUALITY RELATED TO TWO INTEGER SEQUENCES SATISFYING AN ORDER CONDITION

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Abstract

The purpose of this note is to give a brief and clear description of an inequality with applications in computer graphics, including a sketch of the proof. Difficulties related to the formulation and to the proof of the inequality are due to the rather complicated order condition. A complete proof and a detailed description of the application are given in [1]. Roughly speaking, if the merging of two disjoint non-decreasing sequences, which consist of non-negative integers and have the same length, determines at most $h+1$ maximal subsequences of the input sequences, and if the sums of the r-th powers of the members are the same with the both sequences for $1 \leq r \leq h-1$, then the sum of the h-th powers is greater with the sequence which contains the largest integer of the merged sequence.

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1. Preliminaries

Assume that two disjoint non-decreasing sequences (named c-sequence and d-sequence, respectively) are given, both consisting of t non-negative integers:

$$c_1 \leq \ldots \leq c_t , \quad d_1 \leq \ldots \leq d_t , \quad \{c_1, \ldots, c_t\} \cap \{d_1, \ldots, d_t\} = \emptyset .$$

Consider the non-decreasing sequence of the length $2t$ that is obtained by *merging* the c-sequence and the d-sequence. A *maximal c-subsequence* is a maximal subsequence S of the c-sequence which satisfies the following property: there are no elements of d-sequence which arise in the merged sequence between the minimal and maximal member of S. *Maximal d-subsequences* are defined analogously. Thus interior, left and right maximal c-subsequences have the following forms, respectively:

$$d_i < c_j \leq \ldots \leq c_k < d_{i+1} \ , \quad c_1 \leq \ldots \leq c_k < d_1 \ , \quad d_t < c_j \leq \ldots \leq c_t \ .$$

The sequence pair (c, d) satisfies the *order condition* $O(h+1)$ if the total number of (alternate) maximal c- and d-subsequences is equal to $h + 1$.

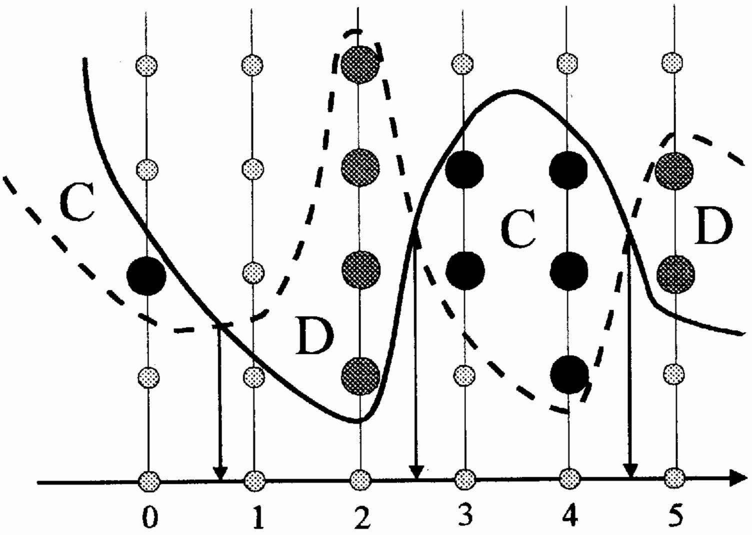

Figure 1. Two curves with 6 intersections determine 7 subintervals corresponding to the c- and d-sequences [1].

This order condition originates from considering the curves which correspond to two functions having at most h intersection points (for example, these may be polynomial functions of a degree not greater than h) within the considered interval. The members of the sequences correspond to the abscissas of integer lattice points lying between the two curves. The c-sequence

[1] The c-sequence and the d-sequence contain abscissas x of those integer lattice points (x, y), which satisfy $f_1(x) < y < f_2(x)$, respectively $f_2(x) < y < f_1(x)$.

is a non-decreasing sequence of abscissas corresponding to the integer lattice points which are below one curve and above the other; the roles of the curves are interchanged with the d-sequence. Consequently, the number of subintervals determined by the intersection points, as well as the number of maximal c- and d-subsequences, is upperbounded by $h+1$ (see Fig. 1).

We say that the c-sequence and the d-sequence satisfy the *equality condition* $E(r)$ if

$$c_1^r + \ldots + c_t^r \quad = \quad d_1^r + \ldots + d_t^r \ .$$

Without loss of generality, it will be assumed that $c_t < d_t$. The following definition is in accordance with this assumption:

The c-sequence and d-sequence satisfy the *inequality condition* $I(h)$ if

$$c_1^h + \ldots + c_t^h \quad < \quad d_1^h + \ldots + d_t^h \ .$$

2. Main result

Theorem 1. *If the c-sequence and the d-sequence satisfy the order condition $O(h+1)$ and the conditions $E(1), \ldots, E(h-1)$, then the inequality $I(h)$ is also satisfied.*

Theorem 1 depends on the parameter h. It will be denoted as $T(h)$ and can be restated in the form

$$T(h): \quad (\, O(h+1) \wedge E(1) \wedge E(2) \wedge \ldots \wedge E(h-1)\,) \implies I(h) \ .$$

3. Sketch of the proof

The proof of Theorem 1 is based on induction on h. The inductive assumption is primarily proved to be true for another sequence pair (y, z) of a common length w, which is derived from the pair (c, d).

In order to define the sequence pair (y, z), the index set $T = \{1, \ldots, t\}$ is partitioned into the subsets $C = \{i \in T |\ c_i > d_i\}$ and $D = \{i \in T |\ d_i > c_i\}$.

The condition $E(r)$ can be rewritten in the form

$$\sum_{i \in C} (c_i^r - d_i^r) \quad = \quad \sum_{i \in D} (d_i^r - c_i^r) \ . \qquad (*)$$

All the summands can be further split into sums of elementary differences in the following way:

$$(c_i^r - d_i^r) = (c_i^r - (c_i - 1)^r) + \ldots + ((d_i + 1)^r - d_i^r), \text{ for each } i \in C,$$

$$(d_i^r - c_i^r) = (d_i^r - (d_i - 1)^r) + \ldots + ((c_i + 1)^r - c_i^r), \text{ for each } i \in D.$$

It is easy to conclude that the number w of elementary differences on the both sides of $(*)$ is equal to the sums which arise for $r = 1$. The non-decreasing sequences $y_1, \ldots, y_w$ and $z_1, \ldots, z_w$ are defined to be the sorted values of x within those elementary differences of the form $(x+1)^1 - x^1$, that are obtained by splitting differences $c_i - d_i$, respectively differences $d_i - c_i$.

Denote by E', I' and O' respectively the equality conditions, the inequality condition and the order condition, which are related to the sequence pair (y, z) and which are defined analogously to the corresponding conditions with the pair (c, d).

The following proof scheme is applied:

1. $E(r) \wedge E'(0) \wedge E'(1) \wedge .. \wedge E'(r-2) \implies E'(r-1) \quad \mathit{for}\ r \geq 2\ ;$
2. $E(1) \wedge \ldots \wedge E(h-1) \implies E'(1) \wedge \ldots \wedge E'(h-2)\ ;$
3. $O(h+1) \wedge E'(1) \wedge \ldots \wedge E'(h-2) \implies O'(h)\ ;$
4. $O'(h) \wedge E'(1) \wedge \ldots \wedge E'(h-2) \implies I'(h-1)\ ;$
5. $E'(1) \wedge \ldots \wedge E'(h-2) \wedge I'(h-1) \implies I(h)\ .$

The proofs of the claims 1 and 5 are rather straightforward. The identity

$$(x+1)^r - x^r = \binom{r}{1} x^{r-1} + \binom{r}{2} + x^{r-2} + \ldots + \binom{r}{r-1} x + 1$$

is applied in two opposite directions.

The claim 2 follows by applying the claim 1, for $2 \leq r \leq h-1$. The claim 4 is proved by direct application of $T(h-1)$ (the inductive assumption) to the derived sequence pair (y, z).

The proof of the claim 3 is by far the most involved part of the proof of Theorem 1. Depending on parity of h, two branches are developed. It is interesting that the statements $T(1), \ldots, T(h-1)$ (applied to the derived sequences) should be (again) applied.

4. An example

Let $t = 6$, $h = 3$ and consider the following sequences: $(c_1, c_2, c_3, c_4, c_5, c_6) = (0, 3, 3, 4, 4, 4)$ and $(d_1, d_2, d_3, d_4, d_5, d_6) = (2, 2, 2, 2, 5, 5)$. These sequences correspond to the curves in Fig. 2.

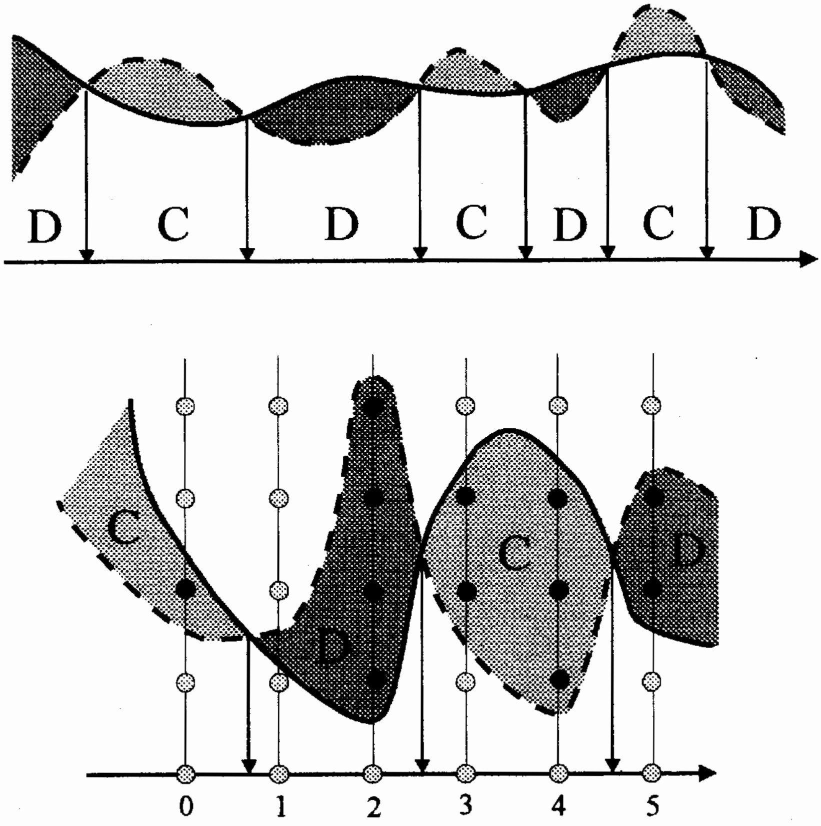

Figure 2. Illustration of the example

It is easy to check that these two sequences satisfy the condition $O(4)$, the maximal c- and d-subsequences being 0 – 2 2 2 2 – 3 3 4 4 4 – 5 5.

Further, the equalities $0+3+3+4+4+4 = 2+2+2+2+5+5 = 18$ and $0^2+3^2+3^2+4^2+4^2+4^2 = 2^2+2^2+2^2+2^2+5^2+5^2 = 66$ mean that the conditions $E(1)$ and $E(2)$ are satisfied. In accordance with Theorem 1, the inequality condition $I(3)$ is satisfied:

$$246 = 0^3+3^3+3^3+4^3+4^3+4^3 < 2^3+2^3+2^3+2^3+5^3+5^3 = 282.$$

The sets C and D are equal to $C = \{2,3,4\}$ and $D = \{1,5,6\}$, respectively. Splitting into elementary differences gives:

$(c_2 - d_2) = (3-2)$; $(c_3 - d_3) = (3-2)$; $(c_4 - d_4) = (4-3)+(3-2)$; $(d_1 - c_1) = (2-1)+(1-0)$; $(d_5 - c_5) = (5-4)$; $(d_6 - c_6) = (5-4)$. Thus $w = 4$, $(y_1, y_2, y_3, y_4) = (2,2,2,3)$ and $(z_1, z_2, z_3, z_4) = (0,1,4,4)$.

The sequence pair (y, z) satisfies the conditions:

$$\begin{aligned} O'(3):&\quad 0 \;-\; 2\,2\,2\,2 \;-\; 3\,3\,4\,4\,4 \;-\; 5\,5;\\ E'(1):&\quad 2+2+2+3 = 0+1+4+4 = 9;\\ I'(2):&\quad 21 = 2^2+2^2+2^2+3^2 < 0^2+1^2+4^2+4^2 = 33. \end{aligned}$$

5. Applications

An efficient general coding scheme for sets of digital curve segments (that are obtained from the corresponding segments of continuous curves by the integer grid approximation) has been proposed in [1]. Theorem 1 is used for proving correctness of the scheme for those sets of digital curve segments that satisfy the condition that h is an upper bound for the number of intersection points, for any two curves of the set. The sets may consist of digital curve segments that result from digitization of curves of different kinds. If the number of intersection points of any two curves in a set is upperbounded by h, then $h+3$ integer parameters $(x_1, m, b_0, b_1, \ldots, b_{h-1})$, where

$$b_r = \sum_{x = x_1}^{x_1+m-1} x^r \lfloor f(x) \rfloor, \qquad \text{for } 0 \le r \le h-1,$$

are sufficient for coding of the digital curve segment of the curve $f(x)$ on the interval $[x_1, x_1+1, \ldots, x_1+m-1]$.

References

[1] Žunić, J. and Acketa, D., A general coding scheme for families of digital curve segments, Graphical Models and Image Process. 60 (1998), 437–460.

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