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A CHARACTERIZATION OF GROUPS IN THE CLASS OF *-REGULAR SEMIGROUPS

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Abstract. In 1979, M. P. Drazin introduced *-regular semigroups as semigroups equipped with an involution in which every element has a generalized (Moore-Penrose) inverse. Since then, many characterizations of *-regular semigroups have been given, e.g. by Nambooripad and Pastijn, and others. In 1982, S. Crvenković described *-regular semigroups in terms of solvability of a certain type of linear equations in the involution semigroups. In this short note we prove that the solutions of all such equations are unique if and only if the binary reduct of the considered *-regular semigroup is a group.

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An algebra $\mathbf{S} = \langle S, \cdot, ^* \rangle$ of type $\langle 2, 1\rangle$ is an *involution semigroup* if $\langle S, \cdot\rangle$ is a semigroup and the following two identities hold in $\mathbf{S}$:

$$(xy)^* = y^*x^*, \tag{1}$$

$$(x^*)^* = x. \tag{2}$$

Sometimes (e.g. in [4]), these structures are called **-semigroups.* In 1979, M. P. Drazin [2] defined **-regular semigroups* as involution semigroups $\mathbf{S}$ in which for each $a \in S$ there exists $x \in S$ such that we have

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa.$$

Drazin showed that in *-regular semigroups such x must be unique, so that it can be denoted by $a^\dagger$. The element $a^\dagger$ is often referred to as the *generalized* (or the *Moore-Penrose*) *inverse* of a. The second of these two names for $a^\dagger$ reveals the motivation for Drazin's definition: it was inspired by the paper of R. Penrose [5], who proved that all complex square matrices have a generalized inverse, where * stands for the conjugate transpose of a matrix. In other words, the involution semigroups of complex square matrices (of a given size) are *-regular.

Later, *-regular semigroups were considered in many papers and characterized in several different ways, see e.g. Nambooripad and Pastijn [3], Nordahl

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and Scheiblich [4], Reilly [6], and others. For example, *-regular semigroups coincide with the involution semigroups in which every $\mathcal{L}$-class contains a *projection* – that is, an idempotent element e fixed by the involution (for which $e^* = e$ holds). However, it is the description of *-regular semigroups from [1] (reviewed below) which will be here in the focus of our interest.

Theorem 1. (Crvenković [1]) *An involution semigroup* $\mathbf{S} = \langle S, \cdot, ^* \rangle$ *is *-regular if and only if for all* $a \in S$ *the equation*

$$aa^*ax = a \tag{3}$$

has at least one solution. In that case, $a^\dagger = (ax)^*$.

Proof. ($\Rightarrow$) Using the results from [3] (or Lemma 3.2 from [1]), one can directly check that $a^\dagger(a^*)^\dagger$ is a solution of the given equation.

($\Leftarrow$) From (3) we have $x^*a^*aa^* = a^*$. Denote $y = (ax)^* = x^*a^*$. It follows $yaa^* = a^*$, which implies $yaa^*y^* = a^*y^*$, thus we have $ya(ya)^* = (ya)^*$. By applying the involution to this equality, we obtain $ya(ya)^* = ya$, yielding

$$ya = (ya)^* = a^*y^*.$$

Multiplying by a^*, we have

$$a^* = yaa^* = a^*y^*a^*,$$

i.e. $aya = a$. Moreover, the following equalities hold:

$$yy^*a^* = x^*a^*y^*a^* = x^*a^* = (ax)^* = y,$$

meaning that $ay(ay)^* = ayy^*a^* = ay$. Hence, in a similar fashion as above, we conclude that $ay = (ay)^*$ holds, as well as $y(ay)^* = yay$, i.e.

$$y = yy^*a^* = yay.$$

Therefore, y is a generalized inverse for a, so $a^\dagger = y = (ax)^*$. □

However, solutions of the equations (3) need not to be unique in *-regular semigroups. Thus it is natural to ask which of them satisfies exactly the uniqueness condition. In the following, we consider some obvious counterexamples.

Example 2. The equation (3) is not uniquely solvable in any *-regular semigroup with a zero 0, provided $a = 0$ (in that case, all elements of the considered *-regular semigroup are solutions). Such are, for example, the Penrose *-regular semigroups of complex square matrices. Aside from this trivial example, consider the *-regular semigroup of the latter type, whose elements are 2×2 matrices. The equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^* \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

is easily seen to be equivalent to the following linear system:

$$\begin{aligned} 4x_1 + 4x_3 &= 1, \\ 4x_2 + 4x_4 &= 1, \end{aligned}$$

whose solution is, clearly, not unique.

On the other hand, if $\langle S, \cdot \rangle$ is a reduct of a group, then all equations (3) have a unique solution. Of course, in that case we have $a^\dagger = a^{-1}$, and $x = (a^* a)^{-1}$ is the only solution of (3). The purpose of this note is to prove that there are, in fact, no other examples of *-regular semigroups with the desired uniqueness property. We are going to reach this conclusion in several steps, as will be shown below.

Theorem 3. *Let* $\mathbf{S} = \langle S, \cdot, ^* \rangle$ *be an involution semigroup such that for all* $a \in S$ *the equation (3) has a unique solution. Then* $\mathbf{S}$ *satisfies the following:*

(1) For all $a \in S$*, the equation* $ax = a$ *has a unique solution.*

(2) All idempotents in $\mathbf{S}$ *are projections.*

(3) If $e, f \in S$ *are idempotents, then* $L_e \le L_f$ *holds if and only if* $R_e \le R_f$*.*

(4) Every two different $\mathcal{L}$*-classes (*$\mathcal{R}$*-classes) of* $\mathbf{S}$ *are incomparable.*

(5) All $\mathcal{H}$*-classes of* $\mathbf{S}$ *are groups.*

(6) All $\mathcal{H}$*-classes of* $\mathbf{S}$ *are fixed by the involution operation* **.*

(7) $\mathbf{S}$ *has a unique idempotent.*

(8) $\langle S, \cdot, ^\dagger \rangle$ *is a group.*

Proof. (1) Assume that there exists an element $a \in S$ such that the equation $ax = a$ has two different solutions $x_1, x_2 \in S$. Then we have $ax_1 = ax_2 = a$, which yields $a^\dagger a x_1 = a^\dagger a x_2 = a^\dagger a$ (note that the operation † is well-defined on $\mathbf{S}$, since by the given condition and Theorem 1, $\mathbf{S}$ must be a *-regular semigroup). Also, the following equalities hold:

$$a^\dagger a = (a^\dagger a)^2 = (a^\dagger a)^3 = a^\dagger a (a^\dagger a)^* a^\dagger a,$$

thus if we put $b = a^\dagger a$, it follows that x_1 and x_2 are two different solutions of the equation $bb^*bx = b$. A contradiction.

(2) Let $e \in S$ be an idempotent. Since $ex = e$ by *(1)* has a unique solution, it follows $e = e^\dagger e$. But then

$$e^* = (e^\dagger e)^* = e^\dagger e = e,$$

as wanted.

(3) Assume that $L_e \leq L_f$ holds. Then there exists $a \in S$ such that $ae = f$, implying $(ae)^* = e^*a^* = f^*$. Using *(2)*, this reduces simply to $ea^* = f$, so that we immediately have $R_e \leq R_f$. The proof of the converse implication is analogous.

(4) Assume that **S** has two different and comparable $\mathcal{L}$-classes. Each of these classes contains an idempotent, since the semigroup reduct of **S** is a regular semigroup. Denote these idempotents by e and f, and let $L_e < L_f$. Then there exists $a \in S$ such that $ae = f$. But in that case we have

$$fe = ae^2 = ae = f,$$

hence (because of $ff = f^2 = f$), the equation $fx = f$ does not have a unique solution in **S**. A contradiction. The statement about $\mathcal{R}$-classes now follows from *(3)* and the facts just proved.

(5) Let $a \in S$ be arbitrary. Since $L_{a^2} \leq L_a$ and $R_{a^2} \leq R_a$, from *(4)* we conclude $L_{a^2} = L_a$ and $R_{a^2} = R_a$, which is equivalent to $H_{a^2} = H_a$, i.e. $a^2 \in H_a$. By Green's Theorem, H_a is a group.

(6) Let e be the unique idempotent in the $\mathcal{H}$-class H_a. By *(2)*, we have $e^* = e$. As the involution * clearly maps $\mathcal{H}$-classes onto $\mathcal{H}$-classes, it immediately follows $H_a^* = H_a$.

(7) Assume that **S** has two different idempotents e and f. By *(6)*, we have $(ef)^* \in H_{ef}$. On the other hand, *(2)* implies

$$(ef)^* = f^*e^* = fe,$$

thus $ef\mathcal{H}fe$. Further on, it must be $L_{ef} \leq L_f$, so $L_{ef} = L_f$ (i.e. $f\mathcal{L}ef$), by *(4)*. Similarly, $e\mathcal{L}fe$ holds, therefore $e\mathcal{L}f$. By a completely analogous argument we obtain $e\mathcal{R}f$, hence $e\mathcal{H}f$. This is impossible unless $e = f$, a contradiction.

(8) As the semigroup reduct of **S** is regular, **S** has exactly as many idempotents as $\mathcal{H}$-classes. Therefore, by *(7)*, **S** has only one $\mathcal{H}$-class. As a consequence of *(5)*, $\langle S, \cdot, ^\dagger \rangle$ is a group. □

Example 4. Of course, if $\langle S, \cdot \rangle$ is a binary reduct of a group, the involution * need not to coincide with its group inverse operation. Therefore, the *-regular semigroup **S** from the above theorem is not necessarily a group itself (where groups are considered as algebras of type $\langle 2, 1 \rangle$). For example, this is illustrated by the multiplicative group of nonzero complex numbers, where * is the complex conjugation. Also, in the Penrose's matrix *-regular semigroups, it is easy to see that the matrix equation

$$AA^*AX = A$$

has a unique solution if and only if A is regular. Regular matrices form a group, the well-known *general linear group* (over the given field). But the involution * yields the conjugate transpose of a matrix (implying that regular matrices are closed under *), which is in general different from its inverse matrix.

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