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ON PARSEVAL EQUALITIES AND BOUNDEDNESS PROPERTIES FOR KONTOROVICH-LEBEDEV TYPE OPERATORS

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Abstract. We describe a class of non-convolution type integral operators, where the integration is with respect to parameters of special functions. This class includes the famous Kontorovich-Lebedev, Mehler-Fock, Olevskii, Fourier-Jacobi transformations, which are quite important, for instance, in the solutions of boundary value problems in the mathematical theory of elasticity. Our techniques involve Mellin-Barnes type representations of the kernels, and the Mellin transform. General boundedness conditions and Parseval equalities are established. A series of examples are presented.

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1. Introduction and preliminary results

We study a special class of integral operators of the form

$$(\mathcal{G}_{\mathcal{H}}f)(x) = \int_0^\infty H(x,\tau)\tau f(\tau)d\tau, \tag{1.1}$$

where the kernel H is given as a Mellin integral [13], i.e.

$$H(x,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{H}(s,\tau)x^{-s}ds,\ s=\gamma+it,\ x\geq 0. \tag{1.2}$$

Note that if $H(x,\tau)=k(x\tau)$ then $\mathcal{H}(s,\tau)=k^{\mathcal{M}}(s)\tau^{-s}$, where $k^{\mathcal{M}}(s)$ denotes the Mellin transform of k

$$k^{\mathcal{M}}(s) = \int_0^\infty k(x)x^{s-1}dx.$$

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Such operators are called Mellin convolution type operators or general transformations of Fourier type. They were investigated in [13, Chapter VIII]. To obtain boundedness properties and inversion properties for such operators it is natural to use L_2-Mellin transform theory.

However, when $H(x,\tau)$ is essentially a function of two variables, the theory of such operators which we call index transformations, still has some gaps.

If we look at the table of the Mellin transforms for hypergeometric functions in [10] most examples there contain τ as a parameter in gamma-functions and they lead to integral transform operators of non-convolution type. Among them one can find the Kontorovich-Lebedev, the Mehler-Fock, the Olevskii and other transformations. An example is

$$\frac{1}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)x^{-s}ds = K_{i\tau}(2\sqrt{x}),\ \gamma>0, \tag{1.3}$$

where $\mathcal{H}(s,\tau)$ is equal to the product of two gamma-functions. It defines the Macdonald function [3] and in turn the following Kontorovich-Lebedev type operator ([2],[4],[12],[14],[15],[17],[18])

$$[KLf](x) = 2\int_0^\infty \tau K_{i\tau}(2\sqrt{x})f(\tau)d\tau. \tag{1.4}$$

In order to derive some generalities first, we impose the following general conditions on $\mathcal{H}$:

C1. For all $\tau \geq 0$ the function $s \mapsto \mathcal{H}(s,\tau)$ is analytic on a strip $-A \leq \mathrm{Re} s \leq A,\ A > 0$, with possible exception of a finite number of simple poles on the imaginary axis.

C2. For all $\tau \geq 0$ and all $-A < a < b < A$

$$\lim_{|t|\to\infty}\max_{a\leq\gamma\leq b}|\mathcal{H}(\gamma+it,\tau)| = 0.$$

C3. For all $\gamma \in (-A,A), \gamma \neq 0$ the Hilbert-Schmidt condition holds

$$\int_{-\infty}^{\infty}\int_0^\infty |\mathcal{H}(\gamma+it,\tau)|^2 d\tau dt < \infty.$$

Because of the Plancherel theorem for the Mellin transform [13] and the Fubini theorem the latter condition is equivalent to

$$\frac{1}{2\pi}\int_0^\infty d\tau\int_{-\infty}^{\infty}|\mathcal{H}(k+it,\tau)|^2 dt = \int_0^\infty\int_0^\infty |H(x,\tau)|^2 x^{2k-1}dxd\tau < \infty$$

for an arbitrary $0 < k < A$. This means that for $0 < k < A$ the operator $\mathcal{G}_{\mathcal{H}} : L_2(\mathbf{R}_+;\tau^2 d\tau) \mapsto L_2(\mathbf{R}_+;x^{2k-1}dx)$ is a Hilbert-Schmidt mapping. Hence $\mathcal{G}_{\mathcal{H}}$ is a bounded and compact operator.

For special classes of functions $\mathcal{H}$ we want to know the best possible weight functions in both L_2 -spaces. For us the perfect set of weight functions is such that a Parseval relation holds. In this case a candidate kernel for the inverse transform is immediately available.

Next, let $\varphi(s)$ be a function which is analytic in the strip $-A < \mathrm{Re} s < A$, again with the possible exception of a finite number of 1-th order poles of the imaginary axis. We will call φ a multiplier for $\mathcal{H}$ if the product $\mathcal{H}(s,\tau)\varphi(s)$ has again the properties C1-C3. Let φ be, in addition, the Mellin transform of some function ϕ on $\mathbf{R}_+$ $(\varphi(s) = \phi^{\mathcal{M}}(s))$. Then, due to the Mellin-Parseval equality [13] under suitable conditions the kernel from (1.2) for the corresponding operator (1.1), $(\mathcal{G}_{\mathcal{H}\varphi})(x)$ becomes

$$H_\varphi(x,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{H}(s,\tau)\varphi(s)x^{-s}ds = \int_0^\infty H(\xi,\tau)\phi\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}.$$

Hence, by formally changing the order of integration and invoking the definition (1.1) we have the following Mellin convolution representation for the operator $(\mathcal{G}_{\mathcal{H}\varphi})(x)$:

$$(1.5)\qquad (\mathcal{G}_{\mathcal{H}\varphi})(x) = \int_0^\infty H_\varphi(x,\tau)\tau f(\tau)d\tau = \int_0^\infty (\mathcal{G}_{\mathcal{H}}f)(\xi)\phi\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}.$$

A variety of examples of the operator transform (1.5) can be obtained as generalizations of the Kontorovich-Lebedev type operator (1.4). The kernel functions will be of hypergeometric type and they are conveniently defined using the Mellin transform. These Mellin transforms are the ratio of products of shifted Euler gamma-functions. Integral representations like (1.3) for them are called Mellin-Barnes integrals. Particular examples are the Meijer G-function and Fox H-function [10]. The relevant tables of Mellin transforms are given in [10].

Before we give a number of such examples, let us first consider a simple illustration of the operator (1.1) by taking

$$\mathcal{H}_s(s,\tau) = \frac{1}{\left(s+\frac{i\tau}{2}\right)\left(s-\frac{i\tau}{2}\right)}.$$

In this case calculation of the integral (1.2) leads to the result

$$H_s(x,\tau) = \begin{cases} -\frac{2}{\tau}\sin\left(\frac{\tau}{2}\log x\right), & 0 < x \le 1, \\ 0, & x \ge 1. \end{cases}$$

A Parseval equality can be obtained directly by contour integration. It corresponds to the Fourier sine transform, viz.

$$\int_0^1 |(\mathcal{G}_{\mathcal{H}_s}f)(x)|^2\frac{dx}{x} = \int_0^\infty \left|(\mathcal{G}_{\mathcal{H}_s}f)(e^{-x})\right|^2 dx = 4\pi\int_0^\infty |f(\tau)|^2 d\tau.$$

Now, as a more serious example, consider

$$\mathcal{H}(s,\tau)=\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right). \tag{1.6}$$

By (1.3) we have $H(x,\tau) = 2K_{i\tau}(2\sqrt{x})$ and according to the values of the integrals (2.16.52.11) and (2.16.28.3) from [9] for any $0 < k < 1/4$ one can evaluate the following iterated integral

$$4\int_0^\infty \frac{\sinh(\pi\tau)}{\tau}\int_0^\infty K_{i\tau}^2(2\sqrt{x})x^{2k-1}dxd\tau = \pi^2\frac{\Gamma^2(2k)\Gamma(1-4k)}{\Gamma^2(1-2k)}.$$

This implies that for any $0 < k < 1/4$ the Kontorovich-Lebedev transform (1.4) is a Hilbert-Schmidt mapping $L_2\left(\mathbf{R}_+;\frac{\tau^3 d\tau}{\sinh(\pi\tau)}\right) \mapsto L_2\left(\mathbf{R}_+;x^{2k-1}dx\right)$. Ultimately, we will show that (1.4) is an isometric isomorphism between the space $L_2\left(\mathbf{R}_+;\frac{\tau d\tau}{\sinh(\pi\tau)}\right)$ and $L_2\left(\mathbf{R}_+;x^{-1}dx\right)$. With suitable modifications the isometric properties remain true for the operator $\mathcal{G}_{\mathcal{H}\varphi}$.

As we observe, the Kontorovich-Lebedev transform (1.4) arises from the kernel (1.6) with the multiplier $\varphi(s)\equiv 1$. In applications, the Kontorovich-Lebedev transform is used to solve boundary value problems formulated in cylindrical coordinates (cf. [12]).

If we put $\varphi(s) = \Gamma(1/2-\mu-s)/\Gamma(1/2+s), \mu\in\mathbf{R}$, then using the relation (8.4.41.10) in [10] under the condition $0<\gamma<1/2-\mu$, we obtain the integral representation for the generalized Legendre function [3], namely

$$|\Gamma((1+i\tau)/2-\mu)|^2x^{-1/2}(1+x)^{\mu/2}P^{\mu}_{(i\tau-1)/2}\left(\frac{2}{x}+1\right)$$

$$=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\frac{\Gamma(1/2-\mu-s)}{\Gamma(1/2+s)}x^{-s}ds. \tag{1.7}$$

The corresponding transform (1.1) is the generalized Mehler-Fock operator [17]

$$[MFf](x)=x^{-1/2}(1+x)^{\mu/2}\int_0^\infty \tau|\Gamma((1+i\tau)/2-\mu)|^2P^{\mu}_{(i\tau-1)/2}\left(\frac{2}{x}+1\right)f(\tau)d\tau. \tag{1.8}$$

The classical Mehler-Fock transform [16] is the one with $\mu=0$. The Mehler-Fock transform is quite important in the theory of elasticity, in particular in the analysis of stress in the vicinity of an external crack (see [12, Chapter 7]).

All integral operators of this type are realized by integrals over parameters of special functions. They may properly be called index transforms, see [17].

As known, cf. [1],[2],[4],[8],[17],[18], the Mehler-Fock transform can be generalized by taking the Gauss hypergeometric function ${}_2F_1$ [3] as a kernel. For

$|z| \le 1$ one has the power series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.9} $$

where $(a)_n$ denotes the Pochammer symbol [3]. Note that in the case $|z| = 1$ we should add the condition $\mathrm{Re}(c - a - b) > 0$ and for $|z| = 1,\ z \ne 1$ we have $-1 < \mathrm{Re}(c - a - b) \le 0$. For $|z| > 1$ the Gauss function is defined by the unique analytic continuation of this series into the plane with a cut along the semi-axis $1 < x < \infty$. This continuation can be obtained if instead of the series (1.9) we consider an equivalent definition of the Gauss function in terms of the Mellin-Barnes integral, namely

$$ \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a,b;c;z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)} (-z)^{-s} ds, \tag{1.10} $$

where $|z| < 1, |\arg(-z)| < \pi$. The series (1.9) can be found back if we evaluate the integral (1.10) as the sum of residues of poles of the integrand, which involve the left-hand poles $s = 0, -1, -2, \dots$ being separated from the right-hand ones $s = a + n, s = b + n, n = 0, 1, 2, \dots$ by the contour of integration. This is possible if we assume, that these poles do not coincide, that is, both a and b are different from $0, -1, -2, \dots$. Hence for $|z| > 1$ we can regard the integral (1.10) as a sum of the residues of the right-hand simple poles (when $a - b \ne n$) of the integrand. Then we obtain,

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right) $$

$$ + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right), \tag{1.11} $$

where $a - b \ne n, n = 0, \pm 1, \pm 2, \dots, |\arg(-z)| < \pi$.

Now consider the product $\mathcal{H}(s,\tau)\varphi(s)$, where we let $\varphi(s) = \Gamma(c - a - s)/\Gamma(s + a)$, $a < c, 0 < \mathrm{Re} s = \gamma < c - a, c \ne 0, -1, -2, \dots$. Then by means of the relation (8.4.49.16) from [10] we obtain the kernel of the Olevskii transformation [2], [8] as

(1.12)

$$ H(x,\tau) = \frac{|\Gamma(c - a + i\tau/2)|^2}{\Gamma(c)} x^{-a}(1+x)^{2a-c} {}_2F_1\left(a + \frac{i\tau}{2}, a - \frac{i\tau}{2}; c; -\frac{1}{x}\right), x > 0. $$

By changing the variable $x = \sinh^{-2} t,\ a = (\alpha + \beta + 1)/2,\ c = 1 + \alpha$ we easily obtain the Jacobi function $\phi_\tau^{\alpha,\beta}(t)$ as the kernel of the Fourier-Jacobi transform [1], [4]. Putting $a = 1/2$ we immediately obtain the generalized Mehler-Fock transform (1.8). We note here, that one more strict generalization of the Olevskii transform was considered in [2], where the kernel (1.2) involves the Appell F_3-function (see [3], vol.1).

Below we list some other examples of the kernel (1.2) as the inverse Mellin transform of the kernel (1.6) with the multiplier φ, i.e. $\mathcal{H}(s,\tau)\varphi(s)$ for special choices of the function $\varphi(s)$. For this we use the table of the Mellin transform from [10]. Note that all these kernels are real-valued functions for real parameters.

We arrive at the following table:

1.$\varphi(s) \equiv 1, \quad H_\varphi(x,\tau) = 2K_{i\tau}(2\sqrt{x});$

2. $\varphi(s) = \frac{\Gamma(1/2-\mu-s)}{\Gamma(1/2+s)}, \quad H_\varphi(x,\tau) = |\Gamma((1+i\tau)/2-\mu)|^2 x^{-1/2}(1+x)^{\mu/2} P^{\mu}_{(i\tau-1)/2}\left(\frac{2}{x}+1\right);$

3. $\varphi(s) = [\Gamma(1/2+s)]^{-1}, \quad H_\varphi(x,\tau) = \frac{1}{\sqrt{\pi}} e^{-x/2} K_{i\tau/2}\left(\frac{x}{2}\right);$

4. $\varphi(s) = \Gamma(1/2-s), \quad H_\varphi(x,\tau) = \frac{\sqrt{\pi}}{\cosh(\pi\tau/2)} e^{x/2} K_{i\tau/2}\left(\frac{x}{2}\right);$

5. $\varphi(s) = \Gamma(s)\Gamma(1/2-s), \quad H_\varphi(x,\tau) = \frac{\pi^{5/2}}{2\cosh(\pi\tau/2)}\left[J^2_{i\tau/2}(\sqrt{x}) + Y^2_{i\tau/2}(\sqrt{x})\right];$

6. $\varphi(s) = \frac{\Gamma(s)}{\Gamma(1/2+s)}, \quad H_\varphi(x,\tau) = \frac{2}{\sqrt{\pi}} K^2_{i\tau/2}(\sqrt{x});$

7. $\varphi(s) = [\Gamma(s+1/2)\Gamma(1-s)]^{-1}, \quad H_\varphi(x,\tau) = \frac{\sqrt{\pi}}{2i\sinh(\pi\tau/2)}\left[J^2_{-i\tau/2}(\sqrt{x}) - J^2_{i\tau/2}(\sqrt{x})\right];$

8. $\varphi(s) = \frac{\Gamma(1/2-s)}{\Gamma(1-s)}, \quad H_\varphi(x,\tau) = \frac{\sqrt{\pi}}{\cosh(\pi\tau/2)} K_{i\tau/2}(\sqrt{x})\left[I_{i\tau/2}(\sqrt{x}) + I_{-i\tau/2}(\sqrt{x})\right];$

9. $\varphi(s) = \frac{\Gamma(c-a-s)}{\Gamma(s+a)}, \quad H_\varphi(x,\tau) = \frac{|\Gamma(c-a+i\tau/2)|^2}{\Gamma(c)} x^{-a}(1+x)^{2a-c} {}_2F_1\left(a+\frac{i\tau}{2}, a-\frac{i\tau}{2}; c; -\frac{1}{x}\right);$

10. $\varphi(s) = [\Gamma(1/2-\mu+s)]^{-1}, \quad H_\varphi(x,\tau) = x^{-1/2} e^{-x/2} W_{\mu,i\tau/2}(x);$

11. $\varphi(s) = \Gamma(1/2-\mu-s), \quad H_\varphi(x,\tau) = |\Gamma((1+i\tau)/2-\mu)|^2 x^{-1/2} e^{x/2} W_{\mu,i\tau/2}(x).$

Remark 1. The definitions of the special functions which are mentioned in this table can be found in [3], [7]. We deal here, for example, with Bessel's functions of the first, second and third kind $J_\nu(z), Y_\nu(z), I_\nu(z)$ respectively and the Whittaker function $W_{\mu,\nu}(x)$.

2. The Kontorovich-Lebedev transform

In this section we will study the Kontorovich-Lebedev type operator (1.4) between two weighted L_2-spaces. We will show it to be bijective. For the time being assume that f is a function from the space $C_c^\infty(\mathbf{R}_+)$ of infinitely smooth functions with a compact support. First let us check the Hilbert-Schmidt condition C3 for the kernel (1.3).

From (2.16.52.6) in [9] we obtain

$$\int_0^\infty \int_0^\infty K^2_{i\tau}(2\sqrt{x}) x^{2k-1} dx d\tau = \frac{\pi}{2^{1+4k}} \int_0^\infty K_0(2\sqrt{x}) x^{2k-1} dx = \frac{\pi}{2^{2+4k}} \Gamma^2(2k) < \infty$$

for any $k > 0$. It fails when $k = 0$ and as we will see below one can arrive at the operator $[KL]$ of (1.4) acting on $L_2(\mathbf{R}_+; x^{-1}dx)$, as a suitable limit of a sequence of Hilbert-Schmidt operators.

For each $\varepsilon \geq 0$ consider the integral

$$I(\varepsilon) = \int_0^\infty x^{\varepsilon-1} |[KLf](x)|^2 \, dx. \tag{2.1}$$

For $f \in C_c^\infty(\mathbf{R}_+)$ in view of (1.4) we have

$$I(\varepsilon) = 4\int_0^\infty \int_0^\infty \tau y f(\tau)\overline{f(y)} d\tau dy \int_0^\infty x^{\varepsilon-1} K_{i\tau}(2\sqrt{x}) K_{iy}(2\sqrt{x}) dx$$

(2.2)

$$= \frac{1}{\Gamma(2\varepsilon)} \int_0^\infty \int_0^\infty \tau y \left| \Gamma\left(\varepsilon + \frac{i(\tau+y)}{2}\right) \Gamma\left(\varepsilon + \frac{i(\tau-y)}{2}\right)\right|^2 f(\tau)\overline{f(y)} d\tau dy,$$

where the integral over x is calculated, for example, in [17, p.45]. Hence

$$I(\varepsilon) = \frac{8\varepsilon}{\Gamma(1+2\varepsilon)} \int_0^\infty \int_0^\infty \frac{f(\tau)\overline{f(y)}}{4\varepsilon^2 + (\tau-y)^2} \left| \Gamma\left(1+\varepsilon + \frac{i(\tau+y)}{2}\right) \Gamma\left(1+\varepsilon + \frac{i(\tau-y)}{2}\right)\right|^2 d\tau dy$$

$$-\frac{8\varepsilon}{\Gamma(1+2\varepsilon)} \int_0^\infty \int_0^\infty \frac{f(\tau)\overline{f(y)}}{4\varepsilon^2 + (\tau+y)^2} \left| \Gamma\left(1+\varepsilon + \frac{i(\tau+y)}{2}\right) \Gamma\left(1+\varepsilon + \frac{i(\tau-y)}{2}\right)\right|^2 d\tau dy$$

$$= I_1(\varepsilon) - I_2(\varepsilon).$$

Change the variable $\tau = y + 2\varepsilon t$ in the integral I_1 and rewrite it as follows

$$I_1(\varepsilon) = \frac{4}{\Gamma(1+2\varepsilon)} \int_0^\infty \overline{f(y)} \Psi(\varepsilon, y) dy, \tag{2.3}$$

where

$$\Psi(\varepsilon, y) = \int_{-\frac{y}{2\varepsilon}}^\infty |\Gamma(1+iy+\varepsilon(1+it))\Gamma(1+\varepsilon(1+it))|^2 \frac{f(y+2\varepsilon t)dt}{1+t^2}.$$

The integral (2.3) is uniformly convergent for $\varepsilon \geq 0$. Indeed, with $f \in C_c^\infty(\mathbf{R})$ we have

$$|\Psi(\varepsilon, y)| \leq C \int_{-\frac{y}{2\varepsilon}}^\infty \frac{dt}{1+t^2} < \pi C$$

and therefore

$$|I_1(\varepsilon)| \leq \pi C \int_0^\infty |f(y)| dy < \infty.$$

So this enables us to pass to the limit under sign of the integral over y for $\varepsilon \to 0$ in view of the Lebesgue theorem and to obtain

$$\lim_{\varepsilon \to 0} I_1(\varepsilon) = 4\pi^2 \int_0^\infty \frac{y}{\sinh(\pi y)} |f(y)|^2 dy.$$

In the same manner change the variable $\tau = -y + 2\varepsilon t$ in the integral I_2 and it becomes

$$I_2(\varepsilon) = \frac{4}{\Gamma(1+2\varepsilon)} \int_0^\infty \overline{f(y)} \hat{\Psi}(\varepsilon, y) dy,$$

where

$$\hat{\Psi}(\varepsilon, y) = \int_{\frac{y}{2\varepsilon}}^\infty |\Gamma(1 - iy + \varepsilon(1+it))\Gamma(1+\varepsilon(1+it))|^2 \frac{f(-y+2\varepsilon t)dt}{1+t^2}.$$

Hence

$$|\hat{\Psi}(\varepsilon, y)| \leq C \int_{\frac{y}{2\varepsilon}}^\infty \frac{dt}{1+t^2} = C\left[\frac{\pi}{2} - \arctan\frac{y}{2\varepsilon}\right] < C_1,$$

where $C_1 > 0$ does not depend on $\varepsilon, y \geq 0$. Consequently,

$$\lim_{\varepsilon \to 0} I_2(\varepsilon) = 4 \int_0^\infty \overline{f(y)} \lim_{\varepsilon \to 0} \hat{\Psi}(\varepsilon, y) dy = 0.$$

Combining now these results we arrive at the desired equality

$$\lim_{\varepsilon \to 0} \int_0^\infty x^{\varepsilon - 1} |[KLf](x)|^2 \, dx = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)} |f(\tau)|^2 d\tau. \tag{2.4}$$

The integral in the left-hand side in (2.4), with ε replaced by 0, is convergent because of Fatou's lemma. Further, for all $\varepsilon, 0 \leq \varepsilon \leq 1$, the integral is less than or equal to the majorant

$$\int_0^\infty \left[I_{[0,1]}(x) + xI_{[1,\infty)}(x)\right] |[KLf](x)|^2 \, \frac{dx}{x} < \infty,$$

and therefore the limit can be taken inder the integral sign because of the Lebesgue dominated convergence theorem.

Thus we obtain a Parseval equality for the Kontorovich-Lebedev transform of the type

$$\int_0^\infty |[KLf](x)|^2 \, \frac{dx}{x} = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)} |f(\tau)|^2 d\tau, \tag{2.5}$$

where both integrals are finite. Now if $f_n(\tau), n = 1, \ldots,$ is some sequence of $C_c^\infty(\mathbf{R}_+)$ functions which converges to an arbitrary function $f(\tau)$ from the space $L_2\left(\mathbf{R}_+; \frac{\tau d\tau}{\sinh(\pi\tau)}\right)$ then via (2.5) it follows that $[KLf_n](x)$ converges in mean over $L_2(\mathbf{R}_+; x^{-1}dx)$-norm to $F(x)$ say and (2.5) remains true. We call $F(x)$ the Kontorovich-Lebedev transform of f. We show that apart from sets of measure zero, there is a one-to-one correspondence between $F(x)$ and $f(\tau)$. Indeed, for $f_n(\tau)$ the integral (1.4) has a finite range of integration and we have

$$\int_0^\xi [KLf_n](x)dx = 2\int_0^\infty \tau f_n(\tau) K(\xi, \tau) d\tau,$$

where

$$K(\xi,\tau)=\int_0^{\xi} K_{i\tau}(2\sqrt{x})dx.$$

If we prove that for each $\xi>0$, $K(\xi,\tau)\in L_2(\mathbf{R}_+;\tau\sinh(\pi\tau))$, then making $n\to\infty$ for almost all $x>0$ we obtain

$$F(x)=2\frac{d}{dx}\int_0^{\infty}\tau f(\tau)K(x,\tau)d\tau, \tag{2.6}$$

which is unique in L_2-sense. For sufficiently large fixed $\Delta>0$ we have

$$\int_0^{\infty}\tau\sinh(\pi\tau)\left|\int_0^{\xi}K_{i\tau}(2\sqrt{x})dx\right|^2 d\tau=\left(\int_0^{\Delta}+\int_{\Delta}^{\infty}\right)\tau\sinh(\pi\tau)|K(\xi,\tau)|^2d\tau.$$

The integral over $[0,\Delta]$ is finite because of the estimate $|K_{i\tau}(2\sqrt{x})|\le K_0(2\sqrt{x})$. For the integral over $[\Delta,\infty)$ and finite range of the variable x apply the asymptotic formula for the Macdonald function with respect to the index (see, for example [17, p. 20])

$$K_{i\tau}(x)=\sqrt{\frac{2\pi}{\tau}}e^{-\pi\tau/2}\sin\left(\tau\log\frac{2\tau}{x}-\tau+\frac{\pi}{4}+\frac{x^2}{4\tau}\right)[1+O(1/\tau)], \tag{2.7}$$

where $\tau\to+\infty$ and $0<x<2\sqrt{\xi}$. So it becomes

$$O\left(\int_{\Delta}^{\infty}d\tau\left|\int_0^{2\sqrt{\xi}}x\sin(\tau\log x)dx\right|^2\right)=O\left(\int_{\Delta}^{\infty}\frac{d\tau}{\tau^2}\right)<\infty.$$

Consequently, the Kontorovich-Lebedev transform is represented by the formula (2.6). If we take $f\in L_2\left(\mathbf{R}_+;\frac{\tau d\tau}{\sinh(\pi\tau)}\right)$ which is zero for $\tau\notin[1/N,N]$, then by the Parseval equality (2.5) we find

$$[KLf](x)=2\lim_{N\to\infty}\int_{1/N}^{N}\tau K_{i\tau}(2\sqrt{x})f(\tau)d\tau, \tag{2.8}$$

where the limit is taken in the norm of the space $L_2(\mathbf{R}_+;x^{-1}dx)$.

Next, for two functions f,h we have, as a consequence from (2.5) and the parallelogram identity,

$$\int_0^{\infty}[KLf](x)\overline{[KLh](x)}\frac{dx}{x}=4\pi^2\int_0^{\infty}\frac{y}{\sinh(\pi y)}f(y)\overline{h(y)}dy.$$

Putting

$$h(y)=\begin{cases}1, & 0\le y\le\tau,\\ 0, & y\ge\tau\end{cases}$$

for almost all $\tau \in \mathbf{R}_+$ we find that the adjoint operator $[KL]^*$ of $[KL]$ is a left inverse, viz.

$$f(\tau) = \frac{1}{2\pi^2} \frac{\sinh(\pi\tau)}{\tau} \frac{d}{d\tau} \int_0^\infty \hat{K}(\tau, x)[KLf](x) \frac{dx}{x}, \tag{2.9}$$

where

$$\hat{K}(\tau, x) = \int_0^\tau y K_{iy}(2\sqrt{x}) dy. \tag{2.10}$$

First, let us show now that for each $\tau > 0$ $\hat{K}(\tau, x) \in L_2(\mathbf{R}_+; x^{-1}dx)$. Indeed, it is sufficient to show that $\hat{K}(\tau, x) \in L_2((0,1); x^{-1}dx)$. For this appeal to the representation of the Macdonald function [3, Vol.2]

$$K_{iy}(x) = \frac{\pi}{2} \frac{I_{-iy}(x) - I_{iy}(x)}{i \sinh(\pi y)},$$

and the series representations for the modified Bessel functions $I_{\pm iy}(x)$ (see [3, Vol. 2]). Substitute them in (2.10), interchange the order of integration and summation using the uniform convergence of the series. Then for $x \to 0$ we deduce the following asymptotic behavior of the kernel (2.10)

$$\begin{aligned} \hat{K}(\tau, x) &= -\mathrm{Im}\left[\int_0^\tau \Gamma(1+iy) x^{iy/2} dy\right] + o(x) \\ &= O\left(\frac{1}{\log x}\right) + o(x). \end{aligned}$$

The wanted estimate follows from integration by parts in the latter integral with the gamma-function and using the boundedness of $\Gamma(1+iy)\psi(1+iy)$, where ψ is the psi-function [3, Vol.I].

Consequently, resuming the obtained results we observe that the left inverse operator takes the form

$$f(\tau) = \frac{1}{2\pi^2} \lim_{N\to\infty} \sinh(\pi\tau) \int_{1/N}^N K_{i\tau}(2\sqrt{x})[KLf](x) \frac{dx}{x}. \tag{2.11}$$

However, in the same manner from the Parseval equality (2.5) for an arbitrary $g \in L_2(\mathbf{R}_+; x^{-1}dx)$ we prove that $[KL]^{-1}$ is the right inverse, i.e. (2.11) is the inverse operator. We summarize our discussion of this section in the following Plancherel theorem for the Kontorovich-Lebedev transform.

Theorem 1. *The Kontorovich-Lebedev operator* (2.8) *is an isometric isomorphism between the Hilbert spaces* $L_2\left(\mathbf{R}_+; \frac{\tau d\tau}{\sinh(\pi\tau)}\right)$ *and* $L_2(\mathbf{R}_+; x^{-1}dx)$. *The Kontorovich-Lebedev operator and its inverse are represented by the formulas* (2.6), (2.9), *respectively.*

Remark 2. Other types of Parseval formulas for the Kontorovich-Lebedev transform can be found in [6],[7],[11], [17],[18].

3. The Parseval theorem for the transform (1.1)

In this section we generalize the Parseval equality (2.5) to the operator $\mathcal{G}_{\mathcal{H}\varphi}$. As examples we will show both the known and new equalities for particular cases of the transform (1.1).

Let us consider two transforms $(\mathcal{G}_{\mathcal{H}\varphi}f)$, $(\mathcal{G}_{\mathcal{H}\psi}f)$ of the function f for two different multipliers φ, ψ. We assume f to be from $C_c^\infty(\mathbf{R}_+)$.

We start with the investigation of the integral

$$I_{\varphi\psi} = \int_0^\infty (\mathcal{G}_{\mathcal{H}\varphi}f)(x)\overline{(\mathcal{G}_{\mathcal{H}\psi}f)(x)}\, \frac{dx}{x}. \tag{3.1}$$

If $\mathcal{G}_{\mathcal{H}\varphi}$, $\mathcal{G}_{\mathcal{H}\psi} \in L_2(\mathbf{R}_+; x^{-1}dx)$ then by the Parseval theorem for the Mellin transform we immediately obtain the equality

$$\int_0^\infty (\mathcal{G}_{\mathcal{H}\varphi}f)(x)\overline{(\mathcal{G}_{\mathcal{H}\psi}f)(x)}\frac{dx}{x} = \frac{1}{2\pi}\int_{-\infty}^\infty (\mathcal{G}_{\mathcal{H}\varphi}f)^{\mathcal{M}}(it)\overline{(\mathcal{G}_{\mathcal{H}\psi}f)^{\mathcal{M}}(it)}dt, \tag{3.2}$$

where $\mathcal{M}$ denotes the Mellin transform of the functions $(\mathcal{G}_{\mathcal{H}\varphi}f)$, $(\mathcal{G}_{\mathcal{H}\psi}f)$ being evaluated at the point $s = it$. As we required above the kernel $H_\varphi(x,\tau)$ is an $L_2(\mathbf{R}_+; x^{2\gamma-1}dx)$-function for all τ. Therefore, the Plancherel theorem for the Mellin transform gives

$$\lim_{a\to\infty}\int_{1/a}^a H_\varphi(x,\tau)x^{s-1}dx = \Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\varphi(s), \tag{3.3}$$

where $s = \gamma + it$. Further, since $f \in C_c^\infty(\mathbf{R}_+)$ one can multiply (1.1) by x^{s-1} and integrate over $x \in [1/a, a]$. After changing the order in the integral on the right, we arrive at the equality

$$\int_{1/a}^a (\mathcal{G}_{\mathcal{H}\varphi}f)(x)x^{s-1}dx = \int_0^\infty \tau f(\tau)\int_{1/a}^a H_\varphi(x,\tau)x^{s-1}dxd\tau. \tag{3.4}$$

The left-hand side of the equality (3.4) tends to $(\mathcal{G}_{\mathcal{H}\varphi}f)^{\mathcal{M}}(s)$ in the mean square over $(\gamma - i\infty, \gamma + i\infty)$ when $a \to \infty$. Applying the generalized Minkowski inequality we motivate the limit passage in mean square over $(\gamma - i\infty, \gamma + i\infty)$ under the sign of the integral in the right-hand side of (3.4). Indeed, we have

$$\left\|\int_0^\infty \tau f(\tau)\left[\mathcal{H}_\varphi(s,\tau)\varphi(s) - \int_{1/a}^a H_\varphi(x,\tau)x^{s-1}dx\right]d\tau\right\|_2$$

$$\le \int_0^\infty |\tau f(\tau)|\left\|\mathcal{H}(s,\tau)\varphi(s) - \int_{1/a}^a H_\varphi(x,\tau)x^{s-1}dx\right\|_2 d\tau$$

$$= \int_{\mathrm{supp} f} |\tau f(\tau)|\left\|\mathcal{H}(s,\tau)\varphi(s) - \int_{1/a}^a H_\varphi(x,\tau)x^{s-1}dx\right\|_2 d\tau \to 0, \tag{3.5}$$

when $a \to \infty$. Thus we find that
(3.6)
$$(\mathcal{G}_{\mathcal{H}\varphi} f)^{\mathcal{M}}(s) = \varphi(s) \int_0^\infty \tau f(\tau) \Gamma\left(s + \frac{i\tau}{2}\right) \Gamma\left(s - \frac{i\tau}{2}\right) d\tau \in L_2(\gamma - i\infty, \gamma + i\infty).$$

Furthermore, from the integral (1.3) we immediately obtain the Mellin dual formula for the product of gamma-functions

$$\Gamma\left(s + \frac{i\tau}{2}\right) \Gamma\left(s - \frac{i\tau}{2}\right) = 2 \int_0^\infty K_{i\tau}(2\sqrt{x}) x^{s-1} dx, \ \gamma > 0. \tag{3.7}$$

So (3.6) can be written in multiplier form

$$(\mathcal{G}_{\mathcal{H}\varphi} f)^{\mathcal{M}}(s) = \varphi(s)[KLf]^{\mathcal{M}}(s), \ \gamma > 0, \tag{3.8}$$

where $[KLf]^{\mathcal{M}}(s)$ is the Mellin transform of the Kontorovich -Lebedev transform (1.4). Further, from Theorem 1 and the Mellin-Parseval equality (see (3.2)) it follows that $[KLf]^{\mathcal{M}}(it) \in L_2(\mathbf{R})$ $(\gamma = 0)$. To establish regularity properties of $\mathcal{G}_{\mathcal{H}\varphi}$, we now prove

Lemma. *(i) For any $f \in C_c^\infty(\mathbf{R}_+)$ the Mellin transform $(\mathcal{G}_{\mathcal{H}\varphi} f)^{\mathcal{M}}(it)$ can be written in the form*

$$(\mathcal{G}_{\mathcal{H}\varphi} f)^{\mathcal{M}}(it) = 2^{1-2it} \pi \Gamma(1 + 2it) \varphi(it) (\Theta f)(t), \tag{3.9}$$

with

$$(\Theta f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\xi x} \hat{f}_s(\operatorname{arccosh} e^{\xi/2}) d\xi,$$

where $\hat{f}_s$ is the Fourier sine transform of f. We have:

(ii) $\Theta : L_2(\mathbf{R}_+) \mapsto L_2(\mathbf{R}_+)$ is a bounded operator;

(iii) for any $\alpha \geq 0$ $\Theta : H^\alpha(\mathbf{R}_+) \mapsto H^\alpha(\mathbf{R}_+)$ is a bounded operator in Sobolev spaces. In particular, if $f \in H^1(\mathbf{R}_+)$ then $(\Theta f)(t)$ is a bounded function;

(iv) the function $t \mapsto (\Theta f)(t)$ can be extended to an analytic function in the lower half-plane.

Proof. Substitute the integral representation, cf. (1.104) from [17],

$$\Gamma\left(s + \frac{i\tau}{2}\right) \Gamma\left(s - \frac{i\tau}{2}\right) = \frac{\Gamma(2s)}{2^{2(s-1)}} \int_0^\infty \frac{\cos(\tau y) dy}{\cosh^{2s} y}$$

in (3.6). Change the order of integration,

$$(\mathcal{G}_{\mathcal{H}\varphi} f)^{\mathcal{M}}(s) = \frac{\sqrt{\pi}}{2^{2s-3/2}} \Gamma(2s) \varphi(s) \int_0^\infty \frac{1}{\cosh^{2s} y}$$

$$\times \sqrt{\frac{2}{\pi}} \frac{d}{dy} \int_0^\infty f(\tau) \sin(\tau y) d\tau dy.$$

After integration by parts, using $\Gamma(2s)2s = \Gamma(1+2s)$, $s = it$ and the substitution $e^{\xi} = \cosh^2 y$ we arrive at the desired representation (3.9). Propositions (ii) and (iv) follow from the L_2-theory and analytic continuation of Fourier integrals.

Let us estimate the norm in the Sobolev space H^{α}, $\alpha \geq 0$. We have,

$$||\Theta f||^2_{H^{\alpha}} = \int_0^{\infty} |\hat{f}_s(\text{arccosh}\, e^{\xi/2})|^2(1+\xi^2)^{\alpha} d\xi$$

$$= 2\int_0^{\infty} |\hat{f}(y)|^2(1 + (2\log(\cosh y))^2)^{\alpha} \tanh y dy$$

$$\leq 2\int_0^{\infty} |\hat{f}_s(y)|^2(1+4y^2)^{\alpha} dy$$

$$\leq 2^{2\alpha+1}\int_0^{\infty} |\hat{f}_s(y)|^2(1+y^2)^{\alpha} dy = 2^{2\alpha+1}||f||^2_{H^{\alpha}}.$$

Thus we obtained (iii). Now Sobolev's lemma says: $\Theta f \in H^1$ implies that $(\Theta f)(t)$ is a bounded and continuous function. This completes the proof. □

Corollary. *Let the function* $\eta(t) = \Gamma(1+2it)\varphi(it)$ *be uniformly bounded on* $\mathbf{R}$. *Then* $(\mathcal{G}_{\mathcal{H}\varphi} f)^{\mathcal{M}}(it) \in L_2(\mathbf{R})$ *for any* $f \in L_2(\mathbf{R}_+)$.

Indeed, this statement follows from the estimate

$$||(\mathcal{G}_{\mathcal{H}\varphi} f)^{\mathcal{M}}||_{L_2(\mathbf{R})} \leq 2\pi \sup_{t\in\mathbf{R}} |\eta(t)|\ ||\Theta f||_{L_2(\mathbf{R})} \leq C||f||_{L_2(\mathbf{R}_+)},$$

in view of the fact that the representation (3.9) can be continuously extended to the whole of L_2.

Suppose now

$$\rho_{\varphi,\psi}(t) = \varphi(it)\overline{\psi(it)},\ t \in \mathbf{R}$$

to be a positive function on the whole real axis. Substitute the expressions of $(\mathcal{G}_{\mathcal{H}\varphi} f)^{\mathcal{M}}(it)$,
$(\mathcal{G}_{\mathcal{H}\psi} f)^{\mathcal{M}}(it)$ from (3.8) in the right-hand side of (3.2), where the integral (3.6) for $[KLf]^{\mathcal{M}}(it)$
($\varphi = 1$) is to be understood in the principal value sense near $\tau = \pm 2t$. As the result we obtain

$$(3.10)\quad I_{\varphi\psi} = \int_0^{\infty} (\mathcal{G}_{\mathcal{H}\varphi} f)(x)\overline{(\mathcal{G}_{\mathcal{H}\psi} f)(x)}\frac{dx}{x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho_{\varphi,\psi}(t)|[KLf]^{\mathcal{M}}(it)|^2 dt.$$

Note that also
(3.10)′

$$I_{\varphi\psi}(f,h) = \int_0^{\infty} (\mathcal{G}_{\mathcal{H}\varphi} f)(x)\overline{(\mathcal{G}_{\mathcal{H}\psi} h)(x)}\frac{dx}{x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho_{\varphi,\psi}(t)[KLf]^{\mathcal{M}}(it)\overline{[KLh]^{\mathcal{M}}(it)} dt,$$

which is the key formula for Section 4.

Now we are ready to prove the following Parseval theorem for the general transformation $(\mathcal{G}_{\mathcal{H}\varphi} f)$.

Theorem 2. *Let the multiplier φ satisfy the conditions of Corollary. Then $\mathcal{G}_{\mathcal{H}\varphi}$ is a bounded operator from $L_2(\mathbf{R}_+)$ into $L_2(\mathbf{R}_+; x^{-1}dx)$.*

Moreover, if $|\varphi(it)| < C$, where $C > 0$ is a constant, then $\mathcal{G}_{\mathcal{H}\varphi}$ is bounded from $L_2\left(\mathbf{R}_+; \frac{\tau d\tau}{\sinh(\pi\tau)}\right)$ into $L_2(\mathbf{R}_+; x^{-1}dx)$.

Next, if $\rho_{\varphi,\psi}(t) \le C,\ t \in \mathbf{R}$, then for $f \in L_2\left(\mathbf{R}_+; \frac{\tau d\tau}{\sinh(\pi\tau)}\right)$ the integral (3.1) is finite and

$$\int_0^\infty (\mathcal{G}_{\mathcal{H}\varphi} f)(x)\overline{(\mathcal{G}_{\mathcal{H}\psi} f)(x)}\frac{dx}{x} \le 4\pi^2 C \int_0^\infty \frac{\tau}{\sinh(\pi\tau)}|f(\tau)|^2 d\tau \tag{3.11}$$

where equality is achieved when $\rho_{\varphi,\psi}$ is a constant. Finally, if $|\varphi(it)| = C \neq 0$, the operator $\mathcal{G}_{\mathcal{H}\varphi}$ forms an isomorphism between the spaces $L_2\left(\mathbf{R}_+; \frac{\tau d\tau}{\sinh(\pi\tau)}\right)$ and $L_2(\mathbf{R}_+; x^{-1}dx)$ with the Parseval equality

$$\int_0^\infty |(\mathcal{G}_{\mathcal{H}\varphi} f)(x)|^2 \frac{dx}{x} = 4(\pi C)^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)}|f(\tau)|^2 d\tau. \tag{3.12}$$

In the latter case the inverse operator is given by the following expression in mean square sense

$$(\mathcal{G}^{-1}_{\mathcal{H}\varphi} g)(\tau) = \frac{1}{4(\pi C)^2} \lim_{N\to\infty} \sinh(\pi\tau) \int_{1/N}^N H_\varphi(x,\tau) g(x)\frac{dx}{x}. \tag{3.13}$$

Proof. Indeed, the boundedness of the operator $\mathcal{G}_{\mathcal{H}\varphi}$ from $L_2(\mathbf{R}_+)$ into $L_2(\mathbf{R}_+; x^{-1}dx)$ follows from Corollary to Lemma. If, in addition, $|\varphi(it)|$ is bounded we use the representation (3.8) and the Parseval equalities for the Mellin and the Kontorovich-Lebedev transforms (see (2.5), (3.10)). This gives the estimate

$$\int_0^\infty |(\mathcal{G}_{\mathcal{H}\varphi} f)(x)|^2 \frac{dx}{x} = \frac{1}{2\pi}\int_{-\infty}^\infty |\varphi(it)|^2 |[KLf]^{\mathcal{M}}(it)|^2 dt$$

$$\le \frac{C}{2\pi}\int_{-\infty}^\infty |[KLf]^{\mathcal{M}}(it)|^2 dtdt$$

$$= C\int_0^\infty |[KLf](x)|^2 \frac{dx}{x} = 4\pi^2 C\int_0^\infty \frac{\tau}{\sinh(\pi\tau)}|f(\tau)|^2 d\tau < \infty.$$

If $\rho_{\varphi,\psi}$ is bounded then in a similar way one can deduce the relations (3.11)-(3.12) from the equality (3.10). It remains to prove that whenever $|\varphi(it)| = C$ the operator $\mathcal{G}_{\mathcal{H}\varphi}$ is an isomorphic mapping of the Hilbert spaces mentioned just before (3.12) and also that the inverse operator is given by (3.13).

From the equality (3.12) it follows, analogously to (2.5), that the sequence $\{\mathcal{G}_{\mathcal{H}\varphi} f_n\}$ is a Cauchy sequence and it converges in mean to some limit function $G(x)$ whenever the sequence $\{f_n\}$ from C_c^∞ is a Cauchy sequence in $L_2(\mathbf{R}_+; \frac{\tau d\tau}{\sinh(\pi\tau)})$.

Further we have

$$\int_0^{\xi} (\mathcal{G}_{\mathcal{H}\varphi} f_n)(x)dx = \int_0^{\infty} \tau f_n(\tau) \int_0^{\xi} H_{\varphi}(x,\tau)dxd\tau. \tag{3.14}$$

According to Schwarz's inequality the right-hand side of the equality (3.14) is uniformly convergent in $n = 1, 2, \ldots$ if we show that the integral

$$\int_0^{\infty} \frac{\sinh(\pi\tau)}{\tau} \left| \int_0^{\xi} \tau H_{\varphi}(x,\tau)dx \right|^2 d\tau < \infty \tag{3.15}$$

for each $\xi > 0$. But

(3.16)

$$\tau \int_0^{\xi} H_{\varphi}(x,\tau)dx = \frac{\xi}{2\pi i}\frac{d}{d\tau} \int_{\gamma-i\infty}^{\gamma+i\infty} \int_0^{\tau} y\Gamma\left(s + \frac{iy}{2}\right)\Gamma\left(s - \frac{iy}{2}\right) \frac{\varphi(s)\xi^{-s}}{1-s} dyds,\ 0 < \gamma < 1.$$

From (1.3) and (2.10)

$$2\hat{K}^{\mathcal{M}}(\tau, \gamma + it) = \int_0^{\tau} y\Gamma\left(\gamma + it + \frac{iy}{2}\right)\Gamma\left(\gamma + it - \frac{iy}{2}\right) dy,\ \ \gamma > 0.$$

Since $\hat{K}(\tau, x) \in L_2(\mathbf{R}_+; x^{-1}dx)$ we immediately arrive at

$$\hat{K}^{\mathcal{M}}(\tau, it) = \lim_{\gamma \to 0} \hat{K}^{\mathcal{M}}(\tau, \gamma + it) \in L_2(\mathbf{R}).$$

We assumed $|\varphi(it)| = C$ and this implies that $\frac{\varphi(it)}{1-it} \in L_2(\mathbf{R})$ too. Consequently, in the contour integral over s in (3.16) we can take $\gamma = 0$. Via the Mellin-Parseval identity we find

$$\begin{aligned} \tau \int_0^{\xi} H_{\varphi}(x,\tau)dx &= \frac{\xi}{\pi}\frac{d}{d\tau} \int_{-\infty}^{\infty} \hat{K}^{\mathcal{M}}(\tau, it) \frac{\varphi(it)\xi^{-it}}{1-it} dt \\ &= 2\xi \frac{d}{d\tau} \int_0^{\infty} \hat{K}(\tau, x)\overline{\hat{\phi}(x\xi)} \frac{dx}{x}, \end{aligned}$$

where $\hat{\phi}(x) \in L_2(\mathbf{R}_+; x^{-1}dx)$ and $\hat{\phi}^{\mathcal{M}}(it) = \overline{\frac{\varphi(-it)}{1+it}}$. From Theorem 1 it follows that the latter integral belongs to $L_2\left(\mathbf{R}_+; \frac{\sinh(\pi\tau)d\tau}{\tau}\right)$. Hence the integral (3.15) is finite. Making $n \to \infty$ in (3.14) we arrive at the equality

$$\int_0^{\xi} G(x)dx = \int_0^{\infty} \tau f(\tau) \int_0^{\xi} H_{\varphi}(x,\tau)dxd\tau$$

and for almost all $x > 0$

$$G(x) = \frac{d}{dx} \int_0^{\infty} \tau f(\tau) \int_0^{x} H_{\varphi}(u,\tau)dud\tau.$$

Put $f_N(\tau) = f(\tau)$, $\tau \in [1/N, N]$ and $f_N = 0$ outside. Then one can perform the differentiation under the sign of the latter integral and obtain

$$G_N(x) = \int_{1/N}^{N} \tau f(\tau) H_\varphi(x,\tau) d\tau = (\mathcal{G}_{\mathcal{H}\varphi} f_N)(x).$$

Via the Parseval equality (3.12)

$$\int_0^\infty |G(x) - (\mathcal{G}_{\mathcal{H}\varphi} f_N)(x)|^2 \frac{dx}{x} = 4(\pi C)^2 \left(\int_0^{1/N} + \int_N^\infty \right) \frac{\tau}{\sinh(\pi\tau)} |f(\tau)|^2 d\tau,$$

which tends to 0 as $N \to \infty$. Hence the transform (1.1)

$$G(x) = (\mathcal{G}_{\mathcal{H}\varphi} f)(x) = \lim_{N\to\infty} \int_{1/N}^{N} \tau f(\tau) H_\varphi(x,\tau) d\tau \tag{3.17}$$

makes sense in $L_2(\mathbf{R}_+; x^{-1}dx)$. As in Section 2, from the equality (3.12) we deduce an inversion formula

$$f(\tau) = \frac{1}{4(\pi C)^2} \frac{\sinh(\pi\tau)}{\tau} \frac{d}{d\tau} \int_0^\infty \int_0^\tau y H_\varphi(x,y) (\mathcal{G}_{\mathcal{H}\varphi} f)(x) \frac{dy dx}{x},$$

which is equivalent to the form of the inversion operator (3.13) if we prove that for each $\tau > 0$

$$G(x,\tau) = \int_0^\tau y H_\varphi(x,y) dy \in L_2(\mathbf{R}_+; x^{-1}dx). \tag{3.18}$$

Similar to (3.16) for almost all $x > 0$ we have the representation

$$G(x,\tau) = \frac{1}{2\pi} \frac{d}{dx} \int_{-\infty}^{\infty} \int_0^\tau y \Gamma\left(\gamma + it + \frac{iy}{2}\right) \Gamma\left(\gamma + it - \frac{iy}{2}\right) \frac{\varphi(\gamma + it) x^{1-\gamma-it}}{1 - \gamma - it} dy dt,$$

and the Parseval equality of type

$$\int_0^\infty |G(x,\tau)|^2 x^{2\gamma-1} dx = \frac{2}{\pi} \int_{-\infty}^{\infty} |\hat{K}^{\mathcal{M}}(\tau, \gamma + it) \varphi(\gamma + it)|^2 dt, \tag{3.19}$$

where $0 < \gamma < A, A > 0$ and to obtain the property (3.18) we let $\gamma \to 0$ in (3.19), as in Section 2. Accordingly, we arrive at the equality

$$\int_0^\infty |G(x,\tau)|^2 \frac{dx}{x} = \frac{2C^2}{\pi} \int_{-\infty}^{\infty} |\hat{K}^{\mathcal{M}}(\tau, it)|^2 dt < \infty.$$

Analogously we find the form of the inversion operator for the transform $\mathcal{G}_{\mathcal{H}\varphi}$. This finishes the proof of Theorem 2.

4. Examples

In this section we give some examples of the general transform $(\mathcal{G}_{\mathcal{H}\varphi}f)(x)$ and the Parseval equalities for them, where $\mathcal{H}(s,\tau)$ is defined by the formula (1.6). As we will see below they may lead to different isometries of Hilbert spaces.

1. The Kontorovich-Lebedev transform. The simplest basic example leads to the Parseval equality (Section 2) for the Kontorovich-Lebedev transform (1.4) when we let $\varphi(s)=\psi(s)=1$.

2. The generalized Mehler-Fock transform. Let us follow our table in Section 1 and take line 2 which corresponds to the generalized Mehler-Fock transform (1.8). We look for the kernel $H_\psi(x,\tau)$ for this case with $\psi(s) = \Gamma(1/2-s)/\Gamma(1/2-\mu+s)$. Then appealing to the formula (8.4.41.12) from [10] we evaluate the corresponding integral (1.2) and obtain

$$H_\psi(x,\tau)=\frac{\pi}{\cosh(\pi\tau/2)}x^{-1/2}(1+x)^{-\mu/2}P^{\mu}_{(i\tau-1)/2}\left(\frac{2}{x}+1\right). \tag{4.1}$$

In our case (cf. (1.8)) $(\mathcal{G}_{\mathcal{H}\varphi}f)(x)=[MFf](x)$ and the multiplier $\varphi(s)$ is defined by line 2 of the table in section 1. Consider the Mehler-Fock transform in the form

(4.2)

$$(\mathcal{G}_{\mathcal{H}\psi}h)(x)=\pi x^{-1/2}(1+x)^{\mu/2}\int_0^\infty\frac{\tau}{\cosh(\pi\tau/2)}P^{\mu}_{(i\tau-1)/2}\left(\frac{2}{x}+1\right)h(\tau)d\tau.$$

where we set $h(\tau)=\pi^{-1}\cosh(\pi\tau/2)|\Gamma((1+i\tau)/2-\mu)|^2f(\tau)$. Observe that for the classical Mehler-Fock transform ($\mu=0$) we have $h\equiv f$. Then comparing with (1.8) we immediately find that $(\mathcal{G}_{\mathcal{H}\psi}h)(x)=[MFf](x)$. Hence, similar to the equality (3.10), applying the Parseval equalities for the Mellin and the Kontorovich-Lebedev transforms (cf. (2.5)), we deduce the following Parseval relation for the generalized Mehler-Fock transformation (1.8)

$$\int_0^\infty|[MFf]|^2\frac{dx}{x}=2\pi\int_0^\infty\frac{\tau|\Gamma((1+i\tau)/2-\mu)|^2}{\sinh(\pi\tau/2)}|f(\tau)|^2d\tau. \tag{4.3}$$

3. The Olevskii-Fourier-Jacobi transform. Take the kernel (1.12) and put $\psi(s)=\Gamma(a-s)/\Gamma(c-a+s)$ to find the kernel $H_\psi(x,\tau)$. The corresponding integral (1.2) is evaluated in [10] (formula (8.4.49.14)) which gives

$$H_\psi(x,\tau)=\frac{\left|\Gamma\left(a+i\frac{\tau}{2}\right)\right|^2}{\Gamma(c)}x^{-a}{}_2F_1\left(a+\frac{i\tau}{2},a-\frac{i\tau}{2};c;-\frac{1}{x}\right),x>0. \tag{4.4}$$

Note that the Gauss function (4.4) is represented by the series (1.9) for the respective parameters and $x>1$. For $0<x\le 1$ one can understand it as an analytic continuation, cf. (1.11).

If we write

(4.5)

$$[{}_2F_1 f](x) = \frac{x^{-a}(1+x)^{2a-c}}{\Gamma(c)} \int_0^\infty \tau \left|\Gamma\left(c-a+i\frac{\tau}{2}\right)\right|^2 {}_2F_1\left(a+\frac{i\tau}{2}, a-\frac{i\tau}{2}; c; -\frac{1}{x}\right) f(\tau) d\tau,$$

$$(4.6)\quad [{}_2\hat{F}_1 h](x) = \frac{x^{-a}}{\Gamma(c)} \int_0^\infty \tau |\Gamma(a+i\tau/2)|^2 {}_2F_1\left(a+\frac{i\tau}{2}, a-\frac{i\tau}{2}; c; -\frac{1}{x}\right) h(\tau) d\tau,$$

and take

$$h(\tau) = f(\tau)\left|\frac{\Gamma(c-a+i\tau/2)}{\Gamma(a+i\tau/2)}\right|^2$$

we get $[{}_2F_1 f](x) = (1+x)^{2a-c}[{}_2\hat{F}_1 h](x)$ and as a consequence a Parseval equality of type

(4.7)

$$\int_0^\infty |[{}_2F_1 f](x)|^2 (1+x)^{c-2a} \frac{dx}{x} = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)} \left|\frac{\Gamma(c-a+i\tau/2)}{\Gamma(a+i\tau/2)}\right|^2 |f(\tau)|^2 d\tau.$$

4. The transform with the Whittaker function. Starting from line 10 of the table in Section 1 introduce the following transformation over the index of the Whittaker function ($\mu \in \mathbf{R}$)

$$(4.8)\qquad [W_\mu f](x) = x^{-1/2} e^{-x/2} \int_0^\infty \tau W_{\mu, i\tau/2}(x) f(\tau) d\tau.$$

The related transform $[\hat{W}_\mu h](x)$ with the kernel from line 11 takes the form

$$(4.9)\qquad [\hat{W}_\mu h](x) = x^{-1/2} e^{x/2} \int_0^\infty \tau |\Gamma((1+i\tau)/2-\mu)|^2 W_{\mu, i\tau/2}(x) h(\tau) d\tau.$$

Hence as a consequence for $h(\tau) = f(\tau)/|\Gamma((1+i\tau)/2-\mu)|^2$ we obtain the following Parseval equality

(4.10)

$$\int_0^\infty e^x |[W_\mu f](x)|^2 \frac{dx}{x} = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)|\Gamma((1+i\tau)/2)-\mu)|^2} |f(\tau)|^2 d\tau.$$

5. Integral transformations over parameters of the hypergeometric ${}_3F_2$-function. As a final example consider a generalization of the Olevskii-Fourier-Jacobi transforms (4.5)-(4.6), which involves as its kernel the hypergeometric function ${}_3F_2$, which is defined by the series

$$(4.11)\qquad {}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \sum_{n=0}^\infty \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!}, \quad |z| < 1.$$

Put $a_1 = a + i\tau/2,\ a_2 = a - i\tau/2,\ a_3 = b,\ b_1 = c,\ b_2 = d$. If

$$\varphi(s) = \frac{\Gamma(a-s)\Gamma(b-a+s)}{\Gamma(c-a+s)\Gamma(d-a+s)},$$

then in view of the formula (8.4.50.2) from [10] we obtain

$$H_\varphi(x,\tau) = \frac{|\Gamma(a+i\tau/2)|^2\Gamma(b)}{\Gamma(c)\Gamma(d)} x^{-a} {}_3F_2\left(a+\frac{i\tau}{2}, a-\frac{i\tau}{2}, b; c, d; -\frac{1}{x}\right),$$

where this kernel is the series like (4.11) for $x > 1$ and for $0 < x \leq 1$ it is an analytic continuation of the corresponding power series for the ${}_3F_2$-function. It can be deduced by calculation of the Mellin-Barnes integral (8.4.50.2) in [10] through the sum of residues in the left-hand poles of the gamma-functions. Thus we arrive at the generalization of the transform (4.6) for $b \neq d$, namely

$$[{}_3F_2 f](x) = \frac{x^{-a}\Gamma(b)}{\Gamma(c)\Gamma(d)} \int_0^\infty \tau |\Gamma(a+i\tau/2)|^2$$

$$(4.12) \qquad \times {}_3F_2\left(a+\frac{i\tau}{2}, a-\frac{i\tau}{2}, b; c, d; -\frac{1}{x}\right) f(\tau) d\tau.$$

The corresponding kernel will be found from the integral like (1.2) for

$$(4.13) \qquad \psi(s) = \frac{\Gamma(c-a-s)\Gamma(d-a-s)}{\Gamma(a+s)\Gamma(b-a-s)}.$$

It is more convenient to calculate the integral explicitly as the sum of residues at the left-hand poles of gamma-functions and $0 < x < 1$. As a result we obtain

$$H_\psi(x,\tau) = \frac{\Gamma(c-a+i\tau/2)\Gamma(d+i\tau/2)\Gamma(-i\tau)}{\Gamma(b+i\tau/2)\Gamma(a-i\tau/2)} x^{i\tau/2}$$

$$\times {}_3F_2\left(c-a+\frac{i\tau}{2}, d+\frac{i\tau}{2}, 1-a+\frac{i\tau}{2}; b+\frac{i\tau}{2}, 1+i\tau; -x\right)$$

$$+\frac{\Gamma(c-a-i\tau/2)\Gamma(d-i\tau/2)\Gamma(i\tau)}{\Gamma(b-i\tau/2)\Gamma(a+i\tau/2)} x^{-i\tau/2}$$

$$(4.14) \qquad \times {}_3F_2\left(c-a-\frac{i\tau}{2}, d-\frac{i\tau}{2}, 1-a-\frac{i\tau}{2}; b-\frac{i\tau}{2}, 1-i\tau; -x\right).$$

For $x \geq 1$ the kernel (4.14) is analytically continued as a sum of residues of the corresponding integrand in the right-hand poles series $s = c-a+n,\ s = d+n,\ n = 0, -1, -2, \ldots$ of the function ψ (4.13) similar to (1.11) for the Gauss function.

Hence making use of the evenness of the integrand one can introduce the transform as follows

$$[{}_3\hat{F}_2 f](x) = \int_{-\infty}^{\infty} \tau \frac{\Gamma(c-a-i\tau/2)\Gamma(d-i\tau/2)\Gamma(i\tau)}{\Gamma(b-i\tau/2)\Gamma(a+i\tau/2)} x^{-i\tau/2}$$

$$\times {}_3F_2\left(c-a-\frac{i\tau}{2}, d-\frac{i\tau}{2}, 1-a-\frac{i\tau}{2}; b-\frac{i\tau}{2}, 1-i\tau; -x\right) f(\tau)d\tau.$$

The Parseval equality will be as follows

$$\int_0^\infty [{}_3F_2 f](x)\overline{[{}_3\hat{F}_2 f](x)}\frac{dx}{x} = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)}|f(\tau)|^2 d\tau.$$

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