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A NEW CHARACTERIZATION OF IDEALS IN BCC-ALGEBRAS

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Abstract. The new characterization of ideals in BCC-algebras is given. A construction of a congruence having a given ideal as its kernel is presented and it is proved that in any variety of BCC-algebras all congruences are uniquely determined by such ideals.

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1. Introduction

In this paper a binary multiplication will be denoted by juxtaposition. We will use dots only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ is written as $(xy \cdot zy) \cdot xz = 0$.

An algebra $\mathbf{G} = (G, \cdot, 0)$ of type $(2,0)$ is called *a BCC-algebra* if it satisfies the following axioms:

(1) $(xy \cdot zy) \cdot xz = 0$,
(2) $xx = 0$,
(3) $0x = 0$,
(4) $x0 = x$,
(5) $xy = yx = 0$ *implies* $x = y$.

BCC-algebras were introduced by Y. Komori [12] in a connection with some problems on BCK-algebras (solved in [14]). Our definition is a dual form of the ordinary definition used in [4], [12] and [13]). In the above convention (introduced in [5]) *any BCK-algebra is a BCC-algebra*, but *there are BCC-algebras which are not BCK-algebras* (cf. [5]). Moreover, any BCC-algebras may be considered as an extension of some BCK-algebra, i.e. any BCC-algebra has at least one nontrivial subalgebra which is a BCK-algebra. The minimal BCC-algebra which is not a BCK-algebra has four elements. Note that (cf. [5]) *a BCC-algebra is a BCK-algebra iff it satisfies* the identity

(6) $xy \cdot z = xz \cdot y$,

which holds in all BCK-algebras. Methods of construction of BCC-algebras from the given BCK-algebras are described in [5] and [6].

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The class of all BCC-algebras is a quasivariety (cf. [13]), but many subclasses of this quasivariety form variety (cf. for example [5]). Also the quasivariety of all BCK-algebras has many well described subclasses which are varieties.

On any BCC-algebra **G** (similarly as in the case of BCK-algebras) one can define the natural order $\leq$ putting

(7) $x \leq y$ *if and only if* $xy = 0$.

It is not difficult to verify that this order is partial and 0 is its smallest element. Moreover, for all $x, y, z \in G$

(8) $xy \cdot zy \leq xz$,

(9) $x \leq y$ *implies* $xz \leq yz$ *and* $zy \leq zx$.

2. Ideals

As is well-known (cf. for example [3], [11]) a non-empty subset A of a BCK-algebra **G** is called *an ideal* if $0 \in A$ and $y,\, xy \in A$ *imply* $x \in A$. In the sequel this ideal will be called *a BCK-ideal* and will be considered also in BCC-algebras. In BCK-algebras any BCK-ideal induces a congruence, but there are BCK-algebras with congruences which are not induced by BCK-ideals. In some varieties of BCK-algebras (for example in the variety of commutative BCK-algebras) all congruences are induced by BCK-ideals.

BCK-ideals do not induces congruences in BCC-algebras. Some congruences in BCC-algebras are induced by *BCC-ideals* defined in [7], i.e. by non-empty subsets A of a BCC-algebra **G** satisfying the following two conditions: (i) $0 \in A$, (ii) $y,\, xy \cdot z \in A$ *imply* $xz \in A$. BCC-ideals are also BCK-ideals. In BCK-algebras the converse statement is true, too [7]. In BCC-algebras (similarly as in BCK-algebras) there are congruences which are not induced by BCC-ideals.

In connection with this fact we define a new type of ideals which are better connected with congruences. The concept of such ideals was suggested in the paper [2].

Definition. A subset I of a BCC-algebra **G** will be called *an ideal* iff

(i) $0 \in I$,
(ii) $ab \in I$ *for* $a \in I$ *and* $b \in G$,
(iii) $b(ba_1 \cdot a_2) \in I$ *for* $a_1, a_2 \in I$ *and* $b \in G$.

Observe that (i) follows from (ii) and (2). From (ii) follows also that any ideal is a subalgebra. Moreover, putting in the above definition $a_1 = a$ and $a_2 = 0$ we obtain

Proposition. *If I is an ideal of BCC-algebra* **G**, *then $b \cdot ba \in I$ for every $a \in I$ and $b \in G$.* □

Corollary 1. *If $a \in I$ and $x \leq a$, then also $x \in I$.*

Proof. Indeed, $x = x0 = x \cdot xa \in I$, by (4), (7) and Proposition. □

Theorem 1. *A subset I of a BCC-algebra $\mathbf{G}$ is an ideal if and only if it is a BCC-ideal.*

Proof. Let I be an ideal of $\mathbf{G}$. Obviously $0 \in I$. If $xa \cdot y = a_1 \in I$ for some $a \in I$, then $x \cdot xa = a_2 \in I$ by Proposition. Now, applying (1) we obtain

$$xy = xy \cdot 0 = xy \cdot ((xy \cdot (xa \cdot y))(x \cdot xa)) = xy \cdot ((xy \cdot a_1) \cdot a_2) \in I\,.$$

Hence I is a BCC-ideal.

Conversely, if I is a BCC-ideal, then $0 \in I$ and $aa \cdot x = 0x = 0 \in I$ for every $a \in I$, which, by the definition of a BCC-ideal, implies $ax \in I$. Thus (i) and (ii) from the definition of an ideal are satisfied. To prove (iii) observe that $xa_1 \cdot xa_1 = 0 \in I$ implies $x \cdot xa_1 \in I$ for every a_1 from a BCC-ideal I. This together with (8) gives $xa_2 \cdot (xa_1 \cdot a_2) \leq x \cdot xa_1 \in I$, and, in the consequence $xa_2 \cdot (xa_1 \cdot a_2) \in I$, because $z \leq a$ and $a \in I$ implies $z \in I$ for any BCC-ideal I. Directly from the definition of a BCC-ideal follows also that $a_2 \in I$ and $xa_2 \cdot (xa_1 \cdot xa_2) \in I$ imply $x(xa_1 \cdot a_2) \in I$, which proves (iii). Hence I is an ideal. □

Theorem 2. *A subset I of a BCK-algebra $\mathbf{G}$ is an ideal if and only if it is a BCK-ideal.*

Proof. Since a BCK-algebra is a BCC-algebra, then every ideal is also a BCC-ideal (Theorem 1) and, in the consequence, a BCK-ideal.

On the other hand, if I is a BCK-ideal of a BCK-algebra $\mathbf{G}$, then $0 \in I$ and $ax \cdot a = aa \cdot x = 0 \in I$ by (6). This, by the definition of a BCK-ideal, implies $ax \in I$ for $a \in I$. Thus (i) and (ii) are satisfied. To prove (iii) note that

$$(x(xa_1 \cdot a_2))a_2 = xa_2 \cdot (xa_1 \cdot a_2) \leq x \cdot xa_1 \leq a_1$$

by (6) and (8). If a_1 and a_2 are from BCK-ideal I, the above implies $(x(xa_1 \cdot a_2)) \cdot a_2 \in I$, and, in the consequence $x(xa_1 \cdot a_2) \in I$, which proves (iii). Hence I is an ideal. □

Note that BCK-ideals of BCC-algebras are not ideals in our sense.

Example 1. Let $G = \{0, 1, 2, 3, 4\}$ and let the multiplication be defined by the table

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

It is not difficult to verify (for detail see [7]) that $(G, \cdot, 0)$ is a BCC-algebra. $A = \{0, 1\}$ is a BCK-ideal of this BCC-algebra, but it is not an ideal in our sense since $4(41 \cdot 1) = 3 \notin A$. □

A special role in the theory of BCC-algebras play initial segments (cf. [8]), i.e. the sets of the form

$$[0, c] = \{x \in G : 0 \leq x \leq c\} = \{x \in G : x \leq c\}.$$

Theorem 3. *An initial segment* $[0, c]$ *of a BCC-algebra* $\mathbf{G}$ *is an ideal if and only if for all* $x \in G$ *holds* $x(xc \cdot c) \leq c$.

Proof. Obviously $0 \in [0, c]$. Since $0 \leq x$ for all $x \in G$, then $cx \leq c$ (by (9)). Similarly for $a \in [0, c]$ we have $a \leq c$ and $ax \leq cx \leq c$, which proves that $[0, c]$ always satisfies (i) and (ii) from the definition of an ideal.

If $[0, c]$ is an ideal, then $x(xa_1 \cdot a_2) \in [0, c]$ for all $x \in G$ and $a_1, a_2 \in [0, c]$. This, for $a_1 = a_2 = c$, implies $x(xc \cdot c) \leq c$.

Conversely, assume that the above condition holds. If $a_2 \in [0, c]$, then $a_2 \leq c$ gives (by (9)) $xc \cdot c \leq xc \cdot a_2$ and $x(xc \cdot a_2) \leq x(xc \cdot c) \leq c$. Analogously, for $a_1 \in [0, c]$ we obtain $xc \leq xa_1$ and $x(xa_1 \cdot a_2) \leq x(xc \cdot a_2)$. Hence

$$x(xa_1 \cdot a_2) \leq x(xc \cdot a_2) \leq x(xc \cdot c) \leq c,$$

which proves $x(xa_1 \cdot a_2) \in [0, c]$. Thus $[0, c]$ is an ideal. □

As a simple consequence of the above theorem and Proposition 2.7 from [8] we obtain

Corollary 2. *An initial segment* $[0, c]$ *of a BCC-algebra* $\mathbf{G}$ *is an ideal if and only if for all* $x, y \in G$ $xc \cdot y \leq c$ *implies* $xy \leq c$. □

Corollary 3. *An initial segment* $[0, c]$ *of a BCK-algebra* $\mathbf{G}$ *is an ideal if and only if for all* $x \in G$ $xc \leq c$ *implies* $x \leq c$. □

3. Congruences

We say that a binary relation R defined on BCC-algebra $\mathbf{G}$ is *compatible* if aRb and cRd imply $ac\,R\,bd$. The set $[0]_R = \{a \in G : aR0\}$ is called *a kernel* of R. Of course, *a congruence* is a compatible eqivalence relation. A congruence kernel is its class containing 0.

Lemma 1. *Let* R *be a reflexive and compatible relation defined on a BCC-algebra* $\mathbf{G}$. *Then the class* $[0]_R$ *is an ideal of* $\mathbf{G}$.

Proof. Since R is reflexive, we have $0 \in [0]_R$. If $a \in [0]_R$ and $b \in G$, then $aR0$ which by the reflexivity and compatibility of R gives $ab\,R\,0b$, i.e. $ab\,R\,0$. Thus $ab \in [0]_R$.

Similarly, for $a_1, a_2 \in [0]_R$ we have $b(ba_1 \cdot a_2)\,R\,b(b0 \cdot 0)$. This (by (4) and (2)) gives $b(ba_1 \cdot a_2) \in [0]_R$. Hence $[0]_R$ is an ideal of $\mathbf{G}$. □

Let **G** be a BCC-algebra. For a non-empty subset H of G we define the binary relation Θ_H in the following way:

(10) $a\Theta_H b$ *if and only if* $ab \in H$ *and* $ba \in H$.

The set $\{b \in G : b\Theta_H a\}$ will be denoted by $[a]_H$.

Lemma 2. *If the above relation* Θ_H *is reflexive, then* $[0]_H = H$.

Proof. Since Θ_H is reflexive, then $z\Theta_H z$ and, in particular, $0\Theta_H 0$. Thus $0 \in H$. If $x \in H$, then $0x = 0 \in H$ and $x0 = x \in H$, which gives $x\Theta_H 0$. Hence $H \subset [0]_H$.

Conversely, if $x \in [0]_H$, then $x\Theta_H 0$, i.e. $x = x0 \in H$ by the definition of Θ_H. Thus $[0]_H \subset H$. Therefore $[0]_H = H$. □

Corollary 4. *If* Θ_H *is an equivalence relation, then*

$$a \in H,\ ab \in H \text{ and } b \leq a \text{ imply } b \in H.$$

Proof. $0 \in H$ by Lemma 2. If $b \leq a$ then $ba = 0 \in H$ which, together with $ab \in H$ gives $a\Theta_H b$. Since Θ_H is an equivalence, a, b belongs to the same class of Θ_H. By Lemma 2, $a \in H = [0]_H$ and in the consequence $b \in H$. □

Theorem 4. *If* H *is an ideal of a BCC-algebra* **G** *then* Θ_H *is a congruence of* **G** *such that* $[0]_H = H$.

Proof. If H is an ideal, then $0 \in H$. Thus the relation Θ_H is reflexive. It is also symmetric. We prove the transitivity.

Assume $x\Theta_H y$ and $y\Theta_H z$. Then, $xy, yx, yz, zy \in H$ by the definition of Θ_H. Putting $a_1 = yx$, $a_2 = zy$, $b = zx$ and applying (1) we obtain

$$zx = zx \cdot 0 = zx \cdot ((zx \cdot yx) \cdot zy) \in H.$$

Similarly, for $a_1 = yz$, $a_2 = xy$ and $b = xz$ we get

$$xz = xz \cdot 0 = xz \cdot ((xz \cdot yz) \cdot xy) \in H.$$

Thus $x\Theta_H z$, which gives the transivity. Hence Θ_H is an equivalence. By Lemma 2 $[0]_H = H$.

It remains to show that Θ_H is compatible. Suppose $x\Theta_H y$ and $z\Theta_H v$. Since $x\Theta_H y$ implies $xy, yx \in H$, then, by (8), for every $z \in G$, we have $xz \cdot yz \leq xy \in H$. This, together with Corollary 1, implies $xz \cdot yz \in H$. Analogously, $yz \cdot xy \in H$. Thus $xz\Theta_H yz$.

By (8) we have also $yz \cdot vz \leq yv$, and in the consequence $yz \cdot yv \leq yz \cdot (yz \cdot vz)$. Since, by the assumption $vz = a_1 \in H$, then

$$yz \cdot yv \leq yz \cdot (yz \cdot vz) = yz \cdot ((yz \cdot vz) \cdot 0) \in H\,.$$

Applying Corollary 1 we obtain $yz \cdot yv \in H$. In a similar way $yv \cdot yz \in H$. Thus $yz\Theta_H yv$. This completes the proof. □

Corollary 5. *If H is an ideal of a BCK-algebra* $\mathbf{G}$, *then* Θ_H *is a congruence of* $\mathbf{G}$ *such that* $[0]_H = H$. □

Corollary 6. *In a finite BCC-algebra all congruences are induced by ideals.* □

Theorem 5. *If a BCC-algebra* $\mathbf{G}$ *belongs to the variety* $\mathcal{V}$, *then every congruence on* $\mathbf{G}$ *is uniquely determined by its kernel.*

Proof. The binary terms xy and yx form a Gödel equivalence system (see [1]) for the variety $\mathcal{V}$ since $xy = 0$ and $yx = 0$ imply $x = y$. Thus by Theorem 0.1 in [1], for each algebra $\mathbf{G}$ from $\mathcal{V}$ and every congruences Θ, Φ on $\mathbf{G}$, $[0]_\Theta = [0]_\Phi$ implies $\Theta = \Phi$, which completes the proof. □

Corollary 7. *If a BCK-algebra* $\mathbf{G}$ *belongs to the variety* $\mathcal{V}$ *then all its congruences are uniquely determined by ideals.* □

Finally, we give one example of a BCC-algebra with the congruence which is not defined by any ideal.

Example 2. Let $G = N \cup A \cup B$, where

$$N = \{0, 1, 2, \ldots\}, \quad A = \{a_n : n \in N\} \ \text{ and } \ B = \{b_n : n \in B\}.$$

On the set G we define the operation $*$ as follows:
for $m, n \in N$,

$$\begin{aligned}
&m * n = \begin{cases} 0, & if \quad m \le n, \\ m - n, & if \quad m > n, \end{cases} \\
&m * a_n = m * b_n = 0, \\
&b_m * n = b_{m+n}, \\
&a_m * n = a_{m+n}, \\
&a_m * a_n = b_m * b_n = \begin{cases} 0, & if \quad n \le m, \\ n - m, & if \quad n > m, \end{cases} \\
&a_m * b_n = a_m * a_{n+1}, \\
&b_m * a_n = b_m * b_{n+1}.
\end{aligned}$$

One can prove (for details see [14]) that the set G with this operation is a BCK-algebra (and in the consequence a BCC-algebra) which is not in any variety of BCK-algebras (BCC-algebras).

The reader will find no difficulty in verifying that the equivalence relation Θ corresponding to the partition $\{N, A, B\}$ is a congruence. Obviously $[0]_\Theta = N$. But the relation Θ_N induced by N has only two equivalence classes: N and $A \cup B$. Hence $\Theta \neq \Theta_N$.

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