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ON EXPONTENTIAL AUTOREGRESSIVE TIME SERIES MODELS

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Abstract. This paper presents some recent time series models (the so-called NAREX(1) models) for exponential variables with first order autoregressive structures. They are analogs of the standard AR(1) model and of the EAR(1), NEAR(1), TEAR(1) and AREX(1) models, introduced by Lawrance, Lewis, Gaver, Mališić and others. Some of their models can be obtained from NAREX models as special cases.

The distribution of the innovation sequences (a probability mixture) and the autoregressive structure of NAREX processes are discussed as well.

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1. Introduction

It is a well-known fact that the standard first-order autoregressive model (AR(1) model) for a stationary sequence of random variables $\{X_t, t \in T = \{0, \pm 1, \pm 2, \ldots\}\}$ is defined by the difference equation

$$X_t = \varepsilon_t + \beta X_{t-1}, \quad t \in T$$

where β is a parameter ($\beta \in (0,1)$) and $\{\varepsilon_t, t \in T\}$ is a sequence of independent and identically distributed (i.i.d.) random variables.

In the Gaussian AR(1) model, the innovation sequence $\{\varepsilon_t, t \in T\}$ is a sequence of Gaussian random variables. However, many naturally occurring time series are clearly non-Gaussian and for this reason several different models for the generation of non-Gaussian time series have been recently constructed. For example, in many positive-valued time series we have an exponential marginal distribution. This provides strong motivation for introducing some new time series models using exponential variables.

Some earlier papers on exponential time series models can be sumarized as follows.

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Gaver and Lewis (1980) considered an AR model with an exponential $\varepsilon(\lambda)$ marginal distribution in the following from:

$$X_t = \begin{cases} \rho\varepsilon_t & \text{w.p.} \quad \rho \\ \rho X_{t-1} + \xi_t & \text{w.p.} \quad (1-\rho) \end{cases}$$

where ρ is a parameter $(0 \le \rho \le 1)$ and ξ_t, $t \ge 0$ are i.i.d. exponential random variables $\varepsilon(\lambda)$ with the parameter $\lambda > 0$. This is an AR(1) model.

Lawrance and Lewis (1981) discussed some new AR models using exponential variables - TEAR(1):

$$X_t = \begin{cases} (1-\rho)\varepsilon_t & \text{w.p.} \quad 1-\rho \\ \alpha(1-\rho)\varepsilon_t + X_{t-1} & \text{w.p.} \quad \rho \end{cases}$$

and NEAR(1)

$$X_t = \begin{cases} \varepsilon_t + \beta X_{t-1} & \text{w.p.} \quad \rho \\ \varepsilon_t & \text{w.p.} \quad 1-\rho \end{cases}$$

Finally, Mališić (1987) considered some new AR models with an exponential $\varepsilon(\lambda)$ marginal distribution - AREX(1) model (and some of its applications)

$$X_t = \begin{cases} \varepsilon_t & \text{w.p.} \quad p_0 \\ \alpha X_{t-1} & \text{w.p.} \quad p_1 \\ \beta X_{t-1} + \varepsilon_t & \text{w.p.} \quad q_1 \end{cases}$$

where $0 \le p_0,\ p_1, q_1 \le 1$, $p_0 + p_1 + q_1 = 1$, $0 < \alpha,\ \beta < 1$.

In this paper we present a new form of time series models where marginal distributions are in fact exponential distributions. The forms of these models (the so-called NAREX models) are quite distinct form the previous forms of time series models using exponential variables.

2. NAREX(1) models

Let the stationary sequence of random variables $\{X_t, t \in T\}$ be defined by the equation

$$X_t = \begin{cases} \alpha X_{t-1} + \varepsilon_t & ; \quad p_0 \\ \beta X_{t-1} + \varepsilon_t & ; \quad p_1 \\ \gamma X_{t-1} & ; \quad p_2, \end{cases} \tag{2.1}$$

where $0 \le p_0,\ p_1, p_2 \le 1$, $p_0 + p_1 + p_2 = 1$, $0 < \alpha, \beta, \gamma < 1$ and ε_n are some i.i.d. random variables. Let us also, suppose that $\{X_t\}$ and $\{\varepsilon_t\}$ are "semi-independent", i.e. that X_t and ε_m are independent if $t < m$.

Our first objective is to obtain the distribution of the i.i.d. sequence $\{\varepsilon_t\}$ which will ensure that the sequence $\{X_n\}$ in (2.1) has an exponential marginal distribution $\varepsilon(\lambda)$.

Let the Laplace-Stieltjes transformation of the X and ε variables be denoted by

$$\phi_X(s) = E(e^{-sX}), \quad \phi_\varepsilon(s) = E(e^{-s\varepsilon}) \tag{2.2}$$

We know that the following applies

$$\phi_X(s) = \frac{\lambda}{\lambda + s} \tag{2.3}$$

if X has an $\varepsilon(\lambda)$ distribution. If we assume stationarity and "semi-independence", then (2.1), (2.2) and (2.3) give

$$\phi_X(s) = p_0\phi_X(\alpha s)\phi_\varepsilon(s) + p_1\phi_X(\beta s)\phi_\varepsilon(s) + p_2\phi_X(\gamma s)$$

$$\phi_\varepsilon(s) = \frac{\phi_X(s) - p_2\phi_X(\gamma s)}{p_0\phi_X(\alpha s) + p_1\phi_X(\beta s)},$$

$$\phi_\varepsilon(s) = \frac{\frac{\lambda}{\lambda+s} - p_2\frac{\lambda}{\lambda+\gamma s}}{p_0\frac{\lambda}{\lambda+\alpha s} + p_1\frac{\lambda}{\lambda+\beta s}} = \frac{(\lambda+\alpha s)(\lambda+\beta s)[(p_0+p_1)\lambda + (\gamma - p_2)s]}{(\lambda+s)(\lambda+\gamma s)[(p_0+p_1)\lambda + (p_0\beta^2 + p_1\alpha)s]}. \tag{2.4}$$

Let α, β, γ, p_0, p_1 and p_2 be chosen in such a way that in (2.4) there are no common fractions. In this case we shall have

$$\phi_\varepsilon(s) = A_0 + A_1\frac{\lambda}{\lambda+s} + A_2\frac{\lambda}{\lambda+\gamma s} + A_3\frac{\lambda}{(p_0+p_1)\lambda + (p_0\beta + p_1\alpha)s}.$$

Further calculation gives

$$A_0 = \frac{\alpha\beta(\gamma - p_2)}{\gamma(p_0\beta + p_1\alpha)}, \quad A_1 = \frac{(1-\alpha)(1-\beta)}{(1-\beta)p_0 + (1-\alpha)p_1},$$

$$A_2 = \frac{-p_2(\gamma-\alpha)(\gamma-\beta)}{\gamma[(\gamma-\beta)p_0 + (\gamma-\alpha)p_1]},$$

$$A_3 = \frac{p_0p_1(p_0+p_1)(\alpha-\beta)^2[\gamma - (p_0\beta + p_1\alpha + p_2)]}{[(1-\alpha)p_1 + (1-\beta)p_0][\gamma-\alpha)p_1 + (\gamma-\beta)p_0]}.$$

It is obvious that A_1, A_2, A_3 and $\frac{A_3}{p_0+p_1}$ are the probabilities whose sum equals to 1 if and only if

1. if $\alpha < p_2 + \alpha p_1 + \beta p_0 < \beta$ (or $\beta < p_2 + \alpha p_1 + \beta p_0 < \alpha$) follow $\gamma \in (p_2 + \alpha p_1 + \beta p_0, \beta)$ (or $\gamma \in (p_2 + \alpha p_1 + \beta p_0, \alpha)$),
2. if $p_2 < \alpha < \frac{\alpha p_1 + \beta p_0}{p_0+p_1} < \beta$ (or $p_2 < \beta < \frac{\alpha p_1 + \beta p_0}{p_0+p_1} < \alpha$) follow $\gamma \in (p_2, \alpha)$ (or $\gamma \in (p_2, \beta)$).

Then ε_n is the following mixture

$$R(\varepsilon_t) = \begin{cases} 0 & ; \quad A_0 \\ \xi_t & ; \quad A_1 \\ \gamma\xi_t & ; \quad A_2 \\ \frac{\alpha p_1+\beta p_0}{p_0+p_1}\xi_t & ; \quad \frac{A_3}{p_0+p_1}, \end{cases}$$

with $\xi_t : \varepsilon(\lambda)$, $\lambda > 0$.

Taking the expectation of both sides in (2.1) we have

$$E\{X_t\} = p_0[\alpha E\{X_{t-1}\} + E\{\varepsilon_t\}] + p_1[\beta E\{X_{t-1}\} + E\{\varepsilon_t\}] + p_2\gamma E\{X_{t-1}\}$$

or

$$E(\varepsilon_t) = \frac{1-(\alpha p_0 + \beta p_1 + \gamma p_2)}{p_0+p_1} \cdot \frac{1}{\lambda}.$$

It follows that the autocovariance function for the NAREX(1) time series model (2.1) is defined by

$$\begin{aligned} \gamma(h) &= E(X_t X_{t-h}) - E(X_t)E(X_{t-h}) = \\ &= (\alpha p_0 + \beta p_1 + \gamma p_2)\gamma(h-1) = \ldots = (\alpha p_0 + \beta p_1 + \gamma p_2)^h \gamma(0) \end{aligned}$$

so that the autocorrelation function of $\{X_t\}$ is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = (\alpha p_0 + \beta p_1 + \gamma p_2)^h, \ h > 0.$$

3. Some special cases

a) If $p_0 = p_2 = 0$, the EAR(1) model will apply

b) If $\alpha = p_2 = 0$, the NEAR(1) model will apply

c) If $\alpha = 0$, $\gamma = \alpha$, $p_2 = p_1$ and $p_1 = q_1$, the AREX(1) model will apply

d) FNAREX(1)

Let $p_2 = 0$. Then

$$X_t = \begin{cases} \alpha X_{t-1} + \varepsilon_t & \text{w.p.} \quad p_0 \\ \beta X_{t-1} + \varepsilon_t & \text{w.p.} \quad p_1 = 1 - p_0 \end{cases}$$

and

$$\phi_\varepsilon(s) = \frac{(\lambda+\alpha s)(\lambda+\beta s)}{(\lambda+s)(\lambda+\delta s)}.$$

We shall have

$$\phi_\varepsilon(s) = \frac{\alpha\beta}{\delta} + \frac{(1-\alpha)(1-\beta)}{1-\delta} \cdot \frac{\lambda}{\lambda+s} + \frac{(\delta-\alpha)(\beta-\delta)}{\delta(1-\delta)} \cdot \frac{\lambda}{\lambda+\delta s}$$

$$R(\varepsilon_t) = \begin{cases} 0 & \text{w.p.} \quad \frac{\alpha\beta}{\delta} \\ \xi_t & \text{w.p.} \quad \frac{(1-\alpha)(1-\beta)}{1-\delta} \\ \delta\xi_t & \text{w.p.} \quad \frac{(\delta-\alpha)(\beta-\delta)}{\delta(1-\delta)} \end{cases}$$

where ξ_t; $\varepsilon(\lambda)$, $\lambda > 0$.

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