NOVI SAD J. MATH.
VOL. 29, NO. 1, 1999, 69-82


# GENERALIZED BOCHNER–SCHWARTZ THEOREM FOR TEMPERED ULTRADISTRIBUTIONS

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**Abstract.** In the paper we prove the generalized Bochner–Schwartz theorem for tempered ultradistributions by using the boundary value characterization of a space of weighted distributions.

*AMS Mathematics Subject Classification (1991):* 46F12, 46F15, 46F05, 46F10

*Key words and phrases:* Tempered, ultradistribution, Bochner–Schwartz

## 1. Introduction

By the well-known Bochner–Schwartz theorem every positive definite tempered distribution is a Fourier transform of a positive tempered measure. In the paper we prove the generalized Bochner–Schwartz theorem for tempered ultradistribtions of the Beurling and Roumieu types ($\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$): *Every positive definite tempered ultradistribution is a Fourier transform of a positive measure which satisfies the appropriate growth condition* (see Theorem 7). We follow the idea of S-Y. Chung and D. Kim who proved the generalized Bochner–Schwartz theorem for Fourier hyperfunctions [4], but our proofs are different, since we deal with the spaces of ultradistributions.

In order to prove our main result, we use the boundary value characterizations of the spaces of tempered ultradistributions obtained in [1] and spaces of weighted distributions defined in this paper. We will give the proofs of the assertions only for the Roumieu type of tempered ultradistributions. By a similar method one can obtain analogous results in the Beurling case.

We use multi-index notation $|\alpha| = \alpha_1+\alpha_2+...+\alpha_d$, $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_d!$, $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_d^{\alpha_d}$, $|x| = \sqrt{x_1^2+x_2^2+\cdots+x_d^2}$, where $d\in\mathbb{N}$, $\alpha = (\alpha_1,\alpha_2,...,\alpha_d)\in\mathbb{N}_0^d$, $x = (x_1,x_2,...,x_d)\in\mathbb{R}^d$,

$$\varphi^{(\alpha)}(x) = (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}\cdots(\partial/\partial x_d)^{\alpha_d}\varphi(x),\quad x\in\mathbb{R}^d,\quad \varphi\in C^\infty(\mathbb{R}^d)$$

Let $\{M_p,\ p\in\mathbb{N}_0\}$ be a sequence of positive numbers, where $M_0 = 0$. The following conditions will be used: (for their detailed analysis see, for example [12])

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(M.1) (*logarithmic convexity*)

$$M_p^2 \le M_{p-1}M_{p+1}, \quad p = 1, 2, \ldots.$$

(M.2) (*stability under ultradifferential operators*) There are constants $A$ and $H$ such that

$$M_p \le AH^p \min_{0\le q\le p} M_q M_{p-q}, \quad p = 0, 1, \ldots.$$

(M.3) (*strong non-quasi-analyticity*) There is a constants $A$ such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \le A\frac{pM_p}{M_{p+1}}, \quad p = 1, 2, \ldots.$$

Throughout he paper we assume that the conditions (M.1), (M.2) and (M.3) are satisfied.

The so-called associated functions for the sequence $\{M_p,\ p \in \mathbb{N}_0\}$ are

$$M(\rho) = \sup_{p\in\mathbb{N}_0} \log\frac{\rho^p}{M_p}, \quad \overline{M}(\rho) = \sup_{p\in\mathbb{N}_0} \log\frac{\rho^p p!}{M_p^2},$$

where $\rho > 0$.

**Remark 1.** *The Gevrey sequence*

$$p^{sp} \quad or \quad (p!)^s \quad or \quad \Gamma(1+sp), \quad p\in\mathbb{N}_0, \quad s>1,$$

*satisfies all the above conditions and* $M(\rho) \sim \rho^{1/s}$, $\overline{M}(\rho) \sim \rho^{1/(2s-1)}$ *(see [6]).*

Recall, a generalized function $u$ is *positive* if $\langle u, \varphi\rangle \ge 0$, for every non-negative test function $\varphi$, and is *positive definite* if $\langle u, \varphi * \widetilde{\varphi}\rangle \ge 0$, for every positive test function $\varphi$, where $\widetilde{\varphi}(x) = \overline{\varphi(-x)}$.

We use the following definition of the Fourier transform of $f \in L^1(\mathbb{R}^d)$ :

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx.$$

A positive measure $\mu$ is *tempered* if

$$\int (1+|x|^2)^{-p} d\mu < \infty,$$

for some $p \ge 0$. A positive measure $\mu$ is $(M_p)$*-tempered* (respectively $\{M_p\}$-tempered) if

$$\int \exp[-M(k|x|)]d\mu < \infty,$$

for every $k > 0$ (respectively for some $k > 0$).

Spaces of tempered ultradistributions of the Beurling and Roumieu types ($\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$) are generalized Gel'fand–Shilov spaces which were introduced in [11] and analyzed in a number of papers (see [8], [1], [13], [3] and references therein).

**Definition 1** **(i) The set of smooth functions $\varphi$ on $\mathbb{R}^d$ which satisfy**

$$(1.1) \qquad |\varphi^{(\alpha)}(x)| \le C_\varphi \frac{M_{|\alpha|}}{m^{|\alpha|}} \exp[-M(n|x|)],$$

**for some $C_\varphi > 0$, every $\alpha \in \mathbb{N}_0^d$ and every $m, n > 0$ (respectively some $m, n > 0$) is denoted by $\mathcal{S}^{(M_p)}$ (respectively by $\mathcal{S}^{\{M_p\}}$).**

**(ii) A sequence $\varphi_j$ of the elements of $\mathcal{S}^{(M_p)}$ (respectively of $\mathcal{S}^{\{M_p\}}$) converges to zero in $\mathcal{S}^{(M_p)}$ (respectively in $\mathcal{S}^{\{M_p\}}$), as $j \to \infty$, if**

$$(1.2) \qquad \sup_{\substack{\alpha \in \mathbb{N}_0^d \\ x \in \mathbb{R}^d}} \frac{m^{|\alpha|}}{M_{|\alpha|}} \left| \varphi_j^{(\alpha)}(x) \exp\left[M(n|x|)\right]\right| \to 0, \quad \text{as } j \to \infty,$$

**for every $m, n > 0$ (respectively for some $m, n > 0$).**

**We denote by $\mathcal{S}'^{(M_p)}$ (respectively by $\mathcal{S}'^{\{M_p\}}$) the strong dual space of $\mathcal{S}^{(M_p)}$ (respectively of $\mathcal{S}^{\{M_p\}}$).**

**Remark 2.** *The following topological inclusions exist (see [1])*

$$\mathcal{G} \hookrightarrow \mathcal{S}^{(M_p)} \hookrightarrow \mathcal{S},\ \mathcal{F} \hookrightarrow \mathcal{S}^{\{M_p\}} \hookrightarrow \mathcal{S},\ \mathcal{D}^{(M_p)} \hookrightarrow \mathcal{S}^{(M_p)},\ \mathcal{D}^{\{M_p\}} \hookrightarrow \mathcal{S}^{\{M_p\}},$$

$$\mathcal{S}' \hookrightarrow \mathcal{S}'^{(M_p)} \hookrightarrow \mathcal{G}',\ \mathcal{S}' \hookrightarrow \mathcal{S}'^{\{M_p\}} \hookrightarrow \mathcal{F}',\ \mathcal{S}'^{(M_p)} \hookrightarrow \mathcal{D}'^{(M_p)},\ \mathcal{S}'^{\{M_p\}} \hookrightarrow \mathcal{D}'^{\{M_p\}}.$$

*Here $\mathcal{D}^{(M_p)}$ and $\mathcal{D}^{\{M_p\}}$ are the test spaces for spaces of the Beurling and Roumieu ultradistributions (see [12]), $\mathcal{G}$ is the test space for the space $\mathcal{G}'$ of the extended Fourier hyperfunctions (defined as in [5]), and $\mathcal{F}$ is a test space for the space $\mathcal{F}'$ of the Fourier hyperfunctions (defined as in [10]).*

## 2. Heat kernel representations of the spaces $\mathcal{D}'$, $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$

Several authors represented various spaces of generalized functions such as distributions, hyperfunctions, ultradistributions, tempered ultradistributions and Fourier hyperfunctions as the initial values of solutions of the heat equation ([14], [5], [1]).

Denote by $E(x,t)$ the heat kernel:

$$(2.1) \qquad E(x,t) = \begin{cases} (4\pi t)^{-d/2} \exp[-|x|^2/4t], & t > 0, \\ 0, & t < 0. \end{cases}$$

The function $E(x,t)$ is entire of order 2, for $t>0$, and has the following properties (see [14]):

(E0) It satisfies the heat equation.

(E1) $\int_{\mathbb{R}^d} E(x,t)dx = 1$, for $t>0$.

(E2) There are positive constants $C$ and $a'$ such that

$$(2.2)\quad \left|\frac{\partial^\alpha}{\partial x^\alpha}E(x,t)\right| \le C^{|\alpha|+1}t^{-(|\alpha|+d)/2}\alpha!^{1/2}\exp[-a'|x|^2/4t],\quad t>0.$$

where $a' \in (0,1)$ can be taken as close as desired to 1.

(E3) $E(\cdot,t)$ is the element of $\mathcal{S}^{(M_p)}$ for every $t>0$.

The following theorem gives the heat kernel representation for the space of distributions.

**Theorem 1.** ([14]) 1. Let $u \in \mathcal{D}'$ and $T>0$. Then there exists a smooth function $U(x,t)$ on $\mathbb{R}^d \times (0,T)$ which satisfies the following:

(i) $(d/dt - \Delta)U(x,t) = 0, \quad (x,t) \in \mathbb{R}^d \times (0,T)$.

(ii) For any compact set $K \subset \mathbb{R}^d$, there exists $N = N(K) > 0$ such that

$$\sup_{x\in K}|U(x,t)| \le Ct^{-N}, \quad 0<t<T.$$

(iii) $\langle u,\varphi\rangle = \lim_{t\to 0^+}\int U(x,t)\varphi(x)dx$, for every $\varphi \in C_0^\infty$.

2. Conversely, if $U(x,t)$ is a smooth function on $\mathbb{R}^d \times (0,T)$, satisfying (i) and (ii), then there exists a unique $u \in \mathcal{D}'$ satisfying (iii).

The next theorem characterizes the space of tempered ultradistributions and is a special case of [1, Theorem 6].

**Theorem 2.** ([1]) Let $u \in \mathcal{S}'^{(M_p)}$ (respectively $\mathcal{S}'^{\{M_p\}}$) and $T$ be a positive constant. A function

$$U(x,t) = \langle u(y), E(x-y,t)\rangle$$

is smooth on $\mathbb{R}^d \times (0,T)$ and satisfies the following:

(i) $(d/dt - \Delta)U(x,t) = 0, \quad (x,t) \in \mathbb{R}^d \times (0,T)$.

(ii) For some $m,n>0$ (respectively for every $m,n>0$), there exists a positive constant $C$ such that

$$(2.3)\quad |U(x,t)| \le C\exp\Big[M(n|x|) + \frac{1}{2}\overline{M}\Big(\frac{m}{t}\Big)\Big],\quad (x,t)\in\mathbb{R}^d\times(0,T).$$

(iii) For any $\psi \in \mathcal{S}^{(M_p)}$ (respectively any $\psi \in \mathcal{S}^{\{M_p\}}$),

$$\langle u, \psi \rangle = \lim_{t \to 0^+} \int_{\mathbb{R}^d} U(x,t)\psi(x)dx. \tag{2.4}$$

2. Conversely, for every smooth function $U(x,t)$ defined on $\mathbb{R}^d \times (0,T)$, satisfying conditions (i) and (ii) for some $m, n > 0$ (respectively for every $m, n > 0$), there exists a unique $u \in \mathcal{S}'^{(M_p)}$ (respectively $u \in \mathcal{S}'^{\{M_p\}}$) satisfying (iii).

## 3. Space of weighted distributions $\mathcal{S}'_M$

In this section we define the space of weighted distributions which will be used in the the proof of the generalized Bochner–Schwartz theorem for tempered ultradistributions of the Roumieu type. In the Beurling case, another space of weighted distributions has to be considered (see Remark 3).

**Definition 2.**

*(i)* *$\mathcal{S}_M$ is the set of smooth functions $\varphi$ on $\mathbb{R}^d$ such that for every $\alpha \in \mathbb{N}_0$, there exists $n > 0$, such that*

$$\sup_{x \in \mathbb{R}^d} |\varphi^{(\alpha)}(x)| \exp[M(n|x|)] < \infty.$$

*(ii)* *A sequence $\varphi_j$ converges to zero in the space $\mathcal{S}_M$, if for every $p \in \mathbb{N}$, there exists $n > 0$, such that*

$$\sup_{x \in \mathbb{R}^d} \sum_{|\alpha| \le p} |\varphi_j^{(\alpha)}(x)| \exp[M(n|x|)] \to 0, \quad as \quad j \to \infty.$$

*(iii)* *We denote by $\mathcal{S}'_M$ the strong dual space of the space $\mathcal{S}_M$.*

Note, the space $\mathcal{S}'_M$ is the subspace of the space of distributions $\mathcal{D}'$ which follows from the following:

1. *The inclusion $\mathcal{D} \to \mathcal{S}_M$ is continuous.*
2. *The set $\mathcal{S}_M \backslash \mathcal{D}$ is nonempty.*
3. *$\mathcal{D}$ is dense in $\mathcal{S}_M$.*

The proof of this is analogous to the proof of Theorem 2 in [13]. The following propositions give characterizations of the spaces $\mathcal{S}_M$ and $\mathcal{S}'_M$.

**Proposition 3.** *1. Let $\varphi \in \mathcal{S}_M$. For every $t > 0$,*

$$U(x,t) = \int_{\mathbb{R}^d} E(x-y,t)\varphi(y)dy, \quad x \in \mathbb{R}, \tag{3.1}$$

*is an element of the space $\mathcal{S}_M$ and $U(x,t)$ converges to $\varphi(x)$ in the space $\mathcal{S}_M$, as $t \to 0^+$.*

*Proof.* Let $\varphi \in \mathcal{S}_M$. We prove that for every $\alpha \in \mathbb{N}_0^d$, there exist $n > 0$, such that

$$\sup_{x\in\mathbb{R}^d}\left|\frac{\partial^\alpha}{\partial x^\alpha}\Big(U(x,t)-\varphi(x)\Big)\exp[M(n|x|)]\right| \to 0, \quad \text{as } t\to 0. \tag{3.2}$$

Let $\delta$ be a small positive number and $\alpha \in \mathbb{N}_0$. We have

$$\begin{aligned}\left|\frac{\partial^\alpha}{\partial x^\alpha}\Big(U(x,t)-\varphi(x)\Big)\right| &= \left|\int_{\mathbb{R}^d} E(x-y,t)(\varphi^{(\alpha)}(y)-\varphi^{(\alpha)}(x))dy\right| \\ &= \left|\int_{\mathbb{R}^d} E(y,t)(\varphi^{(\alpha)}(x-y)-\varphi^{(\alpha)}(x))dy\right| \\ &\le \int_{|y|\le\delta} E(y,t)|\varphi^{(\alpha)}(x-y)-\varphi^{(\alpha)}(x)|dy + \\ &\quad + \int_{|y|\ge\delta} E(y,t)|\varphi^{(\alpha)}(x-y)|dy + \\ &+ \int_{|y|\ge\delta} E(y,t)|\varphi^{(\alpha)}(x)|dy \\ &= I_1+I_2+I_3.\end{aligned}$$

By the mean value theorem,

$$I_1 = \int_{|y|\le\delta} E(y,t)|\varphi^{(\alpha)}(x-y)-\varphi^{(\alpha)}(x)|dy \le$$

$$\int_{|y|\le\delta} E(y,t)|\varphi^{(\alpha+1)}(x-\theta y)|\,|y|\,dy \le C\ \delta\int_{|y|\le\delta} E(y,t)\exp[-M(n|x-\theta y|)]dy, \tag{3.3}$$

for some $\theta\in(0,1)$ and some $n>0$. For a $\delta$ being small enough, $|y|\le\delta$, and $x\in\mathbb{R}^d$, we have

$$M(n|x-\theta y|) \ge M\Big(\frac{n}{2}|x|\Big). \tag{3.4}$$

Now, the inequalities 3.3 and 3.4 and property (E1) imply that there exists $C>0$ such that

$$I_1 \le C\ \delta\exp[-M(\frac{n}{2}|x|)] \le C\delta.$$

There exists $C>0$ such that

$$\begin{aligned} I_2 = &\int_{|y|\ge\delta} E(y,t)|\varphi^{(\alpha)}(x-y)|dy \le \\ &\le C(4\pi t)^{-d/2}\int_{|y|\ge\delta}\exp[-y^2/4t]\exp[-M(n|x-y|)]dy,\end{aligned} \tag{3.5}$$

for some $n>0$.

The condition (M.1) implies that the associated function $M(\cdot)$ satisfies, (see [2]),

$$M(\rho+\delta) \le M(2\rho)+M(2\delta), \quad \rho,\delta>0. \tag{3.6}$$

Taking $2\rho = n|x-y|$ and $2\delta = n|y|$, we get

$$(3.7)\quad M(n|x-y|) \geq M\Big(\frac{n}{2}|x-y| + \frac{n}{2}|y|\Big) - M(n|y|) \geq M\Big(\frac{n}{2}|x|\Big) - M(n|y|).$$

By (M.1) and (M.3) (see [12, Lemma 4.1, (4.7)]) we have for some $C > 0$

$$(3.8)\qquad M(n|y|) \leq C|y| \leq C\,y^2, \quad |y| \geq \tilde{\delta},$$

where $\tilde{\delta}$ is large enough. Therefore from 3.5, 3.7, 3.8, it follows

$$I_2 \leq$$

$$\leq C\,(4\pi t)^{-d/2}\exp\Big[-\frac{\delta^2}{8t}\Big]\exp\Big[-M(\frac{n}{2}|x|)\Big]\int_{|y|\geq\delta}\exp\Big[-\frac{y^2}{8t} + M(n|y|)\Big]dy$$

$$\leq C\,(4\pi t)^{-d/2}\exp\Big[-\frac{\delta^2}{8t}\Big]\exp\Big[-M(\frac{n}{2}|x|)\Big]\Big(\int_{\delta\leq|y|\leq\tilde{\delta}}\exp\Big[-\frac{y^2}{8t} + M(n|y|)\Big]dy +$$

$$+\int_{|y|\geq\tilde{\delta}}\exp\Big[-\frac{y^2}{8t} + M(n|y|)\Big]dy\Big)$$

$$\leq C\,\varepsilon_t\,\exp\Big[-M(\frac{n}{2}|x|)\Big]\Big(C_1 + \int_{|y|\geq\tilde{\delta}}\exp\Big[-\frac{y^2}{8t} + C_2 y^2\Big]\Big)$$

$$\leq \tilde{C}\,\varepsilon_t\,\exp\Big[-M(\frac{n}{2}|x|)\Big],$$

where $\varepsilon_t = (4\pi t)^{-d/2}\exp\Big[-\frac{\delta^2}{8t}\Big]$ tends to zero as $t \to 0$.

Finally, by the properties of $E(x,t)$ we have that there exists $C > 0$ such that

$$I_3 = \int_{|y|\geq\delta} E(y,t)|\varphi^{(\alpha)}(x)|dy \leq |\varphi^{(\alpha)}(x)|\int_{|y|\geq\delta} E(y,t)dy \leq \tilde{\delta}_t C\exp[-M(n|x|)].$$

for some $n > 0$, where $\tilde{\delta}_t = \int_{|y|\geq\delta} E(y,t)dy$ tends to zero as $t \to 0^+$.

From above we obtain that $U(x,t)$ converges to $\varphi(x)$, in the space $\mathcal{S}_M$, as $t$ tends to zero. □

**Proposition 4.** *1. Let $u \in \mathcal{S}'_M$ and $T > 0$. Then*

$$U(x,t) = \langle f(y), E(x-y,t)\rangle$$

*is smooth on $\mathbb{R}^d \times (0,T)$ and it satisfies*

*(i)* $(d/dt - \Delta)U(x,t) = 0.$

*(ii) There exists $N > 0$, such that for every $n > 0$, there exists a constant $C > 0$, such that*

$$(3.9)\qquad |U(x,t)| \leq Ct^{-N}\exp\Big[M(n|x|)\Big] \quad (x,t) \in \mathbb{R}^d \times (0,T).$$

*(iii) For any $\psi \in \mathcal{S}_M$*

$$\langle u, \psi \rangle = \lim_{t \to 0^+} \int_{\mathbb{R}^d} U(x,t)\psi(x)dx,$$

*for every $\psi \in \mathcal{S}_M$ on $\mathbb{R}^d \times (0,T)$.*

*2. Conversely, for every smooth function $U(x,t)$ defined on $\mathbb{R}^d \times (0,T)$, satisfying conditions (i) and (ii), there exists a unique $u \in \mathcal{S}'_M$ satisfying (iii).*

*Proof.* 1. Let $u \in \mathcal{S}'_M$. Obviously, the function $U(x,t) =< f(y), E(x-y,t) >$ belongs to $C^\infty(\mathbb{R}^d \times (0,T))$. Using the properties of $E(x,t)$, for every fixed $t > 0$, and

$$M(h|y|) - M(2h|x-y|) \leq M(2h|x|), \quad h > 0,$$

(which follows from 3.6 by similar argument as in 3.7), we obtain that for every $p \in \mathbb{N}$ and every $h > 0$

$$|U(x,t)| = |\langle u(y), E(x-y,t)\rangle|$$

$$\leq C \sum_{|\alpha|\leq p} \sup_{y\in\mathbb{R}^d} \left| E^{(\alpha)}(x-y,t)\exp[M(h|y|)] \right|$$
$$\leq C \sum_{|\alpha|\leq p} \sup_{y\in\mathbb{R}^d} \tilde{C}^{|\alpha|+1} t^{-(|\alpha|+d)/2} \alpha!^{1/2} \left| \exp\left[\frac{-4|x-y|^2}{(16t/a')}\right] + M(h|y|)] \right|$$
$$\leq C \sum_{|\alpha|\leq p} \sup_{y\in\mathbb{R}^d} \tilde{C}^{|\alpha|+1} t^{-(|\alpha|+d)/2} (4\cdot 16\pi T/a')^{d/2} \left| E(2|x-y|, 16T/a')\exp[M(h|y|)] \right|$$
$$\leq C_1 \sum_{|\alpha|\leq p} t^{-(|\alpha|+d)/2} \sup_{y\in\mathbb{R}^d} |\exp[-M(2h|x-y|)]\exp[M(h|y|)]|$$
$$\leq C_2 t^{-N} \exp[M(2h|x|)].$$

2. Let $T > 0$. By the well-known fact from the theory of distributions there exist smooth functions $v$ and $w$, such that supp $v \subset [0,T/2]$, supp $w \subset [T/4,T/2]$, $|v(x)| \leq x^{N+1}/(N+1)!$ and that

$$(d/dx)^{N+2} v(x) = \delta(x) + w(x). \tag{3.10}$$

Consider

$$g(x,t) = \int_0^\infty U(x,t+s)v(s)ds, \qquad h(x,t) = \int_0^\infty U(x,t+s)w(s)ds. \tag{3.11}$$

Since supp $v \subset [0,T/2]$, we have that for every $n > 0$, there exists $C, \tilde{C} > 0$, such that

$$|g(x,t)| \leq C\exp[M(n|x|)] \int_0^{T/2} (t+s)^{-N} \frac{s^{N+1}}{(N+1)!} ds \leq \tilde{C}\exp[M(n|x|)], \tag{3.12}$$

where $(x,t) \in \mathbb{R}^d \times (0,T)$.

It is easy to see that for every $n > 0$, there exists $C > 0$, such that

$$|h(x,t)| \leq C\exp[M(n|x|)], \qquad (x,t) \in \mathbb{R}^d \times (0,T). \tag{3.13}$$

The functions $g(x,t), h(x,t)$ are smooth on $\mathbb{R}^d \times (0,T)$ and satisfy the heat equation. It can be easily proved that $g(x,t)$ and $h(x,t)$ can be continuously extended on the set $\{(x,t)|x \in \mathbb{R}^d,\ t \in [0,T]\}$. Put

$$g_0(x) = \lim_{t\to 0^+} g(x,0), \qquad h_0(x) = \lim_{t\to 0^+} h(x,0), \qquad x \in \mathbb{R}^d.$$

The functions $g_0(x)$ and $h_0(x)$ are $C^\infty(\mathbb{R}^d \times (0,T))$.

Since supp $w \subset [T/4, T/2]$, function $h(x,t)$ can be analytically continued to $\{(x,t)|x \in \mathbb{R}^d, t > -T/4\}$. Thus $h_0(x)$ is a real analytic function.

From 3.12 and 3.13 it follows that for every $n > 0$, there exists $C > 0$, such that

$$|g_0(x)| \leq C\exp[M(n|x|)] \quad \text{and} \quad |h_0(x)| \leq C\exp[M(n|x|)]. \tag{3.14}$$

The function $g(x,t)$ satisfies the heat equation, thus, it follows from 3.10 that

$$U(x,t) + h(x,t) = \left(\frac{d}{dt}\right)^{N+2} g(x,t) = (-\Delta)^{N+2} g(x,t). \tag{3.15}$$

Define

$$u(x) = (-\Delta)^{N+2} g_0(x) - h_0(x). \tag{3.16}$$

From 3.14 it follows that $g_0, h_0 \in \mathcal{S}'_M$ and that $(-\Delta)^{N+2} g_0 \in \mathcal{S}'_M$. Therefore, $u \in \mathcal{S}'_M$.

Let us prove that $U(x,t) = < f(y), E(x-y) >$. Put

$$A(x,t) = \int_{\mathbb{R}^d} E(x-y,t) g_0(y) dy, \quad t > 0;$$

$$B(x,t) = \int_{\mathbb{R}^d} E(x-y,t) h_0(y) dy, \quad t > 0.$$

The functions $A(x,t)$ and $B(x,t)$ satisfy the heat equation and converge locally uniformly to $g_0(x)$ and $h_0(x)$, respectively, as $t$ converges to zero. Therefore, they can be continuously extended to $\{(x,t)|\ x \in \mathbb{R}^d,\ t \in [0,T]\}$ and

$$\lim_{t\to 0} A(x,t) = g_0(x) = \lim_{t\to 0^+} g(x,t), \quad \text{and} \quad \lim_{t\to 0} B(x,t) = h_0(x) = \lim_{t\to 0^+} h(x,t).$$

Furthermore, the functions $A(x,t)$ and $B(x,t)$ satisfy the following growth conditions:

$$\begin{aligned} |A(x,t)| &\leq C\exp[M(a|x|)] \leq \tilde{C}\exp[ax^2], \qquad t \in [0,T], \\ |B(x,t)| &\leq C\exp[M(a|x|)] \leq \tilde{C}\exp[ax^2], \qquad t \in [0,T]. \end{aligned} \tag{3.17}$$

Let us prove the first inequality. For an arbitrary $\delta > 0$,

$$|A(x,t)| \leq |\int_{\mathbb{R}^d} E(y,t)g_0(x-y)dy|$$
$$\leq C\int_{|y|\leq\delta} E(y,t)\exp[M(n|x-y|)]dy + C\int_{|y|\geq\delta} E(y,t)\exp[M(n|x-y|)]dy$$
$$= I_1 + I_2.$$

Using 3.6 we get that

$$\begin{aligned} I_1 &\leq C\exp[M(2n|x|)]\int_{|y|\leq\delta} E(y,t)\exp[M(2n|y|)]dy \\ &\leq C\exp[M(2n|x|)]\exp[M(2n\delta)]\int_{\mathbb{R}^d} E(y,t)dy \\ &\leq C_1\exp[M(2n|x|)] \leq \tilde{C}\exp[ax^2], \end{aligned}$$

for every $n > 0$, and $a > 0$.

For $\delta > 0$ large enough, by 3.8, we have that for every $n > 0$, $t \in (0,T)$, $0 < b < \min(1, 1/8Tn)$, there exists $C > 0$ such that

$$I_2 \leq C_1(4\pi t)^{-d/2}\exp[M(2n|x|)]\int_{|y|\geq\delta}\exp\Big[-\frac{y^2}{4t} + M(2n|y|)\Big]dy$$
$$\leq C_1(4\pi t)^{-d/2}\exp[M(2n|x|)]\exp\Big[-\frac{\delta^2}{4t}(1-\frac{1}{b})\Big]\int_{|y|>\delta}\exp\Big[-\frac{y^2}{4Tb} + 2ny^2\Big]dy$$
$$\leq C_2\exp[M(2n|x|)] \leq C\exp[2nx^2].$$

By the uniqueness theorem for the initial-value heat equation (see [7, pp. 216]), it follows that the solution of the problem

$$u_t(x,t) - \Delta u(x,t) = 0, \quad x \in \mathbb{R}^d \quad t \in (0,\infty), \qquad u(x,0) = f(x),\ x \in \mathbb{R}^d,$$

is unique, provided that we restrict ourselves to the solutions satisfying

$$|u(x,t)| \leq C\exp[ax^2], \quad x \in \mathbb{R}^d \quad t \in (0,T).$$

Therefore,

$$\begin{aligned} g(x,t) = A(x,t) &= \int_{\mathbb{R}^d} E(x-y,t)g_0(y)dy, \\ h(x,t) = B(x,t) &= \int_{\mathbb{R}^d} E(x-y,t)h_0(y)dy. \end{aligned}$$

From 3.16 and 3.15 it follows

$$\int_{\mathbb{R}^d} E(x-y,t)u(y)dy = \int_{\mathbb{R}^d} E(x-y,t)[(-\Delta)^{N+2}g_0(y) - h_0(y)]dy$$
$$= (-\Delta)^{N+2}\int_{\mathbb{R}^d} E(x-y,t)g_0(y)dy - \int_{\mathbb{R}^d} E(x-y,t)h_0(y)dy$$
$$= (-\Delta)^{N+2}g(x,t) - h(x,t) = U(x,t),$$

i.e. $U(x,t) = \langle u(y), E(x-y,t)\rangle$. □

## 4. Positive and positive definite tempered ultradistributions

In this section we prove our main results. The proofs will be given for the Roumieu case. The proofs of the assertions in the Beurling case are analogous and simpler. First we prove that every tempered ultradistribution and every positive element of the space $\mathcal{S}'_M$ is a measure satisfying an appropriate growth condition. We use the result in order to prove the generalized Bochner-Schwartz theorem for the tempered ultradistributions.

**Proposition 5.** *1. Every positive generalized function in* $\mathcal{S}'_M$ *is an* $\{M_p\}$*-tempered measure.*

*2. Conversely, if* $\mu$ *is a positive* $\{M_p\}$*-tempered measure it defines a positive generalized function* $u$ *in* $\mathcal{S}'_M$*, in a sense that*

$$\langle u, \varphi\rangle = \int \varphi(x) d\mu(x), \quad \varphi \in \mathcal{S}_M.$$

*Proof.* By Proposition 5 it follows that it is enough to prove that each positive element of $\mathcal{S}'^{\{M_p\}}$ belongs to $\mathcal{S}'_M$.

Let $u \in \mathcal{S}'_M$ be positive. Since $\mathcal{S}'_M$ is a subspace of the space of distributions $\mathcal{D}'$ we have that there exists a positive measure $\mu$, such that

$$\langle u, \varphi\rangle = \int \varphi(x) d\mu(x), \quad \varphi \in \mathcal{D}. \tag{4.1}$$

In order to give the sense to 4.1 for all $\varphi \in \mathcal{S}_M$, the measure $\mu$ must satisfy an appropriate growth condition. Let us prove this. Denote by $\psi$ a smooth non-negative function with a compact support, such that $\psi(x) = 1$ for $|x| \leq 1$. Put

$$\varphi_m(x) = \psi\Big(\frac{x}{m}\Big)\exp[-M(k|x|)], \quad x \in \mathbb{R}^d,$$

for some $k > 0$. Then $\varphi_m$ is a sequence of non-negative smooth functions and converges to $\exp[-M(k|x|)]$, in the space $\mathcal{S}_M$, as $m \to \infty$. Since $u \in \mathcal{S}'_M$, there exists $C > 0$ such that

$$\lim_{m\to\infty} \langle u, \varphi_m\rangle \leq C.$$

By Fatou's lemma we have

$$0 \leq \int \exp[-M(k|x|)] d\mu \leq \lim \inf_{m\to\infty} \int \varphi_m(x) d\mu(x) = \lim_{m\to\infty} \langle u, \varphi_m\rangle < \infty.$$

Thus $\mu$ is an $\{M_p\}$–tempered measure. □

**Theorem 6.** *1. Every positive $u \in \mathcal{S}'^{(M_p)}$ (respectively $u \in \mathcal{S}'^{\{M_p\}}$) is an $(M_p)$-tempered (respectively $\{M_p\}$-tempered) measure.*

*2. Conversely, if $\mu$ is an $(M_p)$-tempered (respectively $\{M_p\}$-tempered) measure, then $\mu$ defines positive $u \in \mathcal{S}'^{(M_p)}$ (respectively $u \in \mathcal{S}'^{\{M_p\}}$ ), i.e.*

$$\langle u, \varphi\rangle = \int \varphi(x) d\mu(x), \quad \varphi \in \mathcal{S}^{(M_p)} \quad (\textit{respectively} \quad \varphi \in \mathcal{S}^{\{M_p\}}).$$

*Proof.* Let $u \in \mathcal{S}'^{\{M_p\}}$ be positive. By Theorem 2, for every $m, n > 0$ and arbitrary $T > 0$, there exists a constant $C > 0$, such that

$$|U(x,t)| \leq C\exp\Big[M(n|x|) + \frac{1}{2}\overline{M}\Big(\frac{m}{t}\Big)\Big], \quad (x,t) \in \mathbb{R} \times (0,T), \tag{4.2}$$

where $U(x,t) = \langle u(y), E(x-y,t)\rangle$ and $u(x) = \lim_{t\to 0^+} U(x,t)$.

Therefore, since the heat kernel $E(x,t)$ is non-negative, we have that if u is a positive element in $\mathcal{S}'^{\{M_p\}}$, $U(x,t) = \langle u(y), E(x-y,t)\rangle$ is non-negative for each $t > 0$. Therefore, for $0 < t < a < T$,

$$\begin{aligned} 0 \leq U(x,t) &= \langle u(y), (4\pi t)^{-d/2}\exp\Big[-\frac{|x-y|^2}{4t}\Big]\rangle \leq \\ &\leq \Big(\frac{a}{t}\Big)^{d/2}\langle u(y), (4\pi a)^{-d/2}\exp\Big[-\frac{|x-y|^2}{4a}\Big]\rangle \leq \\ &\leq \Big(\frac{a}{t}\Big)^{d/2}\langle u(y), E(x-y,a)\rangle \leq \Big(\frac{a}{t}\Big)^{d/2} U(x,a). \end{aligned}$$

Since $a$ is an arbitrary constant, such that $0 < a < T$, we have that there exists a constant $C > 0$, such that

$$0 \leq U(x,t) \leq Ct^{-d/2}U(x,T). \tag{4.3}$$

From 4.2 and 4.3 we get that for every $m, n > 0$, there exists $C, C_T > 0$, such that

$$\begin{aligned} 0 \leq U(x,t) &\leq Ct^{-d/2}\exp\Big[M(n|x|) + \frac{1}{2}\overline{M}\Big(\frac{m}{T}\Big)\Big] \leq \\ &\leq C_T t^{-d/2}\exp[M(n|x|)], \quad (x,t) \in \mathbb{R} \times (0,T). \end{aligned} \tag{4.4}$$

This and Proposition 4 imply that $u \in \mathcal{S}'_M$. □

Now we give a generalized Bochner–Schwartz theorem for tempered ultradistributions.

**Theorem 7.** *1. Every positive definite $u \in \mathcal{S}'^{(M_p)}$ (respectively $u \in \mathcal{S}'^{\{M_p\}}$) is the Fourier transform of some positive $(M_p)$-tempered (respectively $\{M_p\}$-tempered) measure $\mu$.*

*2. Conversely, the Fourier transform of any positive and $(M_p)$-tempered (respectively $\{M_p\}$-tempered) measure $\mu$ defines a positive definite tempered ultradistribution $u \in \mathcal{S}'^{(M_p)}$ (respectively $u \in \mathcal{S}'^{\{M_p\}}$), i.e.*

$$\langle u, \varphi \rangle = \int \hat{\varphi}(\xi) d\mu(\xi), \quad \varphi \in \mathcal{S}'^{(M_p)} \quad (\textit{respectively} \quad \varphi \in \mathcal{S}'^{\{M_p\}}),$$

*where $\widehat{\varphi}$ is the Fourier transform of $\varphi$.*

**Remark 3.** *We give the proof of Theorem 7 only in the Roumieu case. In it we use the properties of the space $\mathcal{S}'_M$ of weighted distributions. In order to prove Theorem 7 in the Beurling case, one has to define and use a slightly different space of weighted distributions, whose definition can be obtained formally from Definition 2 by replacing "there exists $n > 0$" in (i) and (ii) by "for all $n > 0$".*

*Proof.* In the proof we use properties of the Fourier transform on the spaces $\mathcal{S}^{\{M_p\}}$ and $\mathcal{S}'^{\{M_p\}}$ (see [9]). If $\varphi \in \mathcal{S}^{\{M_p\}}$ we denote $\tilde{\varphi}(x) = \overline{\varphi(-x)}$, and $\check{\varphi}(x) = \varphi(-x)$. Since

$$\widehat{\check{\psi}} = -\overline{\widehat{\bar{\psi}}} \quad \text{and} \quad \widehat{\tilde{\psi}} = -\widehat{\psi},$$

for every $\psi \in \mathcal{S}^{\{M_p\}}$, we have that

$$\widehat{\varphi * \check{\tilde{\varphi}}} = -\overline{\widehat{\varphi * \tilde{\varphi}}} = -\overline{\widehat{\varphi}\widehat{\tilde{\varphi}}} = -\overline{\widehat{\varphi}}\overline{\widehat{\tilde{\varphi}}} = \widehat{\varphi}\overline{\widehat{\varphi}}. \tag{4.5}$$

Let $u$ be a positive definite tempered ultradistribution of the Roumieu type, and $\varphi \in \mathcal{S}^{\{M_p\}}$. From 4.5, using the equality

$$< \hat{u}, \hat{\varphi} >= (2\pi)^d \langle u, \check{\varphi} \rangle,$$

and the fact that the Fourier transform is an isomorphism of the space $\mathcal{S}^{\{M_p\}}$ onto itself (see [9]), one can prove that the inequality

$$\langle u, \varphi * \tilde{\varphi} \rangle \geq 0, \quad \varphi \in \mathcal{S}^{\{M_p\}},$$

is equivalent to

$$< \hat{u}, \psi\bar{\psi} > \geq 0, \quad \psi \in \mathcal{S}^{\{M_p\}}.$$

Let $V(x,t) =< \hat{u}(y), E(x-y,t) >$, from above it follows

$$\begin{aligned} V(x,t) &=< \hat{u}(y), E(x-y,t) >=< \hat{u}(y), (4\pi t)^{-d/2} \exp\Big[-\frac{|x-y|^2}{4t}\Big] >= \\ &= (4\pi t)^{-d/2} < \hat{u}, \exp\Big[-\frac{|x-y|^2}{8t}\Big] \overline{\exp\Big[-\frac{|x-y|^2}{8t}\Big]} > \geq 0. \end{aligned} \tag{4.6}$$

Therefore, the function $V(x,t)$ is a non-negative smooth function and since $\hat{u} \in \mathcal{S}'^{\{M_p\}}$, it satisfies the heat equation and

$$|V(x,t)| \leq C \exp\Big[M(n|x|) + \frac{1}{2}\overline{M}\Big(\frac{m}{t}\Big)\Big], \quad (x,t) \in \mathbb{R}^d \times (0,T). \tag{4.7}$$

From theorem 2 it follows that $\hat{u} = \lim_{t\to 0} U(x,t)$ is a positive tempered ultradistribution of the Roumieu type. By Theorem 6 we have that $\hat{u}$ is a positive $\{M_p\}$–tempered measure. Since the Fourier transform an isomorphism of the space $\mathcal{S}'^{\{M_p\}}$ onto itself, the assertion of the theorem follows. □

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*Received by the editors February 16, 1998.*