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# FIXED POINTS IN TWO METRIC SPACES

**Y. J. Cho[1], S. M. Kang[1], S. S. Kim[2]**

**Abstract.** We give some fixed point theorems in two complete metric spaces. Thus we improve and extend some results due to D. Delbesco, B. Fisher and V. Popa.

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*Key Words and Phrases:* Delbosco's set, common fixed point

In [4], to give a unified approach for contraction mappings, D. Delbosco considered the set $\mathcal{F}$ of all continuous functions $g : [0, +\infty)^3 \to [0, +\infty)$ satisfying the following conditions:

(g-1) $g(1,1,1) = h < 1$,

(g-2) If $u, v \in [0, +\infty)$ are such that $u \leq g(v, v, u)$ or $u \leq g(u, v, v)$ or $u \leq g(v, u, v)$, then $u \leq hv$,

and proved the following:

**Theorem A.** *Let $(X, d)$ be a complete metric space. If $S$ and $T$ are two mappings from $X$ into itself, satisfying the following conditions:*

$$(A) \qquad d(Sx, Ty) \leq g(d(x, y), d(x, Sx), d(y, Ty))$$

*for all $x, y \in X$, where $g \in \mathcal{F}$, then $S$ and $T$ have a unique common fixed point in $X$.*

Some authors proved many kinds of fixed point theorems for contractive type mappings and expansive mappings by using Delbosco's set ([1]-[3], [7], [8], [10]). On the other hand, in [5] and [6], B. Fisher proved some fixed point theorems in two complete metric spaces as follows:

**Theorem B.** *Let $(X, d)$ and $(Y, e)$ be complete metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$, satisfying the following conditions:*

$$(B) \qquad e(Tx, TSy) \leq c \cdot \max\{d(x, Sy), e(y, Tx), e(y, TSy)\},$$

$$(C) \qquad d(Sy, STx) \leq c \cdot \max\{e(y, Tx), d(x, Sy), e(x, STx)\}$$

*for all $x, y \in X$, where $0 \leq c < 1$, then $ST$ have a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz = w$ and $Sw = z$.*

[1]Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea
[2]Department of Mathematics, Dongeui University, Pusan 614-714, Korea

Recently, in [9], V. Popa extended and improved the results of B. Fisher and Theorem A. In this paper, motivated by Delbosco's set and V. Popa's result, we introduce a new class $\mathcal{G}$ of all functions $g : [0, +\infty)^3 \to [0, +\infty)$ satisfying some conditions and prove some fixed point theorems in two complete metric spaces by using our class. Our results also extend and improve the results of B. Fisher [5], [6] and V. Popa [9].

Let $\mathcal{G}$ be the set of all continuous functions $g : [0, +\infty)^3 \to [0, +\infty)$ satisfying the following conditions:

(g$'$-1) $g(0, 0, 0) = 0$,

(g$'$-2) If $u, v \in [0, +\infty)$ be such that $u^2 \le g(uv, 0, 0)$ or $u^2 \le g(0, uv, 0)$ or $u^2 \le g(0, 0, uv)$, then $u \le cv$ for some $0 \le c < 1$.

**Example 1.** (1) If we define a function $g : [0, +\infty)^3 \to [0, +\infty)$ by

$$g(u, v, w) = c \cdot \max\{u, v, w\}$$

for all $u, v, w \in [0, +\infty)$, where $0 \le c < 1$, then $g \in \mathcal{G}$.

(2) If we define a function $g : [0, +\infty)^3 \to [0, +\infty)$ by

$$g(u, v, w) = c \cdot \max\{uv, uw, vw\}$$

for all $u, v, w \in [0, +\infty)$, where $0 \le c < 1$, then $g \in \mathcal{G}$.

(3) If we define a function $g : [0, +\infty)^3 \to [0, +\infty)$ by

$$g(u, v, w) = auv + buw + cvw$$

for all $u, v, w \in [0, +\infty)$, where $a, b, c \in [0, +\infty)$, then $g \in \mathcal{G}$.

(4) If we define a functions $g : [0, +\infty)^3 \to [0, +\infty)$ by

$$g(u, v, w) = (au^k + bv^k + cw^k)^{\frac{1}{k}}$$

for all $u, v, w \in [0, +\infty)$, where $k > 1$, $0 \le a, b, c < 1$, then $g \in \mathcal{G}$.

Now, we give our theorems as follows:

**Theorem 1.** *Let $(X, d)$ and $(Y, e)$ be two complete metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$ satisfying the following conditions:*

$$\text{(D)} \qquad e^2(Tx, TSy) \le g(d(x, Sy)e(y, Tx), d(x, Sy)e(y, TSy), e(y, Tx)e(y, TSy)),$$

$$\text{(E)} \qquad d^2(Sy, STx) \le g(e(y, Tx)d(x, Sy), e(y, Tx)d(x, STx), d(x, Sy)d(x, STx))$$

*for all $x \in X$ and $y \in Y$, where $g \in \mathcal{G}$, then $ST$ has a unique fixed point $z \in X$ and $TS$ has a unique fixed point $w \in Y$. Further, $Tz = w$ and $Sw = z$.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. Define two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ and $Y$, respectively, as follows:

$$x_n = (ST)^n x_0, \quad y_n = T(ST)^{n-1}x_0$$

for $n = 1, 2, \cdots$. By (D), we have

$$\begin{aligned} d^2(x_n, x_{n+1}) &= d^2((ST)^n x_0, (ST)^{n+1}x_0) \\ &= d^2(S(T(ST)^{n-1}x_0), (ST)(ST)^n x_0) \\ &= d^2(Sy_n, STx_n) \\ &\leq g(e(y_n, Tx_n)d(x_n, Sy_n), e(y_n, Tx_n)d(x_n, STy_n), \\ &\qquad d(x_n, Sy_n)d(x_n, STx_n)) \\ &= g(e(y_n, y_{n+1})d(x_n, x_n), e(y_n, y_{n+1})d(x_n, x_{n+1}), \\ &\qquad d(x_n, x_n)d(x_n, x_{n+1})) \\ &= g(0, e(y_n, y_{n+1})d(x_n, x_{n+1}), 0). \end{aligned}$$

Thus, by (g′-1), we have

$$(F) \qquad d(x_n, x_{n+1}) \leq ce(y_n, y_{n+1})$$

for some $0 \leq c_1 < 1$. Similarly, by (D),

$$\begin{aligned} e^2(y_n, y_{n+1}) &= e^2(T(ST)^{n-1}x_0, T(ST)^n x_0) \\ &= e^2(T(ST)^{n-1}x_0), TS(T(ST)^{n-1}x_0)) \\ &= e^2(Tx_{n-1}, TSy_n) \\ &\leq g(d(x_n, Sy_n)e(y_n, Tx_{n-1}), e(x_{n-1}, Sy_n)e(y_n, TSy_n), \\ &\qquad d(y_n, Tx_{n-1})e(y_n, TSy_n)) \\ &= g(d(x_{n-1}, x_n)e(y_n, y_n), d(x_{n-1}, x_n)e(y_n, y_{n+1}), \\ &\qquad d(y_n, y_n)e(y_n, y_{n+1})) \\ &= g(0, d(x_{n-1}, x_n)e(y_n, y_{n+1}), 0). \end{aligned}$$

Thus, by (g′-2), we have

$$(G) \qquad e(y_n, y_{n+1}) \leq c_2 d(x_{n-1}, x_n)$$

for some $0 \leq c_2 < 1$. Therefore, by (F) and (G),

$$\begin{aligned} d(x_n, x_{n+1}) &\leq c_1 e(y_n, y_{n+1}) \\ &\leq c_1 c_2 d(x_{n-1}, x_n) \\ &\leq \cdots \\ &\leq (c_1 c_2)^n d(x_0, x_1), \end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in $(X, d)$ since $0 < c_1 c_2 < 1$ and so, since $(X, d)$ is complete, it converges to a point $z$ in $X$. Similarly, the sequence $\{y_n\}$ is also a Cauchy sequence in $(Y, e)$ with the limit $w$. By (D) again, we have

$$(H)\qquad \begin{aligned} e^2(Tz, y_{n+1}) &= e^2(Tz, TSy_n) \\ &\le g(d(z, Sy_n)e(y_n, Tz), d(z, Sy_n)e(y_n, TSy_n), \\ &\qquad e(y_n, Tz)e(y_n, TSy_n)) \\ &= g(d(z, x_n)e(y_n, Tz), d(z, x_n)e(y_n, y_{n+1}), \\ &\qquad e(y_n, Tz)e(y_n, y_{n+1})). \end{aligned}$$

Letting $n \to \infty$ in (H), by (g′-1), it follows that

$$e^2(Tz, w) \le g(0, 0, 0) = 0$$

and so, $e(Tz, w) = 0$, i.e., $Tz = w$. On the other hand, by (E) we have

$$(I)\qquad \begin{aligned} d^2(Sw, x_{n+1}) &= d^2(Sw, (ST)^{n+1}x_0) \\ &= d^2(Sw, STx_n) \\ &\le g(e(w, Tx_n)d(x_n, Sw), e(w, Tx_n)d(x_n, STx_n), \\ &\qquad d(x_n, Sw)d(x_n, STx_n)) \\ &= g(e(w, y_{n+1})d(x_n, Sw), e(w, y_{n+1})d(x_n, x_{n+1}), \\ &\qquad d(x_n, Sw)d(x_n, x_{n+1})). \end{aligned}$$

Letting $n \to \infty$ in (I), by (g′-1), we have

$$d^2(Sw, z) \le g(0, 0, 0) = 0$$

and so, $d(Sw, z) = 0$, i.e., $Sw = z$. Therefore, we have $STz = Sw = z$ and $TSw = Tz = w$, which means that the point $z$ is a fixed point of $ST$ and the point $w$ is a fixed point of $TS$.

To prove the uniqueness of the fixed point $z$, let $z'$ be the second fixed point of $ST$. By (D), we have

$$\begin{aligned} d^2(z, z') &= d^2(STz', STz) \\ &\le g(e(Tz', Tz)d(z, STz'), \\ &\qquad e(Tz', Tz)d(z, STz), d(z, STz')d(z, STz)) \\ &= g(e(Tz', Tz)d(z, z'), 0, 0), \end{aligned}$$

which, by (g′-2), implies that

$$(J)\qquad d(z', z) \le c_3 e(Tz', Tz)$$

for some $0 \le c_3 < 1$. Similarly, by (D), we have

$$\begin{aligned} e^2(Tz, Tz') &= e^2(Tz', TSTz) \\ &\le g(d(z', STz)e(Tz, Tz'), d(z', STz)e(Tz, TSTz), \\ &\qquad e(Tz, Tz')e(Tz, TSTz)) \\ &= g(d(z', z)e(Tz, Tz'), 0, 0). \end{aligned}$$

Thus, by (g′-2), it follows that

$$(K) \qquad e(Tz, Tz') \leq c_4 d(z', z)$$

for some $0 \leq c_4 < 1$. Therefore, by (J) and (K),

$$d(z, z') \leq c_3 e(Tz, Tz') \leq c_3 c_4 d(z, z'),$$

which implies that $d(z, z') = 0$, i.e., $z = z'$, since $0 \leq c_3 c_4 < 1$ and so the uniqueness of the fixed point $z$ of $ST$ follows. Similarly, the point $w$ is also a unique fixed point of $TS$. On the other hand, if there exists a positive integer $n$ such that $d(x_n, x_{n+1}) = 0$ or $e(y_n, y_{n+1}) = 0$, then the theorem is evident. This completes the proof. □

As immediate consequences of Theorem 1, we have the following:

**Corollary 2.** [9] *Let $(X, d)$ and $(Y, e)$ be two complete metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$ satisfying the following conditions:*

$$\begin{aligned}
&(L) \qquad e^2(Tx, TSy) &&\leq c_1 \cdot \max\{d(x, Sy)e(y, Tx), d(x, Sy)e(y, TSy), \\
& && \qquad e(y, Tx)e(y, TSy)\}, \\
&(M) \qquad d^2(Sy, STx) &&\leq c_2 \cdot \max\{e(y, Tx)d(x, Sy), e(y, Tx)d(x, STx), \\
& && \qquad d(x, Sy)d(x, STx)\}
\end{aligned}$$

*for all $x \in X$ and $y \in Y$, where $0 \leq c_1, c_2 < 1$, then $ST$ has a unique fixed point in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz = w$ and $Sw = z$.*

*Proof.* Define a function $g : [0, +\infty)^3 \to [0, +\infty)$ by

$$g(u, v, w) = c \cdot \max\{uv, uw, vw\}$$

for all $u, v, w \in [0, +\infty)$, where $0 < c < 1$. Then, from Example 1 (2) follows that $g \in \mathcal{G}$ and, by Theorem 1, the corollary follows. □

**Corollary 3.** *Let $(X, d)$ and $(Y, e)$ be two complete metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$ satisfying the following conditions:*

$$\begin{aligned}
&(N) \qquad e^2(Tx, TSy) &&\leq a_1 d(x, Sy)e(y, Tx) + b_1 d(x, Sy)e(y, TSx) \\
& && \quad + c_1 e(y, Tx)e(y, TSy), \\
&(O) \qquad d^2(Sy, STx) &&\leq a_2 e(y, Tx)d(x, Sy) + b_2 d(x, STx)e(y, Tx) \\
& && \quad + c_3 d(x, Sy)d(x, STx)
\end{aligned}$$

*for all $x \in X$ and $y \in Y$, where $a_1, a_2, b_1, b_2, c_1, c_2 \in [0, +\infty)$ with $(a_1 + b_1 + c_1)(a_2 + b_2 + c_2) < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz = w$ and $Sw = z$.*

*Proof.* Define a function $g : [0, +\infty)^3 \to [0, +\infty)$ by

$$g(u, v, w) = auv + buw + cvw$$

for all $u, v, w \in [0, +\infty)$, where $a, b, c \in [0, +\infty)$. Then, from Example 1 (3), follows that $g \in \mathcal{G}$ and, by Theorem 1, the corollary follows. □

**Corollary 4.** *Let $(X, d)$ and $(Y, e)$ be two complete metric spaces. If $T$ is a mapping from $X$ into $Y$ and $S$ is a mapping from $Y$ into $X$ satisfying the following conditions:*

$$\begin{aligned} &(P) \quad e^2(Tx, TSy) \le a_1 d^2(x, Sy) + b_1 e^2(y, Tx) + c_1 e^2(y, TSy), \\ &(Q) \quad d^2(Sy, STx) \le a_2 e^2(y, Tx) + b_2 d^2(x, Sy) + c_2 d^2(x, STx) \end{aligned}$$

*for all $x \in X$ and $y \in Y$, where $0 \le a_1, a_2, b_1, b_2, c_1, c_2 < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz = w$ and $Sw = z$.*

*Proof.* Define a function $g : [0, +\infty)^3 \to [0, +\infty)$ by

$$g(u, v, w) = au^2 + bv^2 + cw^2$$

for all $u, v, w \in [0, +\infty)$, where $0 < a, b, c < 1$. Then $g \in \mathcal{G}$ and, by Theorem 1, the corollary follows. □

If $(X, d)$ and $(Y, e)$ are the same metric spaces, then by Theorem 1, we have the following:

**Theorem 5.** *Let $(X, d)$ be a complete metric space. If $S$ and $T$ are mappings from $X$ into itself satisfying the following conditions:*

$$\begin{aligned} (R) \quad d^2(Tx, TSy) \le\ & g(d(x, Sy)d(y, Tx), d(x, Sy)d(y, TSy), \\ & d(y, Tx)d(y, TSy)), \\ (S) \quad d^2(Sy, STx) \le\ & g(d(y, Tx)d(x, Sy), d(y, Tx)d(y, STx), \\ & d(x, Sy)d(x, STx)) \end{aligned}$$

*for all $x, y \in X$, where $g \in \mathcal{G}$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $X$. Further, $Tz = w$ and $Sw = z$ and, if $z = w$, then $z$ is the unique common fixed point of $S$ and $T$.*

**Corollary 6.** *Let $(X, d)$ be a complete metric space. If $S$ and $T$ are mappings from $X$ into itself satisfying the following conditions:*

$$\begin{aligned} (T) \quad d^2(Tx, TSy) \le\ & c_1 \cdot \max\{d(x, Sy)d(y, Tx), d(x, Sy)d(y, TSy), \\ & d(y, Tx)d(y, TSy)\}, \\ (U) \quad d^2(Sy, STx) \le\ & c_2 \cdot \max\{d(y, Tx)d(x, Sy), d(y, Tx)d(y, STx), \\ & d(x, Sy)d(x, STx)\} \end{aligned}$$

*for all $x, y \in X$, where $0 \le c_1, c_2 < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $X$. Further, $Tz = w$ and $Sw = z$ and, if $z = w$, then $z$ is the unique common fixed point of $S$ and $T$.*

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