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DANIELL-GRECO-STONE REPRESENTATION TYPE THEOREM FOR AUTOCONTINUOUS FROM ABOVE FUNCTIONALS

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Abstract. A Daniell-Stone type representation of monotone autocontinuous from above functional by integral with respect to monotone autocontinuous from above set function is given.

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*Key words and Phrases:*null-additive set function, autocontinuous from above set function, autocontinuous from above functional

1. Introduction

The representations in the sense of Daniell-Stone of non-additive functionals with integrals with respect to monotone non-additive set functions were investigated by B. Anger [1], G.H. Greco [7], R.C. Basanezi, G.H. Greco [2], D. Deneberg [4], D. Schmeidler [17].

In this paper we shall investigate the representation by special monotone set functions so-called autocontinuous from above set functions. These set functions are special null-additive set functions, which include many important classes of set functions ([19], [20], [11], [12], [13], [14],[15],[16]). We shall prove a general representation theorem using the method of subgraphs of functions given by J. Kindler [10]. As special cases we obtain representations by submodular and supermodular set functions.

2. Autocontinuous from above set functions

Let $\mathcal{L}$ be a lattice of subsets of the given set X such that $\emptyset \in \mathcal{L}$.

Definition 1. *A set function $m, m : \mathcal{L} \to [0, \infty]$ with $m(\emptyset) = 0$ is called null-additive, if we have*

$$m(A \cup B) = m(A)$$

whenever $A, B \in \mathcal{L}, A \cap B = \emptyset$ and $m(B) = 0$.

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For properties of null-additive set functions see E. Pap [15], [16], H. Suzuki [18] and Z. Wang [19].

Definition 2. *A set function m is called autocontinuous from above (resp. from below) if for every $\epsilon > 0$ and every $A \in \mathcal{L}$, there exists $\delta = \delta(A, \epsilon) > 0$ such that*

$$m(A) - \epsilon \leq m(A \cup B) \leq m(A) + \epsilon \quad (\textit{resp.} \;\; m(A) - \epsilon \leq m(A \setminus B) \leq m(A) + \epsilon)$$

whenever $B \in \mathcal{L}, A \cap B = \emptyset$ (resp. $B \subset A, A \setminus B \in \mathcal{L}$) and $m(B) < \delta$ holds.

Let $\mathcal{L}$ be a lattice of subsets of the given set X. For any monotone set function $m : \mathcal{L} \to [0, \infty]$ we define an outer set function m^* on $\mathcal{P}(X)$ by

$$m^*(E) := \inf\{m(F) : E \subset F \in \mathcal{L}\}.$$

In the proof of the main result we shall need the following property of the set function m^* which is invariant with respect to monotonicity and autocontinuity.

Proposition 1. *If m is a monotone and autocontinuous from above set function on a lattice $\mathcal{L}$ of subsets of a given set X, then the corresponding outer set function m^* is also monotone and autocontinuous set function on $\mathcal{P}(X)$.*

Proof. We can easily prove the monotonicity of m^*. Take $A, B \in \mathcal{P}(X)$. In the case $m^*(A) = \infty$, the statement is true. Let $\epsilon > 0$. By the definition of m^* we choose A'_1 from $\mathcal{L}$ such that $A \subset A'_1$ and

$$m(A'_1) \leq m^*(A) + \epsilon. \tag{1}$$

By the autocontinuoity from above of m there exists $\delta = \delta(\epsilon, A'_1) > 0$ such that

$$m(A'_1 \cup B) < m(A'_1) + \frac{\epsilon}{2}$$

whenever $m(B) < \delta, \;\; B \in \mathcal{L}$.
Let $B \in \mathcal{P}(X)$ be such that $m^*(B) < \frac{\delta}{2}$. Then by the definition of m^* we have that for $0 < \delta' < \frac{\delta}{2}$ there exists $B'_1 \in \mathcal{L}$ such that

$$m(B'_1) \leq m^*(B) + \delta'.$$

Therefore

$$m(B'_1) < \delta.$$

Then by the autocontinuity from above of m

$$m(A'_1 \cup B'_1) < m(A'_1) + \frac{\epsilon}{2}. \tag{2}$$

Hence by the monotonicity of m^*, (2) and (1)

$$m^*(A \cup B) \leq m^*(A'_1 \cup B'_1) = m(A'_1 \cup B'_1) < m(A'_1) + \frac{\epsilon}{2} \leq m^*(A) + \epsilon.$$

So we have proved the theorem.

Open problem: Let m be monotone and null-additive set function. Is m^* null-additive too?

3. Integral and functionals

Let $\mathcal{F}$ be a family of functions $f : \mathcal{F} \to [0, \infty]$ which satisfy the conditions:

$$af, \ f \wedge a, \ f - f \wedge a \in \mathcal{F} \quad (f \in \mathcal{F}, a \in [0, \infty)) \quad \textbf{(Stone condition)},$$

$$f \wedge g, \ f \vee g \in \mathcal{F} \ \ (f, g \in \mathcal{F}) \quad \textbf{(Lattice condition)}.$$

The family of all upper level sets of the function f is denoted by $\mathcal{U}_f$, i.e.

$$\mathcal{U}_f := \{ \ \{x : f(x) > t\} : t \in [0, \infty]\} \cup \{ \ \{x : f(x) \geq t\} : t \in [0, \infty]\}.$$

A class of functions $\mathcal{F}_o \subset \mathcal{F}$ is comonotonic (common monotonic) if

$$\cup_{f \in \mathcal{F}_o} \mathcal{U}_f$$

is a chain. A class of functions $\mathcal{F}_o$ is comonotonic iff each pair of functions from $\mathcal{F}_o$ is comonotonic. The equivalent condition for a pair of functions f and g to be comonotonic is that there is no pair $x_1, x_2 \in X$ such that $f(x_1) < f(x_2)$ and $g(x_1) > g(x_2)$.

Let $m : \mathcal{P}(X) \to [0, \infty]$ be a monotone set function with the property $m(\emptyset) = 0$ and $f : X \to [0, \infty]$ a function, then $m(\{x : f(x) > t\})$ is a decreasing function on $[0, \infty]$.
For a monotone set function $m : \mathcal{L} \to [0, \infty]$ and an upper m-measurable function $f : X \to [0, \infty]$ the Choquet integral is defined by

$$\int f dm := \int_0^\infty m(\{x : f(x) > t\}) dt.$$

Let M be a functional $M : \mathcal{F} \to [0, \infty]$. We list the following properties of M for every $f, g \in \mathcal{F}$ and every $a \in [0, \infty)$:

(F1) $f \leq g \ \Rightarrow \ M(f) \leq M(g)$ **(monotonicity)**;

(F2) $M(f + g) = M(f) + M(g)$
for comonotonic f and g such that $f + g \in \mathcal{F}$ **(comonotonic additivity)**;

(F3) $\lim_{a \downarrow 0} M(f - f \wedge a) = M(f)$ **(lower marginal continuity)**;

(F4) $\lim_{a \to \infty} M(f \wedge a) = M(f)$ **(upper marginal continuity)**;

(F5) $(\forall \epsilon > 0)(\exists \delta > 0)(\forall g \in \mathcal{F})(\forall f \in \mathcal{F})$
$(M(g) < \delta \ \Rightarrow \ M(f \vee g) < M(f) + \epsilon)$ **(autocontinuity from above)**;

(F6) $M(f \vee g) = M(f)$ whenever $M(g) = 0$ **(null-additivity)**;

(F7) $M(f \vee g) + M(f \wedge g) \leq M(f) + M(g)$ **(submodularity)**;

(F8) $M(f \vee g) + M(f \wedge g) \geq M(f) + M(g)$ **(supermodularity)**.

Remark 1. Conditions (F1) and (F2) imply the positive homogeneity of M, i.e. $M(af) = aM(f)$ for any $a \in [0, \infty)$.

4. The main result

Let $\mathcal{F}$ be the same as in the previous section.

Definition 3. *For a given class of functions $\mathcal{F}, f : X \to [0, \infty]$, a monotone set function $m : \mathcal{P}(X) \to [0, \infty]$ and a functional $M : \mathcal{F} \to [0, \infty]$ we say that m represents M if we have*

$$M(f) = \int f dm \quad (f \in \mathcal{F}).$$

Now we have the following representation theorem.

Theorem 1. *If M is a functional $M : \mathcal{F} \to [0, \infty]$ with the properties (F1) - (F5), then there exists a monotone autocontinuous from above set function $m : \mathcal{P}(X) \to [0, \infty]$ which represent M.*

Proof. We define two monotone set functions for any $E \in \mathcal{P}(X)$

$$\alpha(E) := \sup\{M(f) : f \in \mathcal{F},\ f \leq \chi_E\}$$

$$\beta(E) := \inf\{M(f) : f \in \mathcal{F},\ \chi_E \leq f\}.$$

We shall prove that the set function $m := \beta$ is autocontinuous from above and it represents the functional M.
We define for $f : X \to [0, \infty]$

$$G_f := \{(x,t) : (x,t) \in X \times [0, \infty),\ t < f(x)\}.$$

Since $\mathcal{F}$ satisfies the lattice condition we have

$$G_f \cup G_g = G_{f \vee g}, \qquad G_f \cap G_g = G_{f \wedge g}.$$

Therefore
$\mathcal{L} := \{G_f : f \in \mathcal{F}\}$ is a lattice of subsets of the set X. We introduce a set function μ on $\mathcal{L}$

$$\mu(G_f) := M(f) \quad (f \in \mathcal{F}).$$

By

$$G_f \subset G_g \quad \Leftrightarrow \quad f \leq g$$

we obtain that μ is a monotone set function. Now we shall use the property (F5) of M. If we suppose $\mu(G_f) < \delta$, then also $M(f) < \delta$ holds. Then, the condition (F5) implies

$$\mu(G_f \cup G_g) = \mu(G_{f \vee g}) = M(f \vee g) < M(f) + \epsilon = \mu(G_f) + \epsilon,$$

i.e. μ is autocontinuous from above.
By Proposition 1 the outer set function μ^* of μ is also monotone and autocontinuous from above. Then the restriction of μ^* to $\mathcal{P}(X)$ given by $\mu^*(G_{\chi_E}) \quad (E \in \mathcal{P}(X))$ is also monotone and autocontinuous from above, since

$$G_{\chi_E} \leq G_{\chi_F} \quad \Leftrightarrow \quad E \subset F,$$

$$G_{\chi_{E \cup F}} = G_{\chi_E} \cup G_{\chi_F}.$$

This set function μ^* is equal to the set function m. Namely, we have for every $E \in \mathcal{P}(X)$

$$m(E) = \inf\{M(f) : \chi_E \leq f \in \mathcal{F}\} = \inf\{\mu(G_f : G_{\chi_E} \subset G_f \in \mathcal{L}\} = \mu^*(G_{\chi_E}).$$

Now we shall prove that m represents the functional M. For that purpose we take for a function $f \in \mathcal{F}$ its approximation by the functions $f_n(x)$

$$f_n(x) := \frac{1}{2^n} \Sigma_{i=1}^{n2^n - 1} \chi_{x > \frac{i}{2^n}}(x).$$

Since we have

$$x \wedge \frac{i+1}{2^n} - x \wedge \frac{i}{2^n} \leq \chi_{x > \frac{i}{2^n}} \leq x \wedge \frac{i}{2^n} - x \wedge \frac{i-1}{2^n},$$

we obtain using Remark 1.

$$f \wedge \frac{i+1}{2^n} - f \wedge \frac{i}{2^n} \leq \alpha(f > \frac{i}{2^n}) \leq m(f > \frac{i}{2^n}) \leq 2^n M(f \wedge \frac{i}{2^n} - f \wedge \frac{i-1}{2^n}).$$

Since the functions $f \wedge \frac{i}{2^n} - f \wedge \frac{i-1}{2^n} \in \mathcal{F}$ are comonotonic we obtain using the property (F2) of M and summing up the last inequalities

$$M(\sum_{i=1}^{n2^n - 1} (f \wedge \frac{i+1}{2^n} - f \wedge \frac{i}{2^n})) \leq \frac{1}{2^n} \sum_{i=1}^{n2^n - 1} m(f > \frac{i}{2^n}) \leq M(\sum_{i=1}^{n2^n - 1} (f \wedge \frac{i}{2^n} - f \wedge \frac{i-1}{2^n})).$$

Hence

$$M(f \wedge n - f \wedge \frac{1}{2^n}) \leq \int f_n dm \leq M(f \wedge (n - \frac{1}{2^n})) \leq M(f).$$

Since the functions $(f - f \wedge 1) \wedge (n-1)$ and $(f \wedge 1 - (f \wedge 1) \wedge \frac{1}{2^n})$ are comonotonic and the following representation is true

$$f \wedge n - f \wedge \frac{1}{2^n} = (f \wedge n - f \wedge 1) + (f \wedge 1 - f \wedge \frac{1}{2^n}) = (f - f \wedge 1) \wedge (n-1) + (f \wedge 1 - (f \wedge 1) \wedge \frac{1}{2^n}).$$

we obtain by the property (F2) of M

$$M(f \wedge n - f \wedge \frac{1}{2^n}) = M((f - f \wedge 1) \wedge (n-1)) + M((f \wedge 1 - (f \wedge 1) \wedge \frac{1}{2^n})).$$

Therefore by the properties (F3) and (F4) of M

$$\lim_{n \to \infty} M(f \wedge n - f \wedge \frac{1}{2^n}) = M(f - f \wedge 1) + M(f \wedge 1) = M(f),$$

i.e., $M(f) = \lim_{n \to \infty} \int f_n dm$.
Now we shall prove that

$$\lim_{n \to \infty} \int f_n dm = \int f dm.$$

By the inequality

$$f \wedge n - f \wedge \frac{1}{2^n} \leq f_n \leq f$$

we have

$$\begin{aligned} \int_{\frac{1}{2^n}}^{n} m(\{x : f(x) \geq t\}) dt &= \int (f \wedge n - f \wedge \frac{1}{2^n}) dm \\ &\leq \int f_n dm \leq \int f dm = \int_0^{\infty} m(\{x : f(x) \geq t\}) dt. \end{aligned}$$

Letting $n \to \infty$ we obtain the desired equality.

Corollary 1. *If M is a functional $M : \mathcal{F} \to [0, \infty]$ with the properties (F1) - (F4) and (F7) (or (F8)), then there exists a monotone submodular (supermodular) autocontinuous from above set function $m : \mathcal{P}(X) \to [0, \infty]$ which represent M.*

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