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AN ANALYTIC INTERPRETATION OF REAL DIMENSION SUBGROUPS

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Abstract

Real dimension subgroups of finitely presented groups are given an analytic interpretation in terms of Chen's iterated path integrals. Some of the algebraic properties of Chen's integrals are analogous to the algebraic properties of Fox derivatives in the free group.

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1. Dimension Subgroups

Dimension subgroups occur naturally in the study of group representations. Their importance stems also from the role these subgroups play in the construction of the Lie algebra associated with a group, a crucial step in the introduction of Lie methods into combinatorial group theory. However, for the purpose of this note, we shall look at the dimension subgroups as related to representations.

Consider a representation ρ of the group G over a field $\mathbb{F}$, that is, a homomorphism $\rho : G \to Aut_{\mathbb{F}}(V)$ where V is a vector space over $\mathbb{F}$. (Throughout this note we shall assume that the field $\mathbb{F}$ is of characteristic 0.) Note that if ρ is a representation of G by automorphisms of V, then V naturally acquires the structure of a module over the group algebra $\mathbb{F}G$, where the module action is obtained by exteding the action $gm = \rho(g)(m)$ by linearity. Actually, the notion of representation of a group by automorphisms of a vectors space is equivalent to the notion of module over the group-algebra. We shall use them interchangeably. If V is a finite-dimensional vector space, then ρ is a representation of G by matrices over the field $\mathbb{F}$, and we can talk about unitriangular representations as one particularly desireable case. (In that case, G is represented by, say, upper triangular matrices with all entries on the diagonal equal to 1.) In order to generalize this desireable situation to the infinite-dimensional case, we observe that if the group is represented by unitriangular matrices, then the $\mathbb{F}G$-module V must have a series of submodules

$$V = V_n \geq V_{n-1} \geq \cdots \geq V_0 = 0$$

such that the group acts trivially on each of the factors V_i/V_{i-1}. The existence of such a series of submodules is the condition that defines an n-stable representation. The notion of an n-stable representation is a generalization of the notion of representation by unitriangular matrices.

One way to study the n-stable representations of a group G is to observe that in the category of representations of G (which is equivalent to the category of $\mathbb{F}G$ modules), the class of n-stable representations forms a subcategory. It is an easy exercise to check that the property of being n-stable is preserved under module homomorphisms. Furthermore, it is not difficult to see that this category has a universal object which is in a technical sense the most general, canonical n-stable representation of the group G: any n-stable representation of G must factor through the canonical one. The kernel of the canonical n-stable representation is contained in the kernel of any n-stable representation. Thus, for example, if we want to know whether the group G may be given a faithful n-stable representation, we could (in principle) compute the kernel of the canonical one and see whether it is trivial. Since the kernel of the generic n-stable representation must be contained in the kernel of any such representation, the existence of a faithful n-stable representation is clearly equivalent to the kernel of the canonical one being trivial. Another way to use the kernel of the generic n-stable representation is to observe that a representation is n-stable only if its kernel contains the kernel of the generic one. We thus have a necessary condition for checking whether a given representation is n-stable.

Given a group G and field $\mathbb{F}$, the kernel of the canonical n-stable representation is called the n-th dimension subgroup of G over $\mathbb{F}$. The n-th dimension subgroup is denoted by $D_{n,\mathbb{F}}(G)$; or simply $D_n(G)$ if the ground field is clear from the context. The task now becomes to find a more explicit description of $D_{n,\mathbb{F}}(G)$.

Consider an n-stable module V over $\mathbb{F}G$, and let Δ denote the augmentation ideal of the group algebra $\mathbb{F}G$, that is, the kernel of the natural projection $\mathbb{F}G \to \mathbb{F}$ induced by $G \to 1$. Thus, Δ is the ideal generated by the elements of the form $g - 1$, where $g \in G$. Now V is stable, so the action of the group algebra is trivial on each of the quotients V_i/V_{i-1}, where V_j are the submodules that witness the stability of V. Thus, for $x \in V_i$ and $g \in G$, we have that $gm \equiv m$ modulo V_{i-1}. In other words, $(g-1)V_i \subseteq V_{i-1}$. This means that the action of $\mathbb{F}G$ on the quotient V_i/V_{i-1} is trivial if and only if $\Delta V_i \subseteq V_{i-1}$. It follows that

$$\begin{aligned} \Delta V &\subseteq V_{n-1} \\ \Delta^2 V &\subseteq V_{n-2} \\ &\cdots \\ \Delta^n V &\subseteq V_0 = 0. \end{aligned}$$

Hence, if V is an n-stable module, then V is annihilated by Δ^n. Conversely, any module annihilated by Δ^n is n-stable, since we may take the stable series to be the sequence $V_i = \Delta^{n-i}V$, $i = 0, \dots, n$. In this case,

$$V = V_n \geq V_{n-1} \geq \cdots \geq V_0 = 0$$

and it is clear that $\mathbb{F}G$ acts trivially on each of the quotients V_i/V_{i-1}, because $\Delta V_i \subseteq V_{i-1}$ is true by definition of the submodules V_j.

From these observations, it is an easy exercise to verify that the kernel of the canonical n-stable representation ρ is precisely the set of elements g of G which are such that $g - 1 \in \Delta^n$. Thus we have:

$$D_n(G) = G \cap (1 + \Delta^n).$$

This is the group-algebra characterization of the dimension subgroups. It is also possible to obtain an internal, entirely group-theoretical characterization of the dimension subgroups. Over a field of characterstic 0, this characterization is known as Jennings's theorem. It states that for every finitely generated group G and any field of characteristic 0, we have:

$$G \cap (1+\Delta^n(G)) = \sqrt{\gamma_n(G)} = \{g \in G \ : \ (\exists m \in \mathbb{N}) \ g^m \in \gamma_n(G)\}.$$

There are at least two different proofs of this result. One is the original Jennings's proof, and the other is outlined in [7] and uses Moran's theorem [6] (*cf.* also [8]). The assumption that the group be finitely generated is not an important restriction to the generality of the theorem. Indeed, to prove that Jennings's theorem is valid for all groups, it suffices to know that it is true for all finitely presentable groups. This is the subject of the following lemma.

Lemma 1. *If Jennings's theorem is true for finitely presentable groups, then it is true for all groups.*

Proof. We shall use the technique of lifting to the free group ring. For the details of this method we refer to [3]. Let $G = F/R$ be a presentation of the group G; thus F is free and R is normal in F. Suppose that $g \in D_n(G)$; this means that $g - 1 \in \Delta^n(G)$. This relation lifts to the relation $w - 1 \in \Delta^n(F) + \Delta_R$ in the free group ring $\mathbb{Q}F$, where $\Delta(F)$ is the augmentation ideal of $\mathbb{Q}F$ and Δ_R is the relative augmenation ideal of R, that is, the kernel of the natural projection $\mathbb{Q}F \mapsto \mathbb{Q}(F/R)$. This involves only finitely many generators and relators, so it follows that there are finitely generated subgroups $F_0 \subseteq F$ and $R_0 \subseteq R \cap F_0$ such that $w-1 \in \Delta^n(F_0) + \Delta_{R_0^{F_0}}$. Since $F_0/R_0^{F_0}$ is finitely presented and the image of w belongs to $D_n(F_0/R_0^{F_0}) = \sqrt{\gamma_n(F_0/R_0^{F_0})}$, it follows that $w^m \in \gamma_n(F_0)R_0^{F_0}$ for some m, and hence g^m, being a homomorphic image of w^m, belongs to $\gamma_n(G)$. Therefore, $D_n(G) = \sqrt{\gamma_n(G)}$. □

Thus, dealing with finitely presentable groups is not a restriction at all, as far Jennings's theorem is concerned; and it has certain advantages, inasmuch as it allows us to introduce analytic tools into the study of dimension subgroups.

The purpose of this note is to provide a further description of dimension sugroups of finitely presented groups. We shall see that it is possible to obtain an analytic interpretation of real dimension subgroups, and that our interpretation in terms of Chen's integrals can be tied to the theory of Fox derivatives in the free group ring.

2. The Analytic Interpretation

To begin with, let us recall the following classical result:

Lemma 2. *(cf. [2], 14.1.) Every finitely presentable group is the fundamental group of a smooth compact connected manifold.*

Thus, every finitely presented group may be realized as the fundamental group of a manifold, so that the question we are interested in can be formulated as follows: How do the dimension subgroups relate to the topological or analytic structure of the manifold?

A straightforward application of Chen's 'π_1 de Rham theorem' which identifies the linear functionals on the augmentation quotients of the group-algebra of the fundamental group of a smooth manifold with certain integration maps, called iterated path integrals, shows that the n-th real dimension subgroup consists precisely of those loops that are neglected by all iterated path integrals of length at most $n-1$. Although the proof of the actual result is quite simple, we need to develop some terminology in order to formulate this characterization of dimension subgroups.

Suppose G is a finitely presentable group and M is a smooth, compact, connected manifold such that $G = \pi_1(M)$. Consider the set $E^1(M, A)$ of A-valued 1-forms on M, where A is an associative algebra. (We shall want to specify that $A = \mathbb{R}$ later, but the following definition is independent of such a specification. In particular, it is occasionally convenient to be able to take A to be an algebra of matrices over reals.) Given a piecewise smooth path $\lambda : [0,1] \mapsto M$ and 1-forms $w_1, \ldots, w_r \in E^1(M, A)$, an *iterated path integral of length* r is defined as

$$\int_\lambda w_1 w_2 \cdots w_r = \int_0^1 \int_{t_1}^1 \cdots \int_{t_{r-1}}^1 f_r(t_r) f_{r-1}(t_{r-1}) \cdots f_1(t_1) dt_r dt_{r-1} \cdots dt_1$$

where $f_j(t_j)dt_j = \lambda^* w_j$.

Let $\int w_1 w_2 \cdots w_r$ denote the function

$$\lambda \to \int_\lambda w_1 w_2 \cdots w_r$$

and let $H^0(B_s(M))$ denote the linear span of those functions $\int w_1 w_2 \ldots w_r$, $r \leq s$, which are homotopy functionals (the value of the integral depends only on the homotopy class of the loop). We can now formulate Chen's theorem:

Theorem 3. *(K.T. Chen, cf. [1],[4]) Let M be a smooth manifold and let Δ denote the augmentation ideal of the group ring $\mathbb{R}\pi_1(M)$. Then*

$$H^0(B_s(M)) \cong Hom(\mathbb{R}\pi_1(M)/\Delta^{s+1}, \mathbb{R}).$$

We are now ready to state the analytic description of real dimension subgroups.

Theorem 4. *Let G be a finitely presentable group and let M be a smooth manifold such that $G = \pi_1(M)$. Then*

$$D_n(G) = \left\{ \lambda \in G \ : \ \int_\lambda w_1 \cdots w_r = 0, \ 0 \leq r < n, w_j \in E^1(M) \right\}.$$

In other words, the n-th real dimension subgroup of the fundamental group of a smooth manifold consists of the homotopy classes of loops λ such that every iterated path integral of length less than n vanishes over λ.

Proof. The lemma follows immediately from Chen's theorem and the identity

$$\begin{aligned} D_n(G) &= G \cap (1 + \Delta^n) \\ &= G \cap (1 + \cap \{\ker(\varphi\rho) \ : \ \varphi \in Hom(\mathbb{R}G/\Delta^n, \mathbb{R})\}) \end{aligned}$$

where ρ is the projection $\rho : \mathbb{R}G \mapsto \mathbb{R}G/\Delta^n$. But, this is clear: for if $g-1 \notin \Delta^n$, then $\rho(g-1)$ is a non-zero element of the finite dimensional vector space $\mathbb{R}G/\Delta^n$ and hence cannot be annihilated by every linear functional φ. □

An application of Jennings's theorem now yields:

Corollary 5. *Let $G = \pi_1(M)$ be as above. Then the following are equivalent:*

1. *G is torsion-free nilpotent of class $\leq n$;*
2. *G admits a faithful n-stable representation;*
3. *For every non-identity element $g \in G$ there are 1-forms $w_1, \ldots, w_r$ $(r < n)$ such that $\int_g w_1 \cdots w_r \neq 0$.*

3. Fox Derivatives

Let F be the free group on the generators $\{x_1, \ldots, x_m\}$. Let $\mathbb{R}F$ denote the free group algebra over the field of reals and ε the augmentation map given by the linear extension of $\varepsilon(w) = 1$ $(w \in F)$. The standard way to define the Fox derivatives in the free group algebra is to consider the maps ∂_i with the following properties:

1. $\partial_i(x_j) = \delta_{ij}$
2. $\partial_i(uv) = u\partial_i(v) + \varepsilon(v)\partial_i(u)$, for all $u, v \in F$
3. ∂_i is a linear mapping.

The operators ∂_i are known as Fox derivatives (for detailed treatment of Fox derivatives and their applications, *cf.* [3]). Since $\Delta = \ker \varepsilon$ is a free $\mathbb{R}F$-module on the generators $\{x_1 - 1, \ldots, x_m - 1\}$, it follows that any element of the algebra $\mathbb{R}F$ can be uniquely written as

$$u = \varepsilon(u) + \Sigma_i \partial_i(u)(x_i - 1).$$

Repeated application of the formula above leads to the following identity, valid in $\mathbb{R}F/\Delta^{n+1}$:

$$\begin{aligned} u \quad \equiv \quad & \varepsilon(u) + \\ & + \Sigma_i \; \varepsilon\partial_i(u)(x_i - 1) + \\ & + \Sigma_{i,j} \; \varepsilon\partial_i\partial_j(u)(x_i - 1)(x_j - 1) + \\ & + \Sigma_{i,j,k} \; \varepsilon\partial_i\partial_j\partial_k(u)(x_i - 1)(x_j - 1)(x_k - 1) + \cdots \\ \cdots \quad & + \Sigma_{i_1,\ldots,i_n} \; \varepsilon\partial_{i_1} \cdots \partial_{i_n}(u)(x_{i_1} - 1) \cdots (x_{i_n} - 1) \end{aligned}$$

It is clear that the elements w that satisfy all the relations $\varepsilon\partial_{i_1}\cdots\partial_{i_k}(w) = 0$, for $0 \le k \le n$, are precisely those that belong to the dimension subgroup $D_{n+1}(F)$. The linear functionals $\varepsilon\partial_{i_1}\cdots\partial_{i_k}$, where $0 \le k \le n$, form a linear basis for the dual of $\mathbb{R}F/\Delta^{n+1}$. In fact, this is precisely the dual basis to the basis consisting of the products $(x_{i_1}-1)\cdots(x_{i_k}-1)$, $0 \le k \le n$. In general, if the group is not free, the Fox derivatives cannot be defined in the standard way. However, Chen's iterated path integrals might be an appropriate generalization. In the free group, instead of dealing directly with the Fox derivatives, one could deal only with the functionals $\varepsilon\partial_{i_1}\cdots\partial_{i_k}$. These functionals may be replaced by the integration maps that form the basis dual to a suitable basis of the truncated group-algebra $\mathbb{R}G/\Delta^n$. The advantage is that some of the algebraic properties of the 'augmentized' Fox derivative functionals are reflected in the algebra of iterated path integrals. For instance, since the Fox derivatives have the property (2) in the definition above, it is easily seen that

$$\varepsilon\partial_i\partial_j((u-1)(v-1)) = \varepsilon\partial_i(u)\varepsilon\partial_j(v),$$

which is exactly analogous to the identity

$$\int_{(\lambda-1)(\mu-1)} w_1 w_2 = \int_\lambda w_1 \int_\mu w_2$$

valid for iterated path integrals. Further, a straightforward induction shows that the identity

$$\varepsilon\partial_{i_1}\cdots\partial_{i_k}(uv) = \Sigma_{j=0}^{k}\varepsilon\partial_{i_1}\cdots\partial_{i_j}(u)\cdot\varepsilon\partial_{i_{j+1}}\cdots\partial_{i_k}(v),$$

is an analog of the integral formula (*cf.* [4])

$$\int_{\lambda\mu} w_{i_1}\cdots w_{i_k} = \Sigma_{j=0}^{k}\int_\lambda w_{i_1}\cdots w_{i_j}\cdot\int_\mu w_{i_{j+1}}\cdots w_{i_k}.$$

Consider, now, the matrix-valued 1-form ω whose only non-zero entries $w_1,\ldots,w_r$ are on the first superdiagonal. The identity above states that the mapping

$$T(g) = I + \int_g \omega + \int_g \omega\omega\cdots\int_g \underbrace{\omega\cdots\omega}_{r}$$

defines a unitriangular representation $G \to GL_{r+1}(\mathbb{R})$ provided that the integral $\int\omega$ is a homotopy functional. (A necessary and sufficient condtion for this is the identity $d\omega + \omega \wedge \omega = 0$. *Cf.* [4].) By analogy, for example, if we consider the 2-generated free group and define the matrix valued operator ∂ with the only non-zero entries ∂_1 and ∂_2 on the first superdiagonal, then this operator induces the representation

$$\rho = I + \varepsilon\partial + \varepsilon\partial^2$$

of the free group into the group of 3×3 unitriangular matrices (it is a homomorphism because of the identities given above for Fox derivatives).

Thus, iterated path integrals play a similar role in the representations of the fundamental group as do the Fox derivatives in the representations of the free group. This 'similarity of formalism' is perhaps a sign that some of the results in combinatorial group theory that employ Fox derivatives on the free group could be made feasible for a broader class of groups by studying the full Hopf algebra structure of the group algebra. For example, Chen's iterated path integrals have been successfully applied by Hain and others in the study of monodromy representations of fundamental groups of varieties, in the context of a generalized form of the Riemann-Hilbert problem (*cf.* [4]).

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