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THE LEBESGUE DECOMPOSITION THEOREM FOR GENERALIZED MEASURES

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Abstract

Measures defined on σ-complete lattice and with values in σ-complete lattice ordered semigroup, generalized measures in the sense of Klement and Weber are considered. A Lebesgue decomposition theorem for such generalized measures on lattice with relative complement is proved.

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1. Introduction

Measures (more generally, additive and exhaustive functions) on distributive lattice and values in semigroup were introduced and investigated in the paper [6]. For such measures, a theorem on uniform boundedness and two theorems on pointwise convergence were proved. On the other hand, Pavlakos in [8] and [9] has investigated the measures defined on the ring and σ-ring and with the values in a partially ordered semigroup. Recently, Klement and Weber [5] have introduced generalized measures as measures

defined on σ-complete lattice and with the range as σ-complete lattice ordered commutative semigroup. It turns out that this notion is very useful as a unified approach to several concepts of measures: σ-additive measure, probability measures on fuzzy events [14], possibility measures [15], fuzzy probability measures [4], fuzzy-valued fuzzy measures [5], $\sigma - \perp$ - decomposable measure [11] and [7], measures on fuzzy events [5], $\oplus$ - decomposable measures [5], Stone and W* algebra - valued positive measures [13].
We shall prove in this paper a Lebesgue decomposition type theorem for the generalized measure with an additional supposition on the domain of the generalized function. Namely, the considered σ-complete lattice in the domain have to be with the relative complement property.

2. Lattice with relative complement

We take the following notions and notations from [5].
Let $(\mathbf{L}, \wedge, \vee, \mathbf{0}, \mathbf{1})$ be a σ-complete lattice with smallest and largest element $\mathbf{0}$ and $\mathbf{1}$, respectively, and let $(\mathbf{S}, \Box, \leq, 0, 1)$ be σ-complete, lattice ordered commutative semigroup with the identity 0 and with the smallest and largest element 0 and 1, respectively.

Definition 1. *A mapping $m : \mathbf{L} \rightarrow \mathbf{S}$ satisfying*

$$m(\mathbf{0}) = 0,$$

$$m(x \wedge y) \Box m(x \vee y) = m(x) \Box m(y),$$

$$(x_n)_{n \in N} \uparrow \Rightarrow \sup_{n \in N} m(x_n) = m(\vee_{n \in N} x_n),$$

is called an **S**-*valued measure on* **L** *or generalized measure.*

For examples see [5].
S-valued measure has the following properties:

(**Is**) $x \leq y \;\Rightarrow\; m(x) \leq m(y)$,

($\Box$-**d**) $m(x \vee y) = m(x) \Box m(y)$ for $x \wedge y = \mathbf{0}$,

($\sigma\Box$-**d**) $m(\vee_{n \in N} x_n) = \sup_{k \in N} (\Box_{n=1}^{k} m(x_k))$ for any sequence (x_n) from **L** such that $x_n \wedge y_m = 0$ for $n \neq m$.

Definition 2. *A lattice* $\mathbf{L}$ *is called the lattice with relative complement if for each element* x *from any interval* $[a, b]$ *there exits an element* y *such that*

$$x \vee y = b \qquad \text{and} \qquad x \wedge y = a.$$

The element y *is called the relative complement of the element* x *on the interval* $[a, b]$.

Remark 1.

(i) The complement, in general, is not unique. For example: $\mathbf{L} = \{\mathbf{0}, a, b, c, \mathbf{1}\}$ and the order $\leq$ is defined so that a, b and c are incomparable. Then the elements b and c are complements of a on the interval $[\mathbf{0}, \mathbf{1}]$.

(ii) For distributive lattice with relative complement, the complement is unique for each element. So, for Boolen algebras the complement always exists and it is unique.

(iii) Each lattice $\mathbf{L}$ can be embedded in a lattice $\mathbf{L}'$ with 0 and 1, and in which each element has a complement (on interval $[\mathbf{0}, \mathbf{1}]$), adding no more than three elements to $\mathbf{L}$.

Proposition 1. *Let* $\mathbf{L}$ *be a lattice with a relative complement. If for the generalized measure* m, *for some* $x \in \mathbf{L}$, $m(y) = 0$, *where* y *is a relative complement of* x *on an interval* $[a, b]$, *then* $m(y') = 0$ *for any other relative complement on* $[a, b]$.

Proof. Since

$$m(b) = m(x \vee y) = m(x) + m(y) = m(x)$$

holds, we have

$$m(x) + m(y') = m(x \vee y') = m(x \vee y) = m(x),$$

i.e.,

$$m(x) + m(y') = m(x).$$

Since the neutral element in $\mathbf{S}$ is unique, we obtain $m(y') = 0$.

We shall restrict to relative complements on the interval $[\mathbf{0}, b]$.

3. Lebesgue decomposition

In this section we suppose that $\mathbf{S}$ has the properties:

$$s + \sup A = \sup(s + A) \qquad (s \in \mathbf{S},\ A \subset \mathbf{S}),$$

monotone completeness, i.e., every majorised increasing directed family in $\mathbf{S}$ has a supremum in $\mathbf{S}$, and $\mathbf{S}$ is of countable type. We suppose that the σ-complete lattice $\mathbf{L}$ is a lattice with relative complement.

We shall need the following

Definition 3. *Let m and g be two generalized measures defined on the lattice $\mathbf{L}$ and with values in $\mathbf{S}$. m is called g - absolutely continuous, denoted as $m \ll g$, if $m(x) = 0$ whenever $x \in \mathbf{L}$ with $g(x) = 0$.*
Let m be with the property:
(a) *if for some $x \in \mathbf{L}$, $m(y) = 0$, where y is a relative complement of x on an interval $[\mathbf{0}, b]$, then $m(y') = 0$ for any other relative complement on $[\mathbf{0}, b]$.*
Then m is called g-singular on $\mathbf{L}$, denoted as $m \perp g$, if for every $x \in L$ there exists $y \in L$, $y \leq x$, such that

$$g(y) = m(u) = 0,$$

where u is the relative complement of y on $[\mathbf{0}, x]$.

We shall need the following

Lemma 1. *Let $m_i : \mathbf{L} \to \mathbf{S}$, $i \in I$, be an increasing directed family of generalized measures which satisfy the condition* **(D)***:*

$$m_i(x \wedge (y \vee z)) = m_i((x \wedge y) \vee (x \wedge z)) \quad (x, y, z \in \mathbf{L})$$

or $\mathbf{L}$ is a distributive lattice, pointwise bounded on $\mathbf{L}$, i.e. for each $x \in \mathbf{L}$ there exists an element a such that

$$m_i(x) \leq a \qquad (i \in I).$$

Then, the function

$$m(x) = \sup\{m_i(x) : i \in I\}$$

is a generalized measure on $\mathbf{L}$.

We have now a version of the Lebesgue decomposition theorem

Theorem 1. *Let m and g be two generalized measures such that m satisfies the condition* (**a**) *and g. If m satisfies the condition* (**D**)*:*

$$m(x \wedge (y \vee z)) = m((x \wedge y) \vee (x \wedge z)) \quad (x, y, z \in \mathbf{L})$$

or **L** *is distributive lattice, then there exist generalized measures m_c and m_s such that*

$$m = m_c \Box m_s, \quad m_c \ll g, \quad m_s \perp g.$$

Proof. The subset

$$\mathbf{L_1} = \{y \in \mathbf{L} : g(y) = 0\}$$

is a σ-complete sublattice of the lattice **L**. For the restriction of the generalized measure m on $\mathbf{L_1}$ we introduce

$$m_s(x) = \sup_{y \in \mathbf{L}_1} m(x \vee y).$$

let

$$\mathbf{L_2} = \{y \in \mathbf{L} : m_s(y) = 0\}.$$

Then we define

$$m_c(x) = \sup_{z \in \mathbf{L}_2} m(x \vee z) \quad (x \in \mathbf{L}).$$

We can prove Using Lemma 1 that m_c and m_s are generalized measures, and that there exist $y \in \mathbf{L_1}$ and $z \in \mathbf{L_2}$ such that for all $x \in \mathbf{L}$

$$m_s(x) = m(x \vee y) = m_s(x \vee y)$$

and

$$m_c(x) = m(x \vee z) = m_c(x \vee z).$$

Using the last two equalities it is easy to check that thus constructed m_s and m_c satisfy the desired conditions.

We have by [2]

Definition 4. *A function $m : \mathbf{L} \to G$, where $(G, +)$ is an Abelian lattice ordered group, is called distributive if it satisfies the condition*

$$m(x \vee y \vee z) = m(x) + m(y) + m(z) - m(x \wedge y) - m(x \wedge z) -$$

$$m(y \wedge z) + m(x \wedge y \wedge z) \quad (x, y, z \in \mathbf{L}).$$

Remark 2. A function m is distributive iff it is modular and satisfies the condition **(D)** from Theorem 1.

Theorem 2. *Let m and g be two generalized measures. If m is distributive, then there exist the distributive generalized measures m_c and m_s such that*

$$m = m_c \Box m_s, \quad m_c \ll g, \quad m_s \perp g.$$

The proof is analogous to the proof of Theorem 1 using Proposition 1.

Remark 3.

(i) If **S** is a lattice ordered group, then for a distributive generalized measure we can assume that the lattice is distributive, without loosing any information. Namely, we can do this by the results from [9],[10], factoring a congruence.

(ii) For an orthomodular lattice **L** and a topological group G the Lebesgue decomposition theorems were proved in papers [6],[8].

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