

NON-NULL HELICOIDAL SURFACES AS NON-NULL BONNET SURFACES

Abdullah İnalçık and Soley Ersoy

Abstract. In this study, we obtain an equivalent of the Codazzi-Mainardi equations for spacelike and timelike surfaces in three dimensional Lorentz space \mathbb{R}_1^3 . Also, we find necessary and sufficient conditions for spacelike and timelike helicoidal surfaces with non-null axis to be Bonnet surfaces.

1. Introduction

Surfaces which admit a one-parameter family of isometric deformations preserving the mean curvature are called Bonnet surfaces. In 1867, Bonnet proved that any surface with constant mean curvature in \mathbb{R}^3 (which is not totally umbilical) is a Bonnet surface (see [2]). Lawson extended Bonnet's results to any surface with constant mean curvature in Riemannian 3-manifold of constant curvature c , $\mathbb{R}^3(c)$, [10]. Also, it is proved that any Bonnet surface of non-constant mean curvature depends on six arbitrary constants. The similar problems for surfaces in space form $\mathbb{R}^3(c)$ and for spacelike surfaces in indefinite space form $\mathbb{R}^3(c)$ were studied by Chen and Li in [4]. A geometric characterization of helicoidal surfaces of constant mean curvature, the helicoidal surfaces as Bonnet surfaces and the tangent developable surfaces as Bonnet surfaces were investigated by Roussos in [15, 16] and [17], respectively. More recently, timelike surfaces in Lorentzian space forms which admit a one-parameter family of isometric deformations preserving the mean curvature were studied by Fujioka and Inoguchi in [8]. As it is known, a helicoidal surface is a kind of some ruled surfaces and rotation surfaces and there are remarkable studies on helicoidal surfaces in \mathbb{R}_1^3 [1, 6, 7, 9, 11].

In these regards, we have investigated spacelike and timelike helicoidal surfaces with non-null axis as Bonnet surfaces and obtained the necessary and sufficient conditions for the existence of these surfaces.

2010 Mathematics Subject Classification: 53A10, 53B30

Keywords and phrases: Bonnet surface; helicoidal surface; Lorentz space form.

2. Preliminaries

Let \mathbb{R}_1^3 be a Lorentzian 3-space with the nondegenerate metric tensor

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

where $\{x_1, x_2, x_3\}$ is a system of the canonical coordinates in \mathbb{R}^3 [13].

Firstly, we introduce the basic knowledge and notions about the geometry of surfaces in Lorentzian spaces. A surfaces M in \mathbb{R}^3 is given by the immersion $X : I \subset \mathbb{R}^2 \rightarrow M \subset \mathbb{R}_1^3$. A surface is said to be spacelike (resp. timelike) if the induced metric on M is positive definite (resp. indefinite). Let $\{e_1, e_2, e_3\}$ be a local orthonormal frame on M , such that e_3 is a unit normal vector field on M . The unit normal vector field e_3 can be regarded as a map $e_3 : M \rightarrow H_+^2$ if M is spacelike and as a map $e_3 : M \rightarrow S_1^2$ if M is timelike. Here $H_+^2 = \{x \in \mathbb{R}_1^3 : \langle x, x \rangle = -1, x_1 > 0\}$ is the hyperbolic space and $S_1^2 = \{x \in \mathbb{R}_1^3 : \langle x, x \rangle = 1\}$ is the de-Sitter space. Also, $\{e_1, e_2\}$ comprise an orthonormal basis of the tangent space of M at x . Let μ_i be the coframe of e_i which is defined by $\mu_i(e_j) = \langle e_i, e_j \rangle = \varepsilon_i \delta_{ij}$, $1 \leq i, j \leq 3$. Then

$$dx = \varepsilon_1 \mu_1 e_1 + \varepsilon_2 \mu_2 e_2$$

where $\varepsilon_i = \langle e_i, e_i \rangle = \pm 1$. The connection forms μ_{ij} are defined by

$$de_i = \sum_{j=1}^3 \varepsilon_j \mu_{ij} e_j \tag{2.1}$$

which satisfy $\mu_{ij} + \mu_{ji} = 0$. Then the structure equations become

$$d\mu_i = \sum_{j=1}^3 \varepsilon_j \mu_{ij} \wedge \mu_{ji} \tag{2.2}$$

and

$$d\mu_{ij} = \sum_{k=1}^3 \varepsilon_k \mu_{ik} \wedge \mu_{kj}.$$

Since μ_3 is a zero form on M , the exterior derivative of μ_3 gives

$$\varepsilon_1 \mu_1 \wedge \mu_{13} + \varepsilon_2 \mu_2 \wedge \mu_{23} = 0. \tag{2.3}$$

By the equation (2.3) and the Cartan Lemma, there exists a symmetric tensor h_{ij} such that

$$\mu_{i3} = \sum_{j=1}^3 \varepsilon_j h_{ij} \mu_j, \quad h_{ij} = h_{ji}. \tag{2.4}$$

The Gauss equation and mean curvature of M are defined as

$$d\mu_{12} = \varepsilon_3 K \mu_1 \wedge \mu_2 \text{ and } H = \frac{1}{2} \varepsilon_3 (\varepsilon_1 h_{11} + \varepsilon_2 h_{22})$$

where K is the Gauss curvature of M [13, 14].

A helicoidal motion group is a non-trivial one-parameter group of rigid motions of \mathbb{R}_1^3 and any element of such a group is called a helicoidal motion of \mathbb{R}_1^3 . Here, trivial cases are pure translation groups. Every helicoidal motion group is completely determined by an axis l and a pitch h . Depending on the line l being

spacelike, timelike or null, there are three types of motion. If the axis is spacelike (resp. timelike), then l is transformed to the x_3 -axis or x_2 -axis (resp. x_1 -axis). Therefore we can always suppose that l is the x_3 -axis (resp. x_1 -axis) if l is spacelike (resp. timelike). If the axis l is null, then we may assume that l is the line spanned by $(1, 0, 1)$. If $G_{l,h} = \{\phi_t : t \in \mathbb{R}\}$ denotes the helicoidal motion group with axis l and pitch h , for $p = (a, b, c) \in \mathbb{R}_1^3$, the image of p under any helicoidal motion of one parameter group $G_{l,h}$ is (see [6, 7, 11, 12]).

$$\begin{aligned} \phi_t(p) &= \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + h \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \text{ if } l \text{ is spacelike,} \\ \phi_t(p) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + h \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \text{ if } l \text{ is timelike,} \\ \phi_t(p) &= \begin{pmatrix} 1 - \frac{t^2}{2} & t & \frac{t^2}{2} \\ -t & 1 & t \\ -\frac{t^2}{2} & t & 1 + \frac{t^2}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + h \begin{pmatrix} \frac{t^3}{3} - t \\ t^2 \\ \frac{t^3}{3} + t \end{pmatrix} \text{ if } l \text{ is null,} \end{aligned}$$

where $t \in \mathbb{R}$. If we take $h = 0$, we obtain the rotations group related to axis l .

A helicoidal surface in Lorentzian space \mathbb{R}_1^3 is a surface invariant by uniparametric group $G_{l,h}$ of helicoidal motion and it is given by immersion $X(s, t) : I \subset \mathbb{R}^2 \rightarrow M \subset \mathbb{R}_1^3$. Here (s, t) is parameter of helicoidal surface. The images of the t -curves are the trajectories of the helicoidal motions, while the s -curves are their orthogonal trajectories parameterized by arc length in the induced metric such a local parameterizations will be called natural parametrization of the helicoidal surface. Notice that the first fundamental form in such parameters can be written as $d\theta^2 = \varepsilon_1 ds^2 + \varepsilon_2 U^2 dt^2$, where $\varepsilon_1 = \langle X_s, X_s \rangle$, $\varepsilon_2 U^2 = \langle X_t, X_t \rangle$ and $U = U(s)$ is a function depending only s . If a helicoidal surface invariant under $G_{l,h}$ meets the axis l at some point and if l is non-null, then the whole axis is contained in the surface. In this paper, we have omitted the case of helicoidal surfaces with null axis.

3. Spacelike helicoidal surfaces as Bonnet surface

Let M be a spacelike surface with unit normal e_3 which is a map from M to H_+^2 and $\{e_1, e_2\}$ are spacelike tangent vectors with corresponding coframe $\{\mu_1, \mu_2\}$. The Weingarten map of M is diagonalizable over \mathbb{R} if and only if $H^2 + K \geq 0$. If $H^2 + K = 0$, M is an umbilic surface. So, we will assume $H^2 + K > 0$ and $J = \sqrt{H^2 + K}$. In this manner, let $\{x; e'_1, e'_2, e'_3 = e_3\}$ be a another typical field of orthonormal principal frame on M with the principal coframe $\{\omega_1, \omega_2\}$ and the corresponding connection forms $\{\omega_{ij}\}$, $1 \leq i, j \leq 3$. Then the first fundamental form of M is

$$I = (\mu_1)^2 + (\mu_2)^2 = (\omega_1)^2 + (\omega_2)^2.$$

There exists a function ϕ on M , such that

$$e'_1 = \cos \phi e_1 + \sin \phi e_2, \quad e'_2 = -\sin \phi e_1 + \cos \phi e_2$$

and

$$\omega_1 = \cos \phi \mu_1 + \sin \phi \mu_2, \quad \omega_2 = -\sin \phi \mu_1 + \cos \phi \mu_2. \tag{3.1}$$

Since e'_1, e'_2 are spacelike principal vector fields, The tensor $h_{12} = h_{21}$ vanishes. Also, by denoting the principal curvatures $h_{11} = k_1$ and $h_{22} = k_2$, we can say

$$\omega_{13} = k_1 \omega_1, \quad \omega_{23} = k_2 \omega_2. \tag{3.2}$$

Obviously, the Gauss and mean curvature are

$$H = -\frac{k_1 + k_2}{2}, \quad K = -k_1 k_2,$$

respectively. Also, we get

$$J = \frac{k_1 - k_2}{2}.$$

THEOREM 3.1. *Let M be a spacelike surface with no umbilic points. Then the Codazzi-Mainardi equations are equal to*

$$\begin{aligned} dH &= H_1 \mu_1 + H_2 \mu_2, \\ d\sigma &= \sin \sigma \left(\frac{H_1}{J} \mu_1 - \frac{H_2}{J} \mu_2 \right) - \cos \sigma \left(\frac{H_2}{J} \mu_1 + \frac{H_1}{J} \mu_2 \right) - *d \ln J - 2\mu_{12}, \end{aligned} \tag{3.3}$$

where $*$ is the Hodge operator whose action on the 1-form is described by

$$*\mu_1 = \mu_2, \quad *\mu_2 = -\mu_1, \quad (*)^2 = -1.$$

Proof. The exterior differentiation of the principal coframe (3.1) gives

$$d\omega_1 = \omega_2 \wedge (-d\phi - \mu_{12}), \quad d\omega_2 = \omega_1 \wedge (d\phi + \mu_{12}).$$

By the structure equations (2.2), the connection form associated to the principal coframe is $\omega_{12} = d\phi + \mu_{12}$, so that

$$d\phi = \omega_{12} - \mu_{12}. \tag{3.4}$$

With the smooth functions p and q , we can write $\omega_{12} = p\omega_1 + q\omega_2$ where p and q are determined uniquely by the structure equations (2.2). Thus, by the aid of the equations (2.2) and (3.2), the Codazzi-Mainardi equations can be reduced to

$$(dk_1 - p(k_1 - k_2)\omega_2) \wedge \omega_1 = 0, \quad (dk_2 - q(k_1 - k_2)\omega_1) \wedge \omega_2 = 0.$$

On the other hand, if we set

$$dH = -\frac{dk_1 + dk_2}{2} = u\omega_1 + v\omega_2,$$

we get

$$dk_1 = (-2Jq - 2u)\omega_1 + 2Jp\omega_2 = 0, \quad dk_2 = 2Jq\omega_1 + (-2v - 2Jp)\omega_2 = 0.$$

Dividing $dk_1 - dk_2$ by $2J$ gives us

$$d \ln J = -\frac{u}{J}\omega_1 + \frac{v}{J}\omega_2 + 2(-q\omega_1 + p\omega_2).$$

If we apply the Hodge operator to the last equation, we get

$$\omega_{12} = -\frac{1}{2J}(v\omega_1 + u\omega_2) - \frac{1}{2} * d \ln J. \quad (3.5)$$

Since $dH = H_1\mu_1 + H_2\mu_2 = u\omega_1 + v\omega_2$, the relationships between u , v and H_1 , H_2 are

$$u = H_1 \cos \phi + H_2 \sin \phi, \quad v = -H_1 \sin \phi + H_2 \cos \phi.$$

By the equations (3.4), (3.5) and the last equation, we find

$$d2\phi = \sin 2\phi \left(\frac{H_1}{J} \mu_1 - \frac{H_2}{J} \mu_2 \right) - \cos 2\phi \left(\frac{H_2}{J} \mu_1 + \frac{H_1}{J} \mu_2 \right) - *d \ln J - 2\mu_{12}.$$

Finally, substituting $\sigma = 2\phi$ into the above equation completes the proof. ■

Now, let us establish a principal coframe of a spacelike helicoidal surface with non-null axis such that

$$\mu_1 = ds, \quad \mu_2 = q(s) dt \quad (3.6)$$

where s is the arc length of curves orthogonal to orbits measured from a fixed orbit and t is time along orbits from a fixed $t = t_0$. The t – constant curves are carried along the orbits by helicoidal motions and foliate the surface. An orthonormal frame $\{e_1, e_2\}$ is determined along these coordinate curves with e_2 tangent to the orbits. By the equations (2.2) and (3.6), it is easily seen that

$$\mu_{12} = \frac{q'(s)}{q(s)} \mu_2 = \eta(s) \mu_2.$$

Hence the μ_1 -curves are geodesic curves and the μ_2 -curves have the geodesic curvature equal to

$$\eta(s) = \frac{d}{ds} \ln (|q(s)|).$$

Moreover, $dJ = \frac{dk_1 - dk_2}{2}$, so we can put $dJ = -J_1\mu_1 + J_2\mu_2$ and we obtain

$$d \ln J = -\frac{J_1}{J} \mu_1 + \frac{J_2}{J} \mu_2.$$

Along the each orbit, k_1 , k_2 , μ and ϕ depend on s , then $H_2 = J_2 = 0$. Hence the relation (3.3) becomes

$$d\sigma = \sin \sigma \left(\frac{H_1}{J} \right) \mu_1 - \cos \sigma \left(\frac{H_1}{J} \right) \mu_2 + \frac{J_1}{J} \mu_2 - 2\eta(s) \mu_2.$$

Since $\sigma = \sigma(s)$, this implies

$$\frac{d\sigma}{ds} = \sin \sigma \left(\frac{dH}{ds} \right) \quad (3.7)$$

and

$$\eta(s) = \frac{1}{2} \left(\frac{dJ}{ds} - \cos 2\phi \frac{dH}{ds} \right).$$

THEOREM 3.2. *A spacelike helicoidal surface M with $H^2 + K > 0$ in \mathbb{R}_1^3 has a one parameter family of non-trivial isometric deformation preserving the mean*

curvature i.e., M is a spacelike Bonnet surfaces if and only if the following relation is satisfied

$$\frac{d}{ds} \left(\frac{dH}{J} \right) + \cos \sigma(s) \left(\frac{dH}{J} \right)^2 + \left(\frac{dH}{J} \right) \frac{d \ln(|q(s)|)}{ds} = 0, \tag{3.8}$$

where $H = H(s)$ is the non-constant mean curvature.

Proof. The criterion of Chern given in [3] for the existence of Bonnet surface is $d\alpha_1 = 0$ and $d\alpha_2 = \alpha_1 \wedge \alpha_2$, where $\alpha_1 = \frac{u}{J}\omega_1 - \frac{v}{J}\omega_2$, $\alpha_2 = \frac{v}{J}\omega_1 + \frac{u}{J}\omega_2$ and $dH = u\omega_1 + v\omega_2$. By substituting u and v in the equations $\omega_1 = \cos \sigma(s) ds + \sin \sigma(s) q(s) dt$, $\omega_2 = -\sin \sigma(s) ds + \cos \sigma(s) q(s) dt$, we get

$$\begin{aligned} \alpha_1 &= \left(\frac{H_1}{J} \cos 2\phi + \frac{H_2}{J} \sin 2\phi \right) ds + \left(\frac{H_1}{J} \sin 2\phi - \frac{H_2}{J} \cos 2\phi \right) q(s) dt, \\ \alpha_2 &= \left(-\frac{H_1}{J} \sin 2\phi + \frac{H_2}{J} \cos 2\phi \right) ds + \left(\frac{H_1}{J} \cos 2\phi + \frac{H_2}{J} \sin 2\phi \right) q(s) dt. \end{aligned}$$

By substituting the equation (3.7) into the exterior derivative of these last two equations, it is seen that Chern’s criterion is verified if and only if the relation (3.8) satisfied. This completes the proof. ■

By the equation (3.7), it is easily seen that

$$\left(\frac{dH}{J} \right) = \frac{1}{\sin \sigma(s)} \frac{d\sigma}{ds}.$$

If we differentiate the last equation with respect to s , we find

$$\frac{d}{ds} \left(\frac{dH}{J} \right) = \frac{d}{ds} \left(\frac{d\sigma}{ds} \right) \frac{1}{\sin \sigma(s)} - \left(\frac{d\sigma}{ds} \right)^2 \frac{\cos \sigma(s)}{\sin^2 \sigma(s)}.$$

By comparing the last equation with (3.8), we get

$$\frac{d}{ds} \left(\frac{d\sigma}{ds} \right) \frac{1}{\sin \sigma(s)} + \left(\frac{d\sigma}{ds} \right) \frac{1}{\sin \sigma(s)} \frac{q'(s)}{q(s)} = 0.$$

The solution of this ordinary differential equation gives us the following remark.

REMARK 3.1. The ordinary differential equation (3.8) is equivalent to

$$q(s) \left(\frac{dH}{J} \right) \sin \sigma(s) = \text{constant}$$

with non-constant mean curvature $H = H(s)$.

4. Timelike helicoidal surface as Bonnet surface

Let M be a timelike surface with unit normal e_3 which is a map from M to de-Sitter space S_1^2 . We can choose a local orthonormal frame field $\{x; e_1, e_2, e_3\}$ on M , such that e_1 is a spacelike tangent vector and e_2 is a timelike tangent vector at x .

Obviously, the normal vector e_3 is spacelike at x . If we take into consideration another field of orthonormal principal frame $\{x; e'_1, e'_2, e'_3 = e_3\}$ with principal coframe $\{\omega_1, \omega_2\}$ and corresponding to the connection forms $\{\omega_{ij}\}$, $1 \leq i, j \leq 3$, the first fundamental form of M is

$$I = (\mu_1)^2 - (\mu_2)^2 = (\omega_1)^2 - (\omega_2)^2. \quad (4.1)$$

The function ϕ exists on M as follows

$$e'_1 = \cosh \phi e_1 + \sinh \phi e_2, \quad e'_2 = \sinh \phi e_1 + \cosh \phi e_2$$

and

$$\omega_1 = \cosh \phi \mu_1 + \sinh \phi \mu_2, \quad \omega_2 = \sinh \phi \mu_1 + \cosh \phi \mu_2. \quad (4.2)$$

The Weingarten map has real eigenvector if and only if $H^2 - K \geq 0$. So, we suppose that $H^2 - K > 0$, that is, M has no umbilic points. Since e'_1, e'_2 are principal vector fields, we can give

$$\omega_{13} = k_1 \omega_1, \quad \omega_{23} = -k_2 \omega_2, \quad h_{12} = h_{21} = 0,$$

where the principal curvatures are $h_{11} = k_1$ and $h_{22} = k_2$ in (2.4). The mean and Gauss curvature of M are

$$H = \frac{k_1 - k_2}{2}, \quad K = -k_1 k_2,$$

respectively. If we define $J = \sqrt{H^2 - K}$, $J = \frac{k_1 + k_2}{2} > 0$ is obtained.

In these regards, we can give the following theorem related to the Codazzi-Mainardi equations for timelike surface.

THEOREM 4.1. *The Codazzi-Mainardi equations for timelike surface with $H^2 - K > 0$ are*

$$dH = H_1 \mu_1 - H_2 \mu_2$$

and

$$d\sigma = -\sinh \sigma \left(\frac{H_1}{J} \mu_1 + \frac{H_2}{J} \mu_2 \right) - \cosh \sigma \left(\frac{H_2}{J} \mu_1 + \frac{H_1}{J} \mu_2 \right) - *d \ln J + 2\mu_{12}, \quad (4.3)$$

where $*$ is the Hodge operator such that

$$*\mu_1 = \mu_2, \quad *\mu_2 = \mu_1, \quad (*)^2 = 1.$$

Proof. By the equation (4.2), we get

$$d\omega_1 = -\omega_2 \wedge (d\phi - \mu_{12}), \quad d\omega_2 = \omega_1 \wedge (-d\phi + \mu_{12}).$$

Thus, the connection form associated to the principal coframe is $\omega_{12} = -d\phi + \mu_{12}$. This implies

$$d\phi = -\omega_{12} + \mu_{12}. \quad (4.4)$$

By the equation (2.2), the Codazzi-Mainardi equations for M are

$$d\omega_{13} = -\omega_{12} \wedge \omega_{23}, \quad d\omega_{23} = \omega_{21} \wedge \omega_{12}$$

and they can be reduced to

$$(dk_1 + p(k_1 + k_2)\omega_2) \wedge \omega_1 = 0, \quad (-dk_2 - q(k_1 + k_2)\omega_1) \wedge \omega_2 = 0,$$

where $\omega_{12} = p\omega_1 + q\omega_2$. Considering $\frac{dk_1 - dk_2}{2} = u\omega_1 - v\omega_2$, we get

$$dk_1 = (2u - 2Jq)\omega_1 - 2Jp\omega_2 = 0, \quad dk_2 = -2Jq\omega_1 + (2v - 2Jp)\omega_2 = 0.$$

If we divide the addition of the last two relation by $2J$, we obtain

$$d \ln J = \frac{u}{J}\omega_1 + \frac{v}{J}\omega_2 + 2(-q\omega_1 - p\omega_2).$$

By use of Hodge operator, we have

$$\omega_{12} = \frac{1}{2J}(v\omega_1 + u\omega_2) + \frac{1}{2} * d \ln J. \tag{4.5}$$

If we compare $dH = u\omega_1 - v\omega_2$ and $dH = H_1\mu_1 - H_2\mu_2$ with ω_1, ω_2 expressed in terms of μ_1 and μ_2 , we get

$$u = H_1 \cosh \phi + H_2 \sinh \phi, \quad v = H_1 \sinh \phi + H_2 \cosh \phi.$$

The equations (4.4), (4.5) and the last equations give us

$$d2\phi = -\sinh 2\phi \left(\frac{H_1}{J}\mu_1 + \frac{H_2}{J}\mu_2 \right) - \cosh 2\phi \left(\frac{H_2}{J}\mu_1 + \frac{H_1}{J}\mu_2 \right) - *d \ln J + 2\mu_{12}.$$

If we take $\sigma = 2\phi$, then we complete the proof. ■

For a timelike helicoidal surface with non-null axis, there is the relation

$$\mu_{12} = \frac{q'(s)}{q(s)}\mu_2 = \eta(s)\mu_2,$$

where the orthonormal corresponding coframe is defined by

$$\mu_1 = ds, \quad \mu_2 = q(s) dt.$$

Since $dJ = \frac{dk_1 + dk_2}{2}$, we can consider $dJ = -J_1\mu_1 - J_2\mu_2$, and then

$$*d \ln J = \frac{J_2}{J}\mu_1 + \frac{J_1}{J}\mu_2.$$

By the fact that k_1, k_2, μ and ϕ depends on s we can say that $H_2 = J_2 = 0$. So we can rewrite the equation (4.3) as follows

$$d\sigma = -\sinh \sigma \left(\frac{H_1}{J} \right) \mu_1 - \cosh \sigma \left(\frac{H_1}{J} \right) \mu_2 - \frac{J_1}{J}\mu_2 - 2\eta(s)\mu_2.$$

By the last equation,

$$\frac{d\sigma}{ds} = -\sinh 2\phi \left(\frac{dH}{ds} \right), \quad \eta(s) = \frac{1}{2} \left(\cosh 2\phi \frac{dH}{ds} + \frac{dJ}{ds} \right)$$

are obtained.

THEOREM 4.2. *Let M be a timelike helicoidal surface with $H^2 - K > 0$ in \mathbb{R}_1^3 . Then M is a Bonnet surface if and only if there exists a relation as follows*

$$\frac{d}{ds} \left(\frac{dH}{ds} \right) - \cosh \sigma(s) \left(\frac{dH}{ds} \right)^2 + \left(\frac{dH}{ds} \right) \frac{d \ln (|q(s)|)}{ds} = 0, \tag{4.6}$$

where H is a non-constant mean curvature.

Proof. In [5], it is proved that every timelike constant mean curvature surface with no umbilic points is a timelike Bonnet surface and also, it is mentioned that M with non-constant mean curvature is timelike Bonnet surface if and only if

$$d\alpha_1 = 0, \quad d\alpha_2 = \alpha_1 \wedge \alpha_2,$$

where $\alpha_1 = \frac{u}{J}\omega_1 + \frac{v}{J}\omega_2$, $\alpha_2 = \frac{v}{J}\omega_1 + \frac{u}{J}\omega_2$ and $dH = u\omega_1 - v\omega_2$. If we substitute the relations

$$\omega_1 = \cosh \sigma(s) ds + \sinh \sigma(s) q(s) dt, \quad \omega_2 = \sinh \sigma(s) ds + \cosh \sigma(s) q(s) dt$$

and (4.1) into the exterior derivative of α_1 and α_2 , we verify the necessary and sufficient condition of a timelike helicoidal surface being a timelike Bonnet surface. The solution of the differential equation (4.6) can be obtained in a similar manner. ■

REMARK 4.1. The ordinary differential equation (4.6) is equivalent to

$$q(s) \left(\frac{dH}{ds} \right) \sinh 2\phi = \text{constant},$$

where H is non-constant.

ACKNOWLEDGEMENT. The authors are very grateful to Prof. Dr. Rafael López for his comments and suggestions to improve the presentation of this paper.

REFERENCES

- [1] Ch. Baba-Hamed, M. Bekkar, *Helicoidal surfaces in the three-dimensional Lorentz-Minkowski space satisfying $\Delta r_i = \lambda_i r_i$* , Int. J. Contemp. Math. Sci., **4** 5–8 (2009), 311–327.
- [2] O. Bonnet, *Mémoire sur la théorie des surfaces applicables*, J. Éc. Polyt., **42** (1867), 72–92.
- [3] S. S. Chern, *Deformation of surfaces preserving principal curvature*, Differ. Geometry and Complex Analysis, H. E. Rauch Memorial Volume, Springer-Verlag (1985), pp. 155–163.
- [4] W. Chen, H. Li, *Bonnet surfaces and isothermic surfaces*, Results Math, **31**, 1–2 (1997), 40–52.
- [5] W. Chen, H. Li, *On the classification of the timelike Bonnet surfaces*, in: *Geometry and Topology of Submanifolds 10, Differential Geometry in Honor of Professor S. S. Chern*, Peking University, China and TU Berlin, Germany, (1999), 18–31.
- [6] M. Choi, Y. H. Kim, H. Liu, D. W. Yoon, *Helicoidal surfaces and their Gauss map in Minkowski 3-space*, Bull. Korean Math. Soc., **47**, 4 (2010), 859–881.
- [7] F. Dillen, W. Kuhnel, *Ruled Weingarten surface in Minkowski 3-space*, Manuscripta Math., **98**, 3 (1999), 307–320.
- [8] A. Fujioka, J. Inouguchi, *Timelike Bonnet surfaces in Lorentzian space forms*, Differential Geom. Appl., **18**, 1 (2003), 103–111.
- [9] Z. H. Hou, F. Ji, *Helicoidal surface under the cubic screw motion in Minkowski 3-space*, J. Math. Anal. Appl., **318**, 2 (2006), 634–647.
- [10] H. Lawson, Jr. Blaine, *Complete minimal surface in S^3* , Ann. of Math., **92** (1970), 335–374.
- [11] R. Lopez, E. Demir, *Helicoidal surfaces in Minkowski space with constant mean curvature and constant Gauss curvature*, Cent. Eur. J. Math., **12**, 9 (2014), 1349–1361.
- [12] P. Mira, J. A. Pastor, *Helicoidal maximal surfaces in Lorentz-Minkowski space*, Monatsh. Math., **140**, 4 (2003), 315–334.
- [13] B. O’Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
- [14] J. Park, *Lorentzian surfaces with constant curvatures and transformations in the 3-dimensional Lorentzian space*, J. Korean Math. Soc., **45**, 1 (2008), 41–61.

- [15] I. M. Roussos, *A geometric characterization of helicoidal surfaces of constant mean curvature*, Publ. Inst. Math. (Beograd) (N.S.) **43(57)** (1988), 137-142.
- [16] I. M. Roussos, *The helicoidal surfaces as Bonnet surfaces*, Tohoku Math. J. (2), **40**, 3 (1988), 485-490.
- [17] I. M. Roussos, *Tangential developable surfaces as Bonnet surfaces*, Acta Math. Sin., **15**, 2 (1999), 269-276.

(received 04.08.2015; in revised form 16.03.2016; available online 21.03.2016)

A.I.: Artvin Coruh University, Faculty of Arts and Sciences, Department of Mathematics, Artvin, Turkey

E-mail: abdullahinalcik@artvin.edu.tr

S.E.: Sakarya University, Faculty of Arts and Sciences, Department of Mathematics, Sakarya, 54187 Turkey

E-mail: sersoy@sakarya.edu.tr