МАТЕМАТИЧКИ ВЕСНИК
65, 3 (2013), 285–298
September 2013

оригинални научни рад
research paper

CONVERGENCE THEOREMS OF TWO-STEP IMPLICIT ITERATIVE PROCESS WITH ERRORS FOR A FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

Gurucharan Singh Saluja

Abstract. The objective of this paper is to study the weak and strong convergence of two-step implicit iteration process with errors to a common fixed point for a finite family of asymptotically nonexpansive mappings in Banach spaces. The results presented in this paper extend and improve the corresponding results of Chang and Cho (2003), Xu and Ori (2001), Zhou and Chang (2002) and Gu and Lu (2006).

1. Introduction

Let C be a nonempty subset of a real Banach space E. Let $T\colon C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a mapping $T\colon C \to C$ is said to be:

(1) asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \tag{1.1}$$

for all $x, y \in C$ and $n \ge 1$;

(2) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \tag{1.2}$$

for all $x, y \in C$ and $n \ge 1$;

(3) semi-compact if any bounded sequence $\{x_n\}$ in C with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some x^* in C;

(4) demiclosed at the origin, if for each sequence $\{x_n\}$ in C the condition $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$.

2010 Mathematics Subject Classification: 47H09, 47H10

Keywords and phrases: Asymptotically nonexpansive mapping; implicit iteration process with errors; common fixed point; strong convergence; weak convergence.

Recall that E is said to satisfy the *Opial's condition* [9] if for each sequence $\{x_n\}$ in E weakly convergent to a point x and for all $y \neq x$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

The examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [9].

Let E be a Hilbert space, let K be a nonempty closed convex subset of E and let $\{T_1, T_2, \ldots, T_N\}\colon K \to K$ be N nonexpansive mappings. In 2001, Xu and Ori [15] introduced the following implicit iteration process $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{n \ (\mathrm{mod}\ N)} x_n, \qquad n \geq 1, \tag{1.3}$$

where $x_0 \in K$ is an initial point, $\{\alpha_n\}_{n\geq 1}$ is a real sequence in $(0,1)$ and proved the weakly convergence of the sequence $\{x_n\}$ defined by (1.3) to a common fixed point $p \in F = \bigcap_{i=1}^N F(T_i)$.

Recently, convergence problems of an implicit (or non-implicit) iterative process to a common fixed point for a finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been considered by several authors (see, e.g., [1–5, 7–8, 10–16])¿

Very recently, Gu and Lu [6] introduced the following implicit iterative sequence $\{x_n\}$ with errors defined by

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 T_1(\hat{\alpha_1} x_0 + \hat{\beta_1} T_1 x_1 + \hat{\gamma_1} v_1) + \gamma_1 u_1,\\
x_2 &= \alpha_2 x_1 + \beta_2 T_2(\hat{\alpha_2} x_1 + \hat{\beta_2} T_2 x_2 + \hat{\gamma_2} v_2) + \gamma_2 u_2,\\
&\ \ \vdots\\
x_N &= \alpha_N x_{N-1} + \beta_N T_N(\hat{\alpha_N} x_{N-1} + \hat{\beta_N} T_N x_N + \hat{\gamma_N} v_N) + \gamma_N u_N,\\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1(\alpha_{\hat{N+1}} x_N + \beta_{\hat{N+1}} T_1 x_{N+1} + \gamma_{\hat{N+1}} v_{N+1}) + \gamma_{N+1} u_{N+1},\\
&\ \ \vdots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N(\alpha_{\hat{2N}} x_{2N-1} + \beta_{\hat{2N}} T_N x_{2N} + \gamma_{\hat{2N}} v_{2N}) + \gamma_{2N} u_{2N},\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1(\alpha_{2\hat{N+1}} x_{2N} + \beta_{2\hat{N+1}} T_1 x_{2N+1} + \gamma_{2\hat{N+1}} v_{2N+1})\\
&\quad + \gamma_{2N+1} u_{2N+1}\\
&\ \ \vdots
\end{aligned} \tag{1.4}$$

for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N\colon K \to K$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ are six sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$ for all $n \geq 1$, x_0 is a given point in K, as well as $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K, which can be written in the following compact form:

$$\begin{aligned}
x_n &= \alpha_n x_{n-1} + \beta_n T_{n \ (\mathrm{mod}\ N)} y_n + \gamma_n u_n,\\
y_n &= \hat{\alpha_n} x_{n-1} + \hat{\beta_n} T_{n \ (\mathrm{mod}\ N)} x_n + \hat{\gamma_n} v_n, \qquad n \geq 1.
\end{aligned} \tag{1.5}$$

Especially, if $\{T_i\}_{i=1}^N : K \to K$ are N nonexpansive mappings, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in [0,1] and x_0 is a given point in K, then the sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + \beta_n T_{n \pmod N} x_{n-1} + \gamma_n u_n, \qquad n \geq 1, \tag{1.6}$$

is called the explicit iterative sequence for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ and they proved weak and strong convergence of iterative sequence $\{x_n\}$ defined by (1.5) and (1.6) in Banach spaces.

Inspired and motivated by Gu and Lu [6] and many others, we introduce the following implicit iterative sequence $\{x_n\}$ with errors defined by

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T^n_{n \pmod N} y_n + \gamma_n u_n, \\ y_n &= \hat{\alpha_n} x_{n-1} + \hat{\beta}_n T^n_{n \pmod N} x_n + \hat{\gamma_n} v_n, \qquad n \geq 1, \end{aligned} \tag{1.7}$$

is called the implicit iterative sequence for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^N$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ are six sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$ for all $n \geq 1$, x_0 is a given point in K, as well as $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K.

Especially, if $\{T_i\}_{i=1}^N : K \to K$ are N asymptotically nonexpansive mappings, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in [0,1] and x_0 is a given point in K, then the sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + \beta_n T^n_{n \pmod N} x_{n-1} + \gamma_n u_n, \qquad n \geq 1, \tag{1.8}$$

is called the explicit iterative sequence for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^N$.

The aim of this paper is to study iterative sequences defined by (1.7) and (1.8) for a finite family of asymptotically nonexpansive mappings in Banach spaces and also establish weak and strong convergence theorems for said iteration schemes and mappings.

2. Preliminaries

PROPOSITION 2.1. *Let C be a nonempty subset of a real Banach space E and $\{T_i\}_{i=1}^N : C \to C$ be N asymptotically nonexpansive mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Then there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that*

$$\|T_i^n x - T_i^n y\| \leq k_n \|x - y\|, \tag{2.1}$$

for all $x, y \in C$, $i = 1, 2, \ldots, N$ and $n \geq 1$.

Proof. Since for each $i = 1, 2, \ldots, N$, $T_i : C \to C$ is an asymptotically nonexpansive mapping, there exists a sequence $\{k_n^{(i)}\} \subset [1,\infty)$ with $k_n^{(i)} \to 1$ as $n \to \infty$ such that

$$\|T_i^n x - T_i^n y\| \leq k_n^{(i)} \|x - y\|,$$

for all $x, y \in C$, $i = 1, 2, \ldots, N$ and $n \geq 1$.

Letting $k_n = \max\{k_n^{(1)}, k_n^{(2)}, \dots, k_n^{(N)}\}$, we have that $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and

$$\|T_i^n x - T_i^n y\| \le k_n^{(i)}\|x - y\| \le k_n\|x - y\|,$$

for all $x, y \in C$, $i = 1, 2, \dots, N$ and $n \ge 1$.

In the sequel we need the following lemmas to prove our main results.

LEMMA 2.1. [13] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \qquad n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

LEMMA 2.2. (Schu [11]) *Let E be a uniformly convex Banach space and $0 < a \le t_n \le b < 1$ for all $n \ge 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying*

$$\limsup_{n\to\infty} \|x_n\| \le r, \quad \limsup_{n\to\infty} \|y_n\| \le r, \quad \lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r,$$

for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

LEMMA 2.3. [3, 5, 12] *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $T\colon C \to C$ be an asymptotically nonexpansive mapping with $F(T) \ne \emptyset$. Then $I - T$ is semi-closed at zero, i.e., for each sequence $\{x_n\}$ in C, if $\{x_n\}$ converges weakly to $q \in C$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)q = 0$.*

LEMMA 2.4. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N\colon C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \ne \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ be six sequences in $[0, 1]$ and $\{k_n\}$ be the sequence defined by (2.1) and $\rho = \sup_{n\ge1} k_n \ge 1$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$;

(ii) $\sum_{n=1}^{\infty}(k_n - 1)\beta_n < \infty$;

(iii) $\tau = \sup\{\beta_n : n \ge 1\} < \frac{1}{\rho^2}$;

(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \hat{\gamma_n} < \infty$.

If $\{x_n\}$ is the implicit iterative sequence defined by (1.7), then for each $p \in F = \bigcap_{i=1}^N F(T_i)$ the limit $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Since $F = \bigcap_{i=1}^N F(T_i) \ne \emptyset$, for any given $p \in F$, it follows from (1.7) and Proposition 2.1 that

$$\begin{aligned}\|x_n - p\| &\le (1 - \beta_n - \gamma_n)\|x_{n-1} - p\| + \beta_n\|T^n_{n \pmod N} y_n - p\| + \gamma_n\|u_n - p\| \\ &= (1 - \beta_n - \gamma_n)\|x_{n-1} - p\| + \beta_n\|T^n_{n \pmod N} y_n - T^n_{n \pmod N} p\| \\ &\quad + \gamma_n\|u_n - p\| \\ &\le (1 - \beta_n)\|x_{n-1} - p\| + \beta_n k_n\|y_n - p\| + \gamma_n\|u_n - p\|, \end{aligned} \tag{2.2}$$

Again it follows from (1.7) and Proposition 2.1 that

$$\begin{aligned}\|y_n - p\| &\le (1 - \hat{\beta}_n - \hat{\gamma_n})\|x_{n-1} - p\| + \hat{\beta}_n\|T^n_{n \pmod N}x_n - p\| + \hat{\gamma_n}\|v_n - p\| \\ &= (1 - \hat{\beta}_n - \hat{\gamma_n})\|x_{n-1} - p\| + \hat{\beta}_n\|T^n_{n \pmod N}x_n - T^n_{n \pmod N}p\| \\ &\quad + \hat{\gamma_n}\|v_n - p\| \\ &\le (1 - \hat{\beta}_n)\|x_{n-1} - p\| + \hat{\beta}_n k_n\|x_n - p\| + \hat{\gamma_n}\|v_n - p\|. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain that

$$\begin{aligned}\|x_n - p\| \le (1 - \beta_n\hat{\beta}_n k_n)\|x_{n-1} - p\| + \beta_n\hat{\beta}_n k_n^2\|x_n - p\| \\ + \beta_n\hat{\gamma_n}k_n\|v_n - p\| + \gamma_n\|u_n - p\|,\end{aligned} \tag{2.4}$$

which implies that

$$(1 - \beta_n\hat{\beta}_n k_n^2)\|x_n - p\| \le (1 - \beta_n\hat{\beta}_n k_n)\|x_{n-1} - p\| + \mu_n, \tag{2.5}$$

where $\mu_n = \beta_n\hat{\gamma_n}k_n\|v_n - p\| + \gamma_n\|u_n - p\|$. By condition (iv) and boundedness of the sequences $\{\beta_n\}$, $\{k_n\}$, $\{\|u_n - p\|\}$ and $\{\|v_n - p\|\}$, we have $\sum_{n=1}^{\infty}\mu_n < \infty$. From condition (iii) we know that

$$\beta_n\hat{\beta}_n k_n^2 \le \beta_n k_n^2 \le \tau < 1 \quad \text{and so} \quad 1 - \beta_n\hat{\beta}_n k_n^2 \ge 1 - \tau\rho^2 > 0; \tag{2.6}$$

hence from (2.5) we have

$$\begin{aligned}\|x_n - p\| &\le \Big(\frac{1 - \beta_n\hat{\beta}_n k_n}{1 - \beta_n\hat{\beta}_n k_n^2}\Big)\|x_{n-1} - p\| + \frac{\mu_n}{1 - \tau\rho^2} \\ &= \Big(1 + \frac{(k_n - 1)\beta_n\hat{\beta}_n k_n}{1 - \beta_n\hat{\beta}_n k_n^2}\Big)\|x_{n-1} - p\| + \frac{\mu_n}{1 - \tau\rho^2} \\ &\le \Big(1 + \frac{(k_n - 1)\beta_n\hat{\beta}_n k_n}{1 - \tau\rho^2}\Big)\|x_{n-1} - p\| + \frac{\mu_n}{1 - \tau\rho^2} \\ &= (1 + A_n)\|x_{n-1} - p\| + B_n,\end{aligned} \tag{2.7}$$

where

$$A_n = \frac{(k_n - 1)\beta_n\hat{\beta}_n k_n}{1 - \tau\rho^2} \qquad \text{and} \qquad B_n = \frac{\mu_n}{1 - \tau\rho^2}.$$

By conditions (ii) and (iii) we have that

$$\begin{aligned}\sum_{n=1}^{\infty} A_n &= \frac{1}{1 - \tau\rho^2}\sum_{n=1}^{\infty}(k_n - 1)\beta_n\hat{\beta}_n k_n \\ &\le \frac{1}{1 - \tau\rho^2}\sum_{n=1}^{\infty}(k_n - 1)\beta_n k_n \le \frac{\rho}{1 - \tau\rho^2}\sum_{n=1}^{\infty}(k_n - 1)\beta_n < \infty\end{aligned}$$

and $B_n = \sum_{n=1}^{\infty}\frac{\mu_n}{1 - \tau\rho^2} < \infty$. Taking $a_n = \|x_{n-1} - p\|$ in inequality (2.7), we have

$$a_{n+1} \le (1 + A_n)a_n + B_n, \qquad \forall n \ge 1,$$

and all the conditions in Lemma 2.1 are satisfied. Therefore the limit $\lim_{n\to\infty}\|x_n - p\|$ exists. Without loss of generality we may assume that

$$\lim_{n\to\infty}\|x_n - p\| = d, \qquad p \in F,$$

where d is some nonnegative number. This completes the proof of Lemma 2.4. ∎

3. Main results

We are now in the position to prove our main results in this paper.

Theorem 3.1. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N\colon C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in C and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ be six sequences in $[0,1]$ and $\{k_n\}$ be the sequence defined by (2.1) and $\rho = \sup_{n\geq 1} k_n \geq 1$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$;

(ii) $\sum_{n=1}^\infty (k_n - 1)\beta_n < \infty$;

(iii) $\tau = \sup\{\beta_n : n \geq 1\} < \frac{1}{\rho^2}$;

(iv) $\sum_{n=1}^\infty \gamma_n < \infty$, $\sum_{n=1}^\infty \hat{\gamma_n} < \infty$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.7) converges strongly to a common fixed point $p \in F = \bigcap_{i=1}^N F(T_i)$ if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{3.1}$$

Proof. The necessity of condition (3.1) is obvious.

Next we prove the sufficiency of Theorem 3.1. For any given $p \in F$, it follows from (2.7) in Lemma 2.4 that

$$\|x_n - p\| \leq (1 + A_n)\|x_{n-1} - p\| + B_n \qquad \forall n \geq 1, \tag{3.2}$$

where

$$A_n = \frac{(k_n - 1)\beta_n\hat{\beta_n}k_n}{1 - \tau\rho^2} \qquad \text{and} \qquad B_n = \frac{\mu_n}{1 - \tau\rho^2}$$

with $\sum_{n=1}^\infty A_n < \infty$ and $\sum_{n=1}^\infty B_n < \infty$. Hence, we have

$$d(x_n, F) \leq (1 + A_n)d(x_{n-1}, p) + B_n \qquad \forall n \geq 1, \tag{3.3}$$

It follows from (3.3) and Lemma 2.1 that the limit $\lim_{n\to\infty} d(x_n, F)$ exists. By the condition (3.1), we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in C. In fact, since $\sum_{n=1}^\infty A_n < \infty$, $1 + x \leq e^x$ for all $x > 0$, and (3.2), therefore we have

$$\|x_n - p\| \leq e^{A_n}\|x_{n-1} - p\| + B_n \qquad \forall n \geq 1. \tag{3.4}$$

Hence, for any positive integers n, m, from (3.4) it follows that

$$\begin{aligned}
\|x_{n+m} - p\| &\leq e^{A_{n+m}}\|x_{n+m-1} - p\| + B_{n+m} \\
&\leq e^{A_{n+m}}\left[e^{A_{n+m-1}}\|x_{n+m-2} - p\| + B_{n+m-1}\right] + B_{n+m} \\
&\leq \dots
\end{aligned}$$

$$\leq e^{\left\{\sum_{i=n+1}^{n+m} A_i\right\}}\|x_n-p\| + e^{\left\{\sum_{i=n+2}^{n+m} A_i\right\}}\sum_{i=n+1}^{n+m} B_i$$
$$\leq Q\|x_n-p\| + Q\sum_{i=n+1}^{n+m} B_i, \tag{3.5}$$

where $Q = e^{\left\{\sum_{n=1}^{\infty} A_n\right\}} < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} B_n < \infty$, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\varepsilon}{4(Q+1)}, \qquad \sum_{i=n+1}^{\infty} B_i < \frac{\varepsilon}{2Q}, \qquad \forall n \geq n_0.$$

Therefore there exists $p_1 \in F$ such that

$$d(x_n, p_1) < \frac{\varepsilon}{2(Q+1)}, \qquad \forall n \geq n_0.$$

Consequently, for any $n \geq n_0$ and for all $m \geq 1$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &\leq Q\|x_n - p_1\| + Q\sum_{i=n+1}^{n+m} B_i + \|x_n - p_1\| \\ &= (Q+1)\|x_n - p_1\| + Q\sum_{i=n+1}^{n+m} B_i \\ &< (Q+1)\cdot\frac{\varepsilon}{2(Q+1)} + Q\cdot\frac{\varepsilon}{2Q} = \varepsilon.\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. By the completeness of C, we can assume that $\lim_{n\to\infty} x_n = p^*$. Since the set of fixed points of an asymptotically nonexpansive mapping is closed, hence F is closed. This implies that $p^* \in F$ and so p^* is a common fixed point of $T_1, T_2, \ldots, T_N$. This completes the proof of Theorem 3.1. ∎

THEOREM 3.2. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in C and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $[0,1]$ and $\{k_n\}$ be the sequence defined by* (2.1) *and $\rho = \sup_{n\geq 1} k_n \geq 1$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=1}^{\infty}(k_n - 1)\beta_n < \infty$;

(iii) $0 < \tau = \sup\{\beta_n : n \geq 1\} < \frac{1}{\rho^2}$;

(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.8) *converges strongly to a common fixed point $p \in F = \bigcap_{i=1}^N F(T_i)$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. Taking $\hat{\beta_n} = \hat{\gamma_n} = 0$ for all $n \geq 1$ in Theorem 3.1, then the conclusion of Theorem 3.2 can be obtained from Theorem 3.1 immediately. This completes the proof of Theorem 3.2.

THEOREM 3.3. *Let E be a real uniformly convex Banach space satisfying Opial condition and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in C and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ be six sequences in $[0,1]$ and $\{k_n\}$ be the sequence defined by* (2.1) *and $\rho = \sup_{n\geq 1} k_n \geq 1$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$;

(ii) $\sum_{n=1}^{\infty}(k_n - 1)\beta_n < \infty$;

(iii) $0 < \tau_1 = \inf\{\beta_n : n \geq 1\} \leq \sup\{\beta_n : n \geq 1\} = \tau_2 < \frac{1}{\rho^2}$;

(iv) $\hat{\beta_n} \to 0$, $(n \to \infty)$;

(v) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \hat{\gamma_n} < \infty$;

(vi) $0 \leq \delta = \sup\{\hat{\beta_n} : n \geq 1\} < \frac{1}{\rho}$;

(vii) there exists constants $L > 0$ and $\alpha > 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$,

$$\|T_i^n x - T_j^n y\| \leq L\|x - y\|^{\alpha}, \qquad \forall n \geq 1,$$

for all $x, y \in C$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.7) *converges weakly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$.*

Proof. First, we prove that

$$\lim_{n\to\infty} \|x_n - T^n_{n \ (\mathrm{mod}\ N)+j} x_n\| = 0, \qquad \forall j = 1, 2, \ldots, N. \tag{3.6}$$

Let $p \in F$. Put $d = \|x_n - p\|$, where d is some nonnegative number. It follows from (1.7) that

$$\begin{aligned} \|x_n - p\| = \|&(1 - \beta_n)[x_{n-1} - p + \gamma_n(u_n - x_{n-1})] \\ &+ \beta_n[T^n_{n \ (\mathrm{mod}\ N)} y_n - p + \gamma_n(u_n - x_{n-1})]\| \to d, \qquad n \to \infty. \end{aligned} \tag{3.7}$$

Again since $\lim_{n\to\infty} \|x_n - p\|$ exists, so $\{x_n\}$ is a bounded sequence in C. By virtue of condition (v) and the boundedness of sequences $\{x_n\}$ and $\{u_n\}$ we have

$$\begin{aligned} &\limsup_{n\to\infty} \|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\| \\ &\qquad \leq \limsup_{n\to\infty} \|x_{n-1} - p\| + \limsup_{n\to\infty} \gamma_n\|u_n - x_{n-1}\| = d, \quad p \in F. \end{aligned} \tag{3.8}$$

It follows from (2.3) and condition (iv) that

$$\limsup_{n\to\infty} \|T^n_{n \ (\mathrm{mod}\ N)} y_n - p + \gamma_n(u_n - x_{n-1})\|$$

$$\begin{aligned}
&\le \limsup_{n\to\infty} \|T^n_{n \pmod N} y_n - p\| + \limsup_{n\to\infty} \gamma_n \|u_n - x_{n-1}\| \\
&= \limsup_{n\to\infty} \|y_n - p\| \\
&\le \limsup_{n\to\infty} [(1-\hat{\beta}_n)\|x_{n-1} - p\| + \hat{\beta}_n k_n \|x_n - p\| + \hat{\gamma}_n \|v_n - p\|] \\
&\le d, \qquad p \in F. \qquad (3.9)
\end{aligned}$$

Therefore from condition (iii), (3.7)–(3.9) and Lemma 2.2 we know that

$$\lim_{n\to\infty} \|T^n_{n \pmod N} y_n - x_{n-1}\| = 0. \qquad (3.10)$$

From (1.7), (3.10) and condition (v), we have

$$\begin{aligned}
\|x_n - x_{n-1}\| &= \|\beta_n [T^n_{n \pmod N} y_n - x_{n-1}] + \gamma_n (u_n - x_{n-1})\| \\
&\le \beta_n \|T^n_{n \pmod N} y_n - x_{n-1}\| + \gamma_n \|u_n - x_{n-1}\| \to 0, \text{ as } n \to \infty, \qquad (3.11)
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \qquad (3.12)$$

and so

$$\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0 \qquad \forall j = 1, 2, \ldots, N. \qquad (3.13)$$

On the other hand, we have

$$\begin{aligned}
\|x_n - T^n_{n \pmod N} x_n\| \le{}& \|x_n - x_{n-1}\| + \|x_{n-1} - T^n_{n \pmod N} y_n\| \\
&+ \|T^n_{n \pmod N} y_n - T^n_{n \pmod N} x_n\|. \qquad (3.14)
\end{aligned}$$

Now, we consider the third term of the right hand side of (3.14). From the Proposition 2.1, (1.7) and the condition (vi) we have

$$\begin{aligned}
\|T^n_{n \pmod N} y_n - T^n_{n \pmod N} x_n\| &\le k_n \|y_n - x_n\| \\
&\le \rho \|\hat{\alpha_n} x_{n-1} + \hat{\beta}_n T^n_{n \pmod N} x_n + \hat{\gamma}_n v_n - x_n\| \\
&\le \rho \Big[\hat{\alpha_n} \|x_{n-1} - x_n\| + \hat{\beta}_n \|T^n_{n \pmod N} x_n - x_n\| + \hat{\gamma}_n \|v_n - x_n\|\Big] \\
&\le \rho \hat{\alpha_n} \|x_{n-1} - x_n\| + \rho\delta \|T^n_{n \pmod N} x_n - x_n\| + \rho \hat{\gamma_n} \|v_n - x_n\|. \qquad (3.15)
\end{aligned}$$

Substituting (3.15) into (3.14), we obtain that

$$\begin{aligned}
&(1-\rho\delta)\|x_n - T^n_{n \pmod N} x_n\| \\
&\quad \le (1 + \rho\hat{\alpha_n})\|x_n - x_{n-1}\| + \|x_{n-1} - T^n_{n \pmod N} y_n\| + \rho\hat{\gamma_n}\|v_n - x_n\|. \qquad (3.16)
\end{aligned}$$

Hence, by virtue of the condition (v), (3.10) and (3.12), we have

$$(1-\rho\delta) \limsup_{n\to\infty} \|x_n - T^n_{n \pmod N} x_n\| \le 0. \qquad (3.17)$$

From the condition (vi), $0 \le \rho\delta < 1$, hence from (3.17) we have

$$\lim_{n\to\infty} \|x_n - T^n_{n \pmod N} x_n\| = 0. \qquad (3.18)$$

By condition (vii), we have

$$\|T^{n-1}_{n\ (\mathrm{mod}\ N)}x_n - T^{n-1}_{(n-1)\ (\mathrm{mod}\ N)}x_{n-1}\| \le L\|x_n - x_{n-1}\|^{\alpha}. \tag{3.19}$$

From (3.10), (3.18), (3.19) and Proposition 2.1, it follows that

$$\begin{aligned}
&\|x_n - T_{n\ (\mathrm{mod}\ N)}x_n\| \\
&\quad \le \|x_n - T^n_{n\ (\mathrm{mod}\ N)}x_n\| + \|T^n_{n\ (\mathrm{mod}\ N)}x_n - T_{n\ (\mathrm{mod}\ N)}x_n\| \\
&\quad \le \|x_n - T^n_{n\ (\mathrm{mod}\ N)}x_n\| + k_1\|T^{n-1}_{n\ (\mathrm{mod}\ N)}x_n - x_n\| \\
&\quad \le \|x_n - T^n_{n\ (\mathrm{mod}\ N)}x_n\| + k_1\Big\{\|T^{n-1}_{n\ (\mathrm{mod}\ N)}x_n - T^{n-1}_{(n-1)\ (\mathrm{mod}\ N)}x_{n-1}\| \\
&\qquad + \|T^{n-1}_{(n-1)\ (\mathrm{mod}\ N)}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\Big\} \\
&\quad \le \|x_n - T^n_{n\ (\mathrm{mod}\ N)}x_n\| + k_1L\|x_n - x_{n-1}\|^{\alpha} \\
&\qquad + k_1\|T^{n-1}_{(n-1)\ (\mathrm{mod}\ N)}x_{n-1} - x_{n-1}\| + k_1\|x_{n-1} - x_n\| \\
&\quad \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T_{n\ (\mathrm{mod}\ N)}x_n\| = 0, \tag{3.20}$$

and so from (3.10) and (3.20), it follows that, for any $j = 1, 2, \dots, N$,

$$\begin{aligned}
&\|x_n - T_{n\ (\mathrm{mod}\ N)+j}x_n\| \\
&\quad \le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n\ (\mathrm{mod}\ N)+j}x_{n+j}\| \\
&\qquad + \|T_{n\ (\mathrm{mod}\ N)+j}x_{n+j} - T_{n\ (\mathrm{mod}\ N)+j}x_n\| \\
&\quad \le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n\ (\mathrm{mod}\ N)+j}x_{n+j}\| + k_1\|x_{n+j} - x_n\| \to 0 \text{ as } n \to \infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T_{n\ (\mathrm{mod}\ N)+j}x_n\| = 0. \tag{3.21}$$

Since E is uniformly convex, every bounded subset of E is weakly compact. Again since $\{x_n\}$ is a bounded sequence in C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $p_1 \in C$. Without loss of generality, we can assume that $n_k = j \pmod N$, where j is some positive integer in $\{1, 2, \dots, N\}$. Otherwise, we can take a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $n_{k_i} = j \pmod N$. For any $q \in \{1, 2, \dots, N\}$, there exists an integer $i_0 \in \{1, 2, \dots, N\}$ such that $n_k + i_0 = q \pmod N$. Hence, from (3.21) we have

$$\lim_{k\to\infty} \|x_{n_k} - T_q x_{n_k}\| = 0. \tag{3.22}$$

By Lemma 2.3 we know that $p_1 \in F(T_q)$. By the arbitrariness of $q \in \{1, 2, \dots, N\}$, we know that $p_1 \in F = \bigcap_{j=1}^{N} F(T_j)$.

Finally, we prove that the sequence $\{x_n\}$ converges weakly to p_1. In fact, suppose this is not true. Then there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $p_2 \in C$ and $p_1 \ne p_2$. Then by the same method as given above, we can also prove that $p_2 \in F = \bigcap_{j=1}^{N} F(T_j)$.

Taking $p = p_1$ and $p = p_2$ and using the same method given in the proof of Lemma 2.4, we can prove that the following two limits exist and

$$\lim_{n\to\infty} \|x_n - p_1\| = d_1, \qquad \lim_{n\to\infty} \|x_n - p_2\| = d_2$$

where d_1 and d_2 are two nonnegative numbers. By virtue of the Opial condition of E, we have

$$\begin{aligned} d_1 &= \limsup_{n_k\to\infty} \|x_{n_k} - p_1\| < \limsup_{n_k\to\infty} \|x_{n_k} - p_2\| = d_2 \\ &= \limsup_{n_j\to\infty} \|x_{n_j} - p_2\| < \limsup_{n_j\to\infty} \|x_{n_j} - p_1\| = d_1. \end{aligned}$$

This is a contradiction. Hence $p_1 = p_2$. This implies that $\{x_n\}$ converges weakly to p_1. This completes the proof of Theorem 3.3. ∎

THEOREM 3.4. *Let E be a real uniformly convex Banach space satisfying Opial condition and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N \colon C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in C and let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $[0,1]$ and $\{k_n\}$ be the sequence defined by (2.1) and $\rho = \sup_{n\ge1} k_n \ge 1$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=1}^{\infty}(k_n - 1)\beta_n < \infty$;

(iii) $0 < \tau_1 = \inf\{\beta_n : n \ge 1\} \le \sup\{\beta_n : n \ge 1\} = \tau_2 < \frac{1}{\rho^2}$;

(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(v) there exists constants $L > 0$ and $\alpha > 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$,

$$\|T_i^n x - T_j^n y\| \le L\|x - y\|^{\alpha}, \qquad \forall n \ge 1,$$

for all $x, y \in C$.

Then the explicit iterative sequence $\{x_n\}$ defined by (1.8) converges weakly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$.

Proof. Taking $\hat{\beta_n} = \hat{\gamma_n} = 0$ for all $n \ge 1$ in Theorem 3.3, then the conclusion of the Theorem 3.4 can be obtained from Theorem 3.3 immediately. This completes the proof of Theorem 3.4.

THEOREM 3.5. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N \colon C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and there exists an T_l, $1 \le l \le N$ which is semi-compact (without loss of generality, we can assume that T_1 is semi-compact). Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in C and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ be six sequences in $[0,1]$ and $\{k_n\}$ be the sequence defined by (2.1) and $\rho = \sup_{n\ge1} k_n \ge 1$ satisfying the conditions (i)–(vii) as in Theorem 3.3. Then the implicit iterative sequence $\{x_n\}$ defined by (1.7) converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$ in C.*

Proof. For any given $p \in F = \bigcap_{i=1}^{N} F(T_i)$, by the same method as given in proving Lemma 2.4 and (3.22), we can prove that

$$\lim_{n\to\infty} \|x_n - p\| = d, \tag{3.23}$$

where $d \geq 0$ is some nonnegative number, and

$$\lim_{k\to\infty} \|x_{n_k} - T_q x_{n_k}\| = 0, \tag{3.24}$$

for all $q = 1, 2, \dots, N$.

Especially, we have

$$\lim_{k\to\infty} \|x_{n_k} - T_1 x_{n_k}\| = 0. \tag{3.25}$$

By the assumption of the theorem, T_1 is semi-compact, therefore it follows from (3.25) that there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \to x^* \in C$. Hence from (3.24) we have that

$$\|x^* - T_q x^*\| = \lim_{k\to\infty} \|x_{n_{k_i}} - T_q x_{n_{k_i}}\| = 0,$$

for all $q = 1, 2, \dots, N$, which implies that $x^* \in F = \bigcap_{i=1}^{N} F(T_i)$.

Take $p = x^*$ in (3.23), similarly we can prove that $\lim_{n\to\infty} \|x_n - x^*\| = d_1$, where $d_1 \geq 0$ is some nonnegative number. From $x_{n_{k_i}} \to x^*$ we know that $d_1 = 0$, i.e., $x_n \to x^*$. This completes the proof of Theorem 3.5. ■

THEOREM 3.6. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N} : C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists an T_l, $1 \leq l \leq N$ which is semi-compact (without loss of generality, we can assume that T_1 is semi-compact). Let $\{u_n\}$ be a bounded sequence in C and let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $[0,1]$ and $\{k_n\}$ be the sequence defined by (2.1) and $\rho = \sup_{n\geq 1} k_n \geq 1$ satisfying the conditions (i)–(v) as in Theorem 3.4. Then the explicit iterative sequence $\{x_n\}$ defined by (1.8) converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N\}$ in C.*

Proof. Taking $\hat{\beta_n} = \hat{\gamma_n} = 0$ for all $n \geq 1$ in Theorem 3.5, then the conclusion of the Theorem 3.6 can be obtained from Theorem 3.5 immediately. This completes the proof of Theorem 3.6.

REMARK 3.1. Since $0 \leq (k_n - 1)\beta_n \leq k_n - 1$, therefore it is easy to see that if condition (ii) is replaced by $(ii)'$:

$(ii)'$ $\sum_{n=1}^{\infty}(k_n - 1) < \infty$,

then also the conclusion of Theorem 3.1 - 3.6 holds true.

REMARK 3.2. Theorem 3.3 improves and extends Theorem 3.1 of Chang and Cho [2] in its two ways:

(1) The key condition "$\sum_{n=1}^{\infty}(k_n - 1) < \infty$" is replaced by more weak condition "$\sum_{n=1}^{\infty}(k_n - 1)\beta_n < \infty$".

(2) The implicit iteration process $\{x_n\}$ in [2] is replaced by the more general implicit or explicit iteration process $\{x_n\}$ with bounded errors defined by (1.7) or (1.8).

Remark 3.3. Theorem 3.3 improves and extends Theorem 1 of Zhou and Chang [16] in its two ways:

(1) The key condition "$\sum_{n=1}^{\infty}(k_n-1)<\infty$" is replaced by more weak condition "$\sum_{n=1}^{\infty}(k_n-1)\beta_n<\infty$".

(2) The condition (v) in [16, Theorem 1]: there exists a constant $L>0$ such that for any $i,j\in\{1,2,\dots,N\}$, $i\neq j$

$$\|T_i^n x - T_j^n y\| \le L\|x-y\|, \qquad \forall n\ge 1,$$

for all $x,y\in C$ is replaced by the more general condition (vii) in Theorem 3.3.

(3) The implicit iteration process $\{x_n\}$ in [16] is replaced by the more general implicit or explicit iteration process $\{x_n\}$ with bounded errors defined by (1.7) or (1.8).

Remark 3.4. Theorems 3.1–3.6 generalize and improve the corresponding results of Bauschke [1], Halpern [7], Lions [8], Reich [10], Wittmann [14], Xu and Ori [15] in the following aspects:

(1) The Hilbert space is extended to that of Banach space satisfying Opial's or semi-compactness condition.

(2) The class of nonexpansive mappings is extended to that of asymptotically nonexpansive mappings.

(3) The implicit iteration process $\{x_n\}$ is replaced by the more general implicit or explicit iteration process $\{x_n\}$ with bounded errors defined by (1.7) or (1.8).

Remark 3.5. Our results also extend the corresponding results of Gu and Lu [6] to the case of more general class of nonexpansive mappings considered in this paper.

Example 3.1. Let $X=\mathbb{R}$ and $C=[0,1]$. Define $T\colon C\to C$ by $T(x)=x/2$, $x\in[0,1]$. Hence

$$|Tx-Ty| = 1/2|x-y| \le |x-y|$$

for all $x,y\in C$. Therefore T is a nonexpansive mapping and hence it is an asymptotically nonexpansive mapping with constant sequence $\{1\}$. But the converse is not true in general.

Example 3.2. Let $X=\ell_2=\{\bar{x}=\{x_i\}_{i=1}^{\infty} : x_i\in C, \sum_{i=1}^{\infty}|x_i|^2<\infty\}$, and let $\bar{B}=\{\bar{x}\in\ell_2 : \|x\|\le 1\}$. Define $T\colon \bar{B}\to\ell_2$ by

$$T\bar{x} = (0, x_1^2, a_2x_2, a_3x_3, \dots),$$

where $\{a_j\}_{j=1}^{\infty}$ is a real sequence satisfying: $a_2>0$, $0<a_j<1$, $j\neq 2$, and $\prod_{j=2}^{\infty}a_j=1/2$. Then

$$\|T^n\bar{x}-T^n\bar{y}\| \le 2\Big(\prod_{j=2}^{n}a_j\Big)\|\bar{x}-\bar{y}\| \le k_n\|\bar{x}-\bar{y}\|$$

where $k_n = 2\Big(\prod_{j=2}^{n} a_j\Big)$ and $\bar{x}, \bar{y} \in X$. Since $\lim_{n\to\infty} k_n = \lim_{n\to\infty} 2\Big(\prod_{j=2}^{n} a_j\Big) = 1$, it follows that T is an asymptotically nonexpansive mapping. But it is not a nonexpansive mapping.

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(received 15.06.2011; available online 01.01.2012)

Department of Mathematics and Information Technology, Govt. Nagarjuna P.G. College of Science, Raipur - 492010 (C.G.), India.

E-mail: saluja_1963@rediffmail.com, saluja1963@gmail.com