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GENERALIZED HOLOMORPHIC FUNCTIONS AND CLIFFORD ANALYSIS

ABSTRACT. With the help of a Clifford Algebra and The Dirac operator, in the multidimensional space a generalized Cauchy–Riemann system is constructed whose Cauchy-kernel can be represented explicitly. In the two-dimensional case it is a classical system and can be considered as Maxwell or Dirac stationary equations with two independent variables. A classification of Beltrami type equations is given determined by elements of the Clifford algebra. Some boundary value problems are studied.

რეზიუმე. მრავალგანზომილებიან სივრცეში კლიფორდის ალგებრისა და დირაკის ოპერატორის გამოყენებით აგებულია განზოგადებული კომპლექსური განტოლებათა ისეთი სისტემა, რომლისთვისაც კომპლექსური სინგულარული გული ცხადი სახით არის წარმოდგენილი. ორგანზომილებიან შემთხვევაში ის კლასიკური სისტემაა და შეიძლება განხილულ იქნეს, როგორც მაქსველის ან დირაკის განტოლებები სტაციონარულ შემთხვევაში ორი დამოუკიდებელი სივრცითი ცვლადით. მოცემულია ბელტრამის ტიპის განტოლებების კლასიფიკაცია, განსაზღვრული კლიფორდის ალგებრის ელემენტებით. შესწავლილია ზოგიერთი სასაზღვრო ამოცანა.

INTRODUCTION

The generalized Cauchy-Riemann equations (GCRE) and the Beltrami equations are significant generalizations of the classical Cauchy-Riemann equations (CRE) for which a complete theory with important applications to the shell theory and the differential geometry was constructed by I. Vekua [14]. As is known, in investigating GCRE or CRE the corresponding Cauchy type integral representations play an essential role. Unlike CRE, for GCRE with variable coefficients the Cauchy kernel cannot be written explicitly but for them unique existence theorems are proved [14]. As to the case of constant coefficients, the Cauchy kernel can be written explicitly even in the multidimensional space. Almost all classical partial differential equations of

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mathematical physics are put in Clifford analysis and can be obtained by using the Dirac operator.

For multidimensional CRE the Cauchy kernel is written explicitly and for them the well-known properties of holomorphic functions of one complex variable [9], namely the Cauchy integral theorem and formula, the maximum principle, the Liouville theorem and others, are established [3], [6], [7], [8], [5].

GCRE and the generalized Beltrami's equation written with the help of the Dirac operator can be considered in various Clifford algebras. In such a way were investigated [10,11] in the multidimensional space elliptic, hyperbolic and parabolic equations related to the Helmholtz, Klein-Gordon and Maxwell equations of electromagnetic fields, the Dirac equations of relativity quantum mechanics and heat equations were investigated [10,11]. One can construct multidimensional GCRE in a form which permits to write the corresponding Cauchy kernel explicitly and to establish well-known properties of holomorphic functions [12].

Below we will see that classical GCRE can be considered as Maxwell or Dirac equations with two independent space variables in the stationary cases. Some boundary value problems will be investigated also.

Let $R_{(n)}$, $R_{(n,n-1)}$, $R_{(n)}^0$ ($n \geq 1$) be Clifford algebras with the basis e_A , $A(\alpha_1 \dots \alpha_k)$, $0 \leq \alpha_1 < \dots < \alpha_k \leq n$, and with the multiplication rules:

$$e_0^2 = e_0, \quad e_j^2 = -e_0, \quad j = 1, 2, \dots, n-1;$$

$$e_j e_k + e_k e_j = 0, \quad k \neq j, = 1, 2, \dots, n, \quad (1)$$

$$e_n^2 = -e_0, \quad \text{in the case of } R_n, \quad (2)$$

$$e_n^2 = e_0, \quad \text{in the case of } R_{(n,n-1)}, \quad (3)$$

$$e_n^2 = 0, \quad \text{in the case of } R_{(n)}^0, \quad (4)$$

where e_0 is the identity element. It is remarkable also considering the space $R_{(n,n-1)}^0$ ($n \geq 2$) for which

$$e_j^2 = -e_0, \quad j = 1, \dots, n-2, \quad e_{n-1}^2 = e_0, \quad e_n^2 = 0. \quad (5)$$

Any element of these spaces can be written as

$$u = \sum_A u_A e_A, \quad (6)$$

where u_A are real; the element u is said to be vectorial if

$$u = \sum_0^n u_k e_k. \quad (7)$$

In the case $n \geq 2$, for u two conjugates are defined:

$$\bar{u} = \sum_A u_A \bar{e}_A, \quad \tilde{u} = \sum_A u_A \tilde{e}_A \quad (8)$$

$$\bar{e}_0 = e_0, \quad \bar{e}_j = \tilde{e}_j = -e_j, \quad j = 1, \dots, n; \quad (9)$$

$$\bar{e}_A = \bar{e}_{\alpha_k} \dots \bar{e}_{\alpha_1}, \quad \tilde{e}_A = \bar{e}_{\alpha_1} \dots \bar{e}_{\alpha_k}.$$

The product of any two elements is defined as

$$uv = \sum_{A,B} u_A v_B e_A e_B. \quad (10)$$

Note that using the well-known Pauli matrices we can by the mathematical induction method represent the bases of the above spaces for any n explicitly in a matrix form [12]. One can see that $R_{(1)}$ is the space of complex numbers, $R_{(1,1)}$ is the space of double numbers and $R_{(1)}^0$ is the space of dual numbers. They can be successfully used for considering elliptic, hyperbolic and parabolic equations in the two-dimensional case.

1. DIRAC OPERATOR, GCRE AND GENERALIZED BELTRAMI EQUATIONS

Consider a modification of the Dirac operator [3]

$$\bar{\partial} = \sum_0^n \frac{\partial}{\partial x_k} e_k = \frac{\partial}{\partial x_0} e_0 + D, \quad (1.1)$$

$$\partial = \frac{\partial}{\partial x_0} e_0 - D,$$

where D is the Dirac operator. It is clear that $\bar{\partial}$ in $R_{(1)}$ will be the classical Cauchy–Riemann operator $\frac{\partial}{\partial \bar{z}}$. Therefore by virtue of (1), (2) GCRE and generalized Beltrami equations [14], [2] in the case $n = 1$ can be written in the form

$$\bar{\partial}u + h\bar{u} = 0, \quad (1.2)$$

$$\bar{\partial}u + q\partial u + h\bar{u} = 0. \quad (1.3)$$

with $u, q, h \in R_{(1)}$. These equations can obviously be considered in various Clifford algebras for any $n \geq 1$. Since in the case $n \geq 2$ when for u two different conjugates are defined by (8), one can consider in the multidimensional space the equations

$$\bar{\partial}u + \tilde{u}h = 0, \quad (1.4)$$

$$\bar{\partial}u + q\partial u + \tilde{u}h = 0. \quad (1.5)$$

Although there is not much outward difference between the equations (1.2) and (1.4), in the case of a constant vectorial coefficient h the Cauchy kernel for (1.4) can be written explicitly, while for (1.2) this is impossible

[12]. Namely, (1.4) has 2^n linearly independent fundamental solutions represented as

$$\overset{A}{u} = (\partial\varphi)e_A - \varphi\tilde{e}_A h, \quad A(\alpha_1 \dots \alpha_k)$$

where $\varphi(r)$ is a fundamental solution of the equation

$$\partial\bar{\partial}\varphi - |h|^2\varphi = 0, \quad r^2 = |x|^2 = x\bar{x}, \quad x = \sum_0^n x_k e_k, \quad (1.6)$$

which is the Helmholtz equation in $R_{(n)}$ and the Klein–Gordon one in $R_{(n,n-1)}$, i.e., $\varphi(r)$ is known.

If u is of the form (6), then (1.4) and (1.5) will be systems with 2^n equations for 2^n unknowns, but if u is of the form (7), then they will be undetermined systems with $\frac{n(n+1)}{2} + 1$ equations for $(n+1)$ unknowns. Using Clifford algebras, we can see that the classification of the system (1.5) for both cases is the same. Let

$$\lambda = \sum_0^n \lambda_k e_k, \quad q = \sum_0^n q_k e_k,$$

and consider a quadratic form with respect to λ :

$$M(\lambda) = |\lambda + q\bar{\lambda}|^2 \equiv \text{Re}[(\lambda + q\bar{\lambda})(\bar{\lambda} + \lambda\bar{q})]; \quad (1.7)$$

this form will be called characteristic one for (1.5).

Definition. The system (1.5) will be called elliptic if $M(\lambda)$ is definite with respect to $\lambda_0, \dots, \lambda_n$ will be called hyperbolic if $M(\lambda)$ is indefinite and will be called parabolic if $M(\lambda)$ is degenerate. Let $\lambda, q \in R_{(n)}$. Then by virtue of (1) and (2), $M(\lambda)$ is represented as

$$M(\lambda) = [\lambda_0(1+q_0) + \sum_1^n \lambda_k q_k]^2 + \sum_1^n [\lambda_0 q_k + \lambda_k(1-q_0)]^2 + \sum_{1=k<j}^n (\lambda_k q_j - \lambda_j q_k)^2.$$

If all addends vanish in this expression for some $\lambda_0, \dots, \lambda_n$, then we will have

$$\lambda_k [1 - |q|^2] = 0, \quad k = 0, 1, \dots, n.$$

Thus, if

$$|q|^2 = \sum_0^n q_k^2 \neq 1, \quad (1.8)$$

then $M(\lambda)$ is definite and (1.5) in $R_{(n)}$ is elliptic.

Consider now (1.5) in the space $R_{(n,n-1)}$. By virtue of (1), (3), we can easily define $M(\lambda)$ and obtain that $M(\lambda)$ is indefinite if

$$|q|^2 = \sum_0^{n-1} q_k^2 - q_n^2 \neq 1, \quad (1.9)$$

and in this case (1.5) is hyperbolic. If $|q|^2 = 1$, then $M(\lambda)$ is degenerate and (1.5) is parabolic. Hence, the equations (1.4), (1.5) under the condition (1.8) in the space $R_{(n)}$ are elliptic while under the condition (1.9) in the space $R_{(n,n-1)}$ they are hyperbolic.

The equation

$$\bar{\partial}u + q\partial u = 0, \quad |q| \neq 1 \quad (1.10)$$

in $R_{(1)}$ is the classical Beltrami equation, and therefore in $R_{(1,0)}$ it can be called hyperbolic Beltrami equation. As is known [14], a definite differential quadratic form with two independent variables can be reduced to the canonical form by using the transformation defined by solutions of the Beltrami equation (1.10) in $R_{(1)}$. It is remarkable that the indefinite differential quadratic form can analogously be reduced to the canonical form by using a solution of the equation (1.10) in $R_{(1,0)}$.

The equation (1.10) for $n \geq 1$ will be called multidimensional elliptic Beltrami equation in $R_{(n)}$ and hyperbolic Beltrami equation in $R_{(n,n-1)}$ with $|q| \neq 1$. Without loss of generality one can suppose

$$|q| \leq q_0 < 1. \quad (1.11)$$

If $|q| \geq q_0 > 1$, then in considering the conjugation of the equation (1.10) defined by the second equality of (8), we obtain that (1.11) is satisfied.

Note that if $q \neq \pm 1$ is a real constant, then the solution of (1.5) can be represented as

$$u = w[(1 - q)x_0, (1 + q)x_1, \dots, (1 + q)x_n],$$

where $w = w(y)$, $y = (1 - q)x_0e_0 + (1 + q)\sum_1^n x_k e_k$ is the solution of the same equation as (1.4) with the coefficient $h/1 - q^2$, i.e. the problems solvable for (1.4) can be solved for the equation (1.5) as well.

The space $R_{(3,2)}$ is especially interesting as far as it is defined over a four-dimensional Minkowski space which is particularly important in the special relativity theory. Consider (1.4) in this space. Let

$$u = u_0e_0 - u_1e_1 - u_2e_2 - \varphi e_3 - \psi e_1e_2 - u_3e_3e_1 - u_4e_2e_3 - u_5e_1e_2e_3 \quad (1.12)$$

$$h = h_0e_0 - h_1e_1 - h_2e_2 - h_3e_3.$$

Then by (1), (3), (1.1) one can obtain eight equations with eight unknowns

which can be written in the vector form as

$$\begin{aligned} \operatorname{div} E - \frac{\partial \varphi}{\partial t} + (E \cdot A) - \varphi h_3 &= 0, \\ \operatorname{grad} \psi + \operatorname{rot} E + \frac{\partial H}{\partial t} + [E \times A] + \psi A + H h_3 &= 0, \\ \operatorname{div} H + \frac{\partial \psi}{\partial t} - (H \cdot A) - \psi h_3 &= 0, \\ \operatorname{grad} \varphi + \operatorname{rot} H - \frac{\partial E}{\partial t} - [H \times A] - \varphi A + E h_3 &= 0, \end{aligned} \quad (1.13)$$

where $E(u_0, u_1, u_2)$, $H(u_3, u_4, u_5)$, $A(h_0, h_1, h_2)$ are three-component vectors, $t \equiv x_3$, the operators div , grad , rot are taken with respect to x_0, x_1, x_2 . Note that the Dirac equations of the relativity quantum mechanics presented for instance in [4], can be written like (1.13). If $\varphi = \psi = 0$, $A = 0$, $h_3 = 0$, then (1.13) will be the Maxwell equations, i.e. from the Dirac equations one can get as a particular case the Maxwell equations. It is clear that solutions of the equations (1.13) are at the same time solutions of the Klein–Gordon equation (1.6). From the equations (1.13) one can obtain as particular cases the Moisil–Theodorescu equations, GCRE and also the metaparabolic equations [1], [13]; that is, the Dirac or generalized Maxwell equations involve all the above-mentioned equations.

It is also interesting to consider the space $R_{(n, n-2)}$. In particular, the equation $\bar{\partial}u = 0$ in $R_{(3,1)}$ can be called ultrahyperbolic because $u(x)$ is at the same time solution of the following ultrahyperbolic equation [4]

$$\partial \bar{\partial}u = \frac{\partial^2 u}{\partial x_0^2} + \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = 0 \quad (1.14)$$

for which some correctly posed initial and boundary value problems can be solved by using the Fourier integral transformation [12].

The aim of the present paper is not to consider parabolic systems. We only note that by means of the operator $\bar{\partial}$ acting in the spaces $R_{(n)}^0$ and $R_{(n, n-1)}^0$, two kinds of parabolic systems can be obtained [3,10]. One of them is related to the classical heat equation, i.e., the equation with the Laplace operator in the principal part, and the second one is related to a parabolic equation with the D'Alambertian (wave operator) in the principal part.

2. BOUNDARY VALUE PROBLEMS IN THE ELLIPTIC CASE

The classical Riemann and Riemann-Hilbert problems [9] considered in some domains for the equation (1.4) with constant coefficients in $R_{(n)}$ ($n \geq 1$), can be solved explicitly [11]. For example, in the half-space $x_n > 0$, the vanishing at infinity solution of the equation (1.4) with 2^{n-1} boundary conditions

$$\operatorname{Re}[ue_A] = f_A \quad \text{for } x_n = 0, \quad (2.1)$$

where A takes 2^{n-1} different values from $(\alpha_1 \dots \alpha_k)$, and $0 \leq \alpha_1 < \dots < \alpha_k \leq n$, f_A are given scalar Hölder-continuous functions, can be constructed uniquely in quadratures. But if one considers such a problem in the domain $x_n > 0$, $x_{n-1} > 0$, then the solution cannot be defined uniquely for any h . For example, let $n = 1$. Then (1.4) will be the generalized Cauchy-Riemann equations in the domain $x_0 \geq 0$, $x_1 \geq 0$:

$$\begin{aligned} \frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_1} + u_0 h_0 + u_1 h_1 &= 0, \\ \frac{\partial u_0}{\partial x_1} - \frac{\partial u_1}{\partial x_0} - u_0 h_1 + u_1 h_0 &= 0, \end{aligned} \quad (2.2)$$

with the boundary conditions

$$u_0(x_0, 0) = 0, \quad x_0 > 0; \quad u_0(0, x_1) = 0, \quad x_1 > 0. \quad (2.3)$$

Vanishing at infinity solution of this homogeneous problem cannot be zero for some h . Really, since $u_0(x_0, x_1)$ is in the given domain a solution of the Helmholtz equation (1.6) which vanishes at infinity, one has $u_0(x_0, x_1) \equiv 0$. Therefore it follows from (2.2) that

$$u_1 = c e^{h_0 x_0 - h_1 x_1}, \quad x_0 \geq 0, \quad x_1 \geq 0,$$

where c is a real constant. Thus, if $h_0 < 0$, $h_1 > 0$ the problem (2.2), (2.3) has also nonzero solutions vanishing at infinity. Clearly, if we assume that $u_1(0, 0) = 0$, then the solution is uniquely defined, i.e., $u_1 \equiv 0$.

Supplementary conditions for obtaining uniquely defined solutions of (1.4) are more complicated when $n \geq 2$. Let $n = 2$, $h = h_0 e_0 - h_1 e_1 - h_2 e_2$, $u = u_0 e_0 - u_1 e_1 - u_2 e_2 - \varphi e_1 e_2$. Then (1.4) represents the generalized Moisil-Theodorescu system

$$\begin{aligned} \operatorname{div} U + (U \cdot H) &= 0, \\ \operatorname{grad} \varphi + \operatorname{rot} U + [U \times H] + \varphi H &= 0, \end{aligned} \quad (2.4)$$

where $U(u_0, u_1, u_2)$, $H(h_0, h_1, h_2)$. Consider these equations in the domain $x_1 \geq 0$, $x_2 \geq 0$, $x_0 \in \mathbb{R}$ under the boundary conditions

$$\begin{aligned} u_0(x_0, x_1, 0) = u_2(x_0, x_1, 0) &= 0, \quad x_1 \geq 0, \\ u_0(x_0, 0, x_2) = u_2(x_0, 0, x_2) &= 0, \quad x_2 \geq 0. \end{aligned} \quad (2.5)$$

Again, $u_0(x_0, x_1, x_2) = u_2 \equiv 0$ as a vanishing at infinity solution of (1.6). Then by virtue of (2.4) for u_1 , φ we have equations whose solution can be represented in the form

$$\varphi + i u_1 = w(z) e^{-h_1 x_1}, \quad x_1 > 0,$$

where $w(z)$ is a generalized holomorphic function of $z = x_0 + i x_2$ in the half-plane $x_2 > 0$:

$$\frac{\partial w}{\partial \bar{z}} + \frac{1}{2}(h_0 + i h_2) \bar{w} = 0. \quad (2.6)$$

Consequently, if $h_1 > 0$, then there exists an infinite number of vanishing at infinity solutions. Given a supplementary condition, for example, $\varphi(x_0, x_1, 0) = 0$, we obtain the uniqueness of the vanishing at infinity solution. Various correctly posed initial and boundary value problems for the above-mentioned system can be found in [12].

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