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Rational approximations to $\sqrt[3]{2}$ and other algebraic numbers revisited

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Résumé. Dans cet article, nous améliorons des mesures effectives d'irrationalité pour certains nombres de la forme $\sqrt[3]{n}$ en utilisant des approximations obtenues à partir de fonctions hypergéométriques. Ces résultats sont très proche du mieux que peut donner cette méthode. Nous obtenons ces résultats grâce à des informations arithmétiques très précises sur les dénominateurs des coefficients de ces fonctions hypergéométriques.

Des améliorations de bornes pour les fonctions de Chebyshev $\theta(k,l;x) = \sum_{\substack{p \equiv l \bmod k; \\ p, \text{prime},\ p \leq x}} \log p$. et $\psi(k,l;x) = \sum_{\substack{n \equiv l \bmod k; \\ n \leq x}} \Lambda(n)$ $(k = 1, 3, 4, 6)$ sont aussi présentés.

Abstract. In this paper, we establish improved effective irrationality measures for certain numbers of the form $\sqrt[3]{n}$, using approximations obtained from hypergeometric functions. These results are very close to the best possible using this method. We are able to obtain these results by determining very precise arithmetic information about the denominators of the coefficients of these hypergeometric functions.

Improved bounds for the Chebyshev functions in arithmetic progressions $\theta(k,l;x)$ and $\psi(k,l;x)$ for $k = 1, 3, 4, 6$ are also presented.

1. Introduction

In this article, we shall consider some refinements of a method due to Alan Baker [1, 2] for obtaining effective irrationality measures for certain algebraic numbers of the form $z^{m/n}$. As an example, he showed that for any integers p and q, with $q \neq 0$,

$$\left| 2^{1/3} - \frac{p}{q} \right| > \frac{10^{-6}}{|q|^{2.955}}.$$

This method has its basis in the work of Thue. There are two infinite families of hypergeometric polynomials in $\mathbb{Q}[z]$, $\{X_{m,n,r}(z)\}_{r=0}^{\infty}$ and

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$\{Y_{m,n,r}(z)\}_{r=0}^{\infty}$, such that $\{Y_{m,n,r}(z)/X_{m,n,r}(z)\}_{r=0}^{\infty}$ is a sequence of good approximations to $z^{m/n}$. Under certain conditions on z, these approximations are good enough to enable us to establish an effective irrationality measure for $z^{m/n}$ which is better than the Liouville measure.

Since it is easy to obtain sharp estimates for the other quantities involved, the most important consideration in applying this method is the size of the denominators of these hypergeometric polynomials.

Chudnovsky [5] improved on Baker's results by showing that, if p is a sufficiently large prime divisor of the least common denominator of $X_{m,n,r}(z)$ and $Y_{m,n,r}(z)$, then p must lie in certain congruence classes mod n and certain subintervals of $[1, nr]$.

In the case of $z^{m/n} = 2^{1/3}$, he was able to show that for any $\epsilon > 0$ there exists a positive integer $q_0(\epsilon)$ such that

$$\left| 2^{1/3} - \frac{p}{q} \right| > \frac{1}{|q|^{2.4297\ldots+\epsilon}}$$

for all integers p and q with $|q| > q_0(\epsilon)$. Moreover, since his estimates for the relevant quantities are asymptotically correct, this exponent is the best that one can obtain from this hypergeometric method although "off-diagonal" or the method of "ameliorating factors" (à la Hata) still might yield improvements.

Shortly after this work, Easton [6] obtained explicit versions for the cube roots of various positive integers. For $2^{1/3}$, he showed that

$$\left| 2^{1/3} - \frac{p}{q} \right| > \frac{2.2 \cdot 10^{-8}}{|q|^{2.795}}$$

for all integers p and q with $q \neq 0$.

It is the purpose of this paper to establish effective irrationality measures which come quite close to Chudnovsky's. In the particular case of $2^{1/3}$,

$$\left| 2^{1/3} - \frac{p}{q} \right| > \frac{0.25}{|q|^{2.4325}}$$

for all integers p and q with $q \neq 0$.

This paper was initially written and circulated in 1996. Independently, Bennett [3] obtained a result, which in the cubic case, is slightly weaker than the theorem stated here. E.g., for $2^{1/3}$, he showed that

$$\left| 2^{1/3} - \frac{p}{q} \right| > \frac{0.25}{|q|^{2.45}}$$

for all integers p and q with $q \neq 0$.

In fact, this subject has been the topic of even more work. As part of his Ph.D. Thesis (see [8]), Heimonen has also obtained effective irrationality measures for numbers of the form $\sqrt[n]{a/b}$, as well as of the form $\log(a/b)$.

His results are not as sharp as those of the author, but they are still substantially better than Easton's.

The general method used in each of these three papers is essentially the same. However, there are substantial differences in the presentations due to the fact that the approach of Bennett and Heimonen shows more apparently the role that Padé approximations play in this area, while the author deals explicitly with hypergeometric polynomials.

Actually, the referee has pointed out that other work in this area has been done, producing results not much weaker than our own. And this work preceeded the results of Bennett, Heimonen and the author. We are referring to the work of Nikishin [12] and, especially, Korobov [9]. In particular, in 1990, Korobov showed that

$$\left| \sqrt[3]{2} - p/q \right| > q^{-2.5},$$

for all natural numbers p and q with $q \neq 1, 4$. The reader looking for a more accessible reference to these works is referred to [7, pp. 38–39].

The main differences between this version of the paper and the previous version are Theorem 2.3 and improvements in computer hardware. This has resulted in replacing 0.93 with 0.911 in the exponents on e in the expressions for E and Q in Theorem 2.1 (which requires a larger value of c_1), along with the consequent improvements to Corollary 2.2 including new results for $\sqrt[3]{41}$ and $\sqrt[3]{57}$.

The main incentive for publication of this paper now is completeness. Several articles have since appeared in the literature (e.g., [11] and [15]) which depend on results in this article. Furthermore, the lemmas in this article, which are either new or sharpen results currently in the literature, are important in forthcoming articles by the author and others. They are accompanied by an analysis showing that they are best-possible or else what the best-possible results should be. And lastly, the main theorem itself, along with its corollary, is an improvement on the present results in the literature.

We structure this paper as follows. Section 2 contains the statements of our results. In Section 3, we state and prove the arithmetic results that we obtain for the coefficients of the hypergeometric polynomials. Section 4 is devoted to the proof of Theorem 2.3, as this theorem will be required in Section 5, where we obtain the analytic bounds that we will require for the proof of Theorem 2.1. Section 6 contains the diophantine lemma that allows us to obtain an effective irrationality measure from a sequence of good approximations. At this point, we have all the pieces that we need to prove Theorem 2.1, which is done in Section 7. Finally, Corollary 2.2 is proven in Section 8.

Finally, I'd like to thank Gary Walsh for his encouragement and motivation to resume my work in this area. Also, Clemens Heuberger deserves my thanks for his careful reading of an earlier version of this paper and accompanying suggestions. And, of course, I thank the referee for their time and effort as well as their suggestions for improvements.

2. Results

Theorem 2.1. *Let a and b be integers satisfying $0 < b < a$. Define c_1, d, E and κ by*

$$d = \begin{cases} 0 & \text{if } 3 \nmid (a-b), \\ 1 & \text{if } 3 \parallel (a-b) \text{ and} \\ 3/2 & \text{otherwise,} \end{cases}$$

$$E = e^{-0.911} 3^d \left(a^{1/2} - b^{1/2} \right)^{-2},$$

$$\kappa = \frac{\log \left\{ e^{0.911} 3^{-d} \left(a^{1/2} + b^{1/2} \right)^2 \right\}}{\log E} \quad \text{and}$$

$$c_1 = 10^{40(\kappa+1)} a.$$

If $E > 1$ then

$$\left| (a/b)^{1/3} - p/q \right| > \frac{1}{c_1 |q|^{\kappa+1}} \tag{1}$$

for all integers p and q with $q \neq 0$.

Remark. c_1 grows quite rapidly as the absolute values of the arguments of the exponential functions in the definition of E approach their best possible value of $\pi\sqrt{3}/6 = 0.9068\ldots$.

In the earlier version of this paper with 0.911 replaced by 0.93, we could have taken $c_1 = 10^{7(\kappa+1)} a$. It is feasible to prove Theorem 2.1 with 0.911 replaced by 0.91, but then we would have to take $c_1 = 10^{86(\kappa+1)} a$.

The rate of growth is even more rapid as we continue to approach 0.9068. For example, with 0.907, $c_1 > 10^{2400(\kappa+1)} a$.

As an application of Theorem 2.1, we give effective irrationality measures for all numbers of the form $\sqrt[3]{n}$ where n is a cube-free rational integer with $2 \leq n \leq 100$ and for which the hypergeometric method yields an improvement over the Liouville bound.

Corollary 2.2. *For the values of n given in Table 1, we have*

$$\left| \sqrt[3]{n} - p/q \right| > \frac{c_2}{|q|^{\kappa+1}},$$

for all integers p and q with $q \neq 0$ where c_2 and κ are the values corresponding to n in Table 1.

n	c_2	κ	n	c_2	κ	n	c_2	κ
2	0.25	1.4325	25	0.07	1.7567	60	0.08	1.5670
3	0.37	1.6974	26	0.03	1.4860	61	0.06	1.5193
4	0.41	1.4325	28	0.03	1.4813	62	0.04	1.4646
5	0.29	1.7567	30	0.10	1.6689	63	0.02	1.3943
6	0.01	1.3216	31	0.14	1.9288	65	0.02	1.3929
7	0.08	1.6717	36	0.08	1.3216	66	0.04	1.4610
9	0.08	1.6974	37	0.01	1.2472	67	0.06	1.5125
10	0.15	1.4157	39	0.08	1.1848	68	0.08	1.5562
11	0.22	1.8725	41	0.41	1.9956	70	0.12	1.6314
12	0.28	1.9099	42	0.12	1.4186	76	0.08	1.5154
13	0.35	1.8266	43	0.01	1.2890	78	0.03	1.5729
15	0.19	1.4964	44	0.21	1.8164	83	0.09	1.6898
17	0.01	1.1996	49	0.13	1.6717	84	0.37	1.8797
18	0.37	1.9099	50	0.11	1.1962	90	0.09	1.3751
19	0.02	1.2718	52	0.26	1.8901	91	0.009	1.2583
20	0.009	1.1961	57	0.15	1.9825	98	0.38	1.4813
22	0.07	1.2764	58	0.12	1.6526	100	0.35	1.4158

TABLE 1. Results for $\sqrt[3]{n}$

Remark. If α be an irrational element of $\mathbb{Q}(\sqrt[3]{n})$, then we can write

$$\alpha = \frac{a_1 \sqrt[3]{n} + a_2}{a_3 \sqrt[3]{n} + a_4},$$

where $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ with $a_1 a_4 - a_2 a_3 \neq 0$. In this way, we can use Corollary 2.2 to obtain effective irrationality measures for any such α (see Section 8 of [5]).

These values of a and b were found from the convergents p/q in the continued-fraction expansion of $\sqrt[3]{n}$ by setting a/b to be either $(p/q)^3/n$ or its reciprocal, whichever is greater than one. For each cube-free positive integer less than or equal to 100, we searched through all the convergents with $q < 10^{100}$.

In this way, we obtain measures for $\sqrt[3]{5}$, $\sqrt[3]{11}$ and $\sqrt[3]{41}$ — values of n within the range considered by Chudnovsky, but not treated by him — as well as an improved irrationality measure for $\sqrt[3]{7}$. Bennett also found the same a and b for these n (along with $n = 41$ and 57, which we also consider here). However, his version of our Theorem 2.1 was not sufficiently strong to allow him to obtain effective irrationality measures for $n = 41$ and 57 which improve on Liouville's theorem, so these remain as new results here.

Given the scale of the search, the table is almost certainly complete for $n \leq 100$.

The values of a and b listed in Table 1 produced the minimal values of $\kappa < 2$ satisfying the conditions of Theorem 2.1 for the given value of n.

A key element in translating the sharp result contained in Proposition 3.2 into tight numerical results is a strong bound for

$$\theta(x;k,l) = \sum_{\substack{p \equiv l \bmod k; p,\text{prime} \\ p \leq x}} \log p.$$

Ramaré and Rumely [13] provide good bounds. However, due to recent computational work of Rubinstein [14], we are able to improve these bounds considerably for some k. So we present here the following results on $\theta(x;k,l)$, and the closely-related $\psi(x;k,l)$, for $k = 1, 3, 4$ and 6.

Theorem 2.3. (a) *For* $1 \leq x \leq 10^{12}$,
$\max_{1 \leq y \leq x} \max\left(|\theta(y) - y|, |\psi(y) - y|\right) \leq 2.052818\sqrt{x}$,
$\max_{1 \leq y \leq x} \max\left(|\theta(y;3,\pm 1) - y/2|, |\psi(y;3,\pm 1) - y|\right) \leq 1.798158\sqrt{x}$,
$\max_{1 \leq y \leq x} \max\left(|\theta(y;4,\pm 1) - y/2|, |\psi(y;4,\pm 1) - y|\right) \leq 1.780719\sqrt{x}$ *and*
$\max_{1 \leq y \leq x} \max\left(|\theta(y;6,\pm 1) - y/2|, |\psi(y;6,\pm 1) - y|\right) \leq 1.798158\sqrt{x}$.
(b) *For each* $(k,l), x_0$ *and* ϵ *given in Table* 2,

$$\left|\theta(x;k,l) - \frac{x}{\varphi(k)}\right|, \left|\psi(x;k,l) - \frac{x}{\varphi(k)}\right| \leq \epsilon x,$$

for $x \geq x_0$.

	10^5	10^6	10^7	10^8	10^9	10^{10}
$(1,0)$	0.00474	0.00168	0.000525	0.0001491	0.0000459	0.0000186
$(3,1)$	0.00405	0.00148	0.000401	0.0001260	0.0000371	0.0000351
$(3,2)$	0.00217	0.00068	0.000180	0.0000428	0.0000351	0.0000351
$(4,1)$	0.00494	0.00169	0.000471	0.0001268	0.0000511	0.0000511
$(4,3)$	0.00150	0.00036	0.000197	0.0000511	0.0000511	0.0000511
$(6,1)$	0.00405	0.00148	0.000401	0.0001260	0.0000371	0.0000351
$(6,5)$	0.00217	0.00068	0.000180	0.0000428	0.0000351	0.0000351

TABLE 2. Analytic epsilons for $x \geq x_0$

Only the results for $\theta(x;3,2)$ and $\psi(x;3,2)$ will be used here, but we record the additional inequalities in this theorem for use in ongoing work and by other researchers as they improve the current bounds of Ramaré and Rumely [13] by a factor of approximately 30.

Unless otherwise noted, all the calculations mentioned in this paper were done using the Java programming language (release 1.4.2) running on an IBM-compatible computer with an Intel P4 CPU running at 1.8 GHz with 256 MB of memory. Source code for all programs can be provided upon request. Many of these computations were also checked by hand, using MAPLE, PARI/GP and UBASIC. No discrepancies beyond round-off error were found.

3. Arithmetic properties of hypergeometric polynomials

We use ${}_2F_1(a,b;c;z)$ to denote the hypergeometric function

$$ {}_2F_1(a,b;c;z) = 1 + \sum_{k=1}^{\infty} \frac{a(a+1)\cdots(a+k-1)b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)k!} z^k. $$

For our purposes here, we are interested in the following functions, which we define for all positive integers m, n and r with $(m,n)=1$. Let

$$ \begin{aligned} X_{m,n,r}(z) &= z^r {}_2F_1\left(-r, -r-m/n; 1-m/n; z^{-1}\right), \\ Y_{m,n,r}(z) &= {}_2F_1\left(-r, -r-m/n; 1-m/n; z\right) \text{ and} \\ R_{m,n,r}(z) &= \frac{(m/n)\cdots(r+m/n)}{(r+1)\cdots(2r+1)} {}_2F_1\left(r+1-m/n, r+1; 2r+2; 1-z\right). \end{aligned} $$

This differs from (4.3) of [5] where the expressions for $X_r(z)$ and $Y_r(z)$ have been switched. The same change must be made in (4.4) of [5] too.

Notations. We let $D_{m,n,r}$ denote the smallest positive integer such that $D_{m,n,r}Y_{m,n,r}(z)$ has rational integer coefficients.
To simplify the notation in the case of $m=1$ and $n=3$, which is of particular interest in this paper, we let $X_r(z), Y_r(z), R_r(z)$ and D_r denote $X_{1,3,r}(z), Y_{1,3,r}(z), R_{1,3,r}(z)$ and $D_{1,3,r}$, respectively.
We will use $v_p(r)$ to denote the largest power of a prime p which divides into the rational number r.
Finally, we let $\lfloor \cdot \rfloor$ denote the floor function which maps a real number to the greatest integer less than that number.

We first need a refined version of Chudnovsky's Lemma 4.5 in order to establish our criterion for the prime divisors of $D_{m,n,r}$.

Lemma 3.1. *Suppose that m, n, p, u and v are integers with $0 < m < n$ and $(m,n) = (p,n) = 1$. For each positive integer, i, define the integer*

$1 \le k_i \le p^i$ *by* $k_i n \equiv m \bmod p^i$. *Then*

$$v_p\left(\prod_{j=u}^{v}(nj-m)\right) = \sum_{i=1}^{\infty}\left(\left\lfloor\frac{v-k_i}{p^i}\right\rfloor - \left\lfloor\frac{u-1-k_i}{p^i}\right\rfloor\right)$$
$$= \sum_{i=1}^{\infty}\left(\left\lfloor\frac{-u+k_i}{p^i}\right\rfloor - \left\lfloor\frac{-v-1+k_i}{p^i}\right\rfloor\right).$$

Remark. It would be more typical to state the above lemma with the condition $0 \le k_i < p^i$ rather than $1 \le k_i \le p^i$. The proof below holds with either condition. However, the above formulation suits our needs in the proof of Proposition 3.2 below better.

Proof. For each positive integer i, we will count the number of j's in $u \le j \le v$ with $nj - m \equiv 0 \bmod p^i$. That is, with $nj - k_i n \equiv 0 \bmod p^i$. And, since $(n, p) = 1$, with $j \equiv k_i \bmod p^i$. The remainder of the proof is identical to Chudnovsky's proof of his Lemma 4.5 [5], upon replacing his p with p^i. $\square$

Proposition 3.2. *Let* m, n *and* r *be positive integers with* $0 < m < n$ *and* $(m, n) = 1$.

The largest power to which a prime p *can divide* $D_{m,n,r}$ *is at most the number of positive integers* i *for which there exist a positive integer* l_i *satisfying* $(l_i, n) = 1, l_i p^i \equiv -m \bmod n$ *such that*

$$\frac{l_i p^i + m}{n} \le r \bmod p^i \le \frac{(n-l_i)p^i - m - n}{n}.$$

Furthermore, all such i *satisfy* $p^i \le nr$.

Remark. From the calculations done in the course of this, and other, work (see, for example, the notes following Lemmas 3.3, 3.4 and 5.1), it appears that the conditions given in this Proposition provide the exact power to which a prime divides $D_{m,n,r}$. However, I have not been able to prove this.

Proof. Let $a_{r,h}$ denote the coefficient of z^h in $Y_{m,n,r}(z)$ and let p be a prime number. From our definition of $Y_{m,n,r}(z)$ above, we can write

$$a_{r,h} = \binom{r}{h}\frac{C_{r,h}}{B_{r,h}},$$

where

$$B_{r,h} = \prod_{i=1}^{h}(in-m) \quad \text{and} \quad C_{r,h} = \prod_{i=r-h+1}^{r}(in+m).$$

We first show that if p divides $D_{m,n,r}$ then $(p, n) = 1$.

If p does divide $D_{m,n,r}$ then p must divide $B_{r,h}$ for some $0 \le h \le r$. So it must divide some number of the form $in - m$ where $1 \le i \le r$. But, if p divides such a number and also divides n, then it must also divide m.

However, our hypothesis that $(m,n)=1$ does not allow this and so, if p divides $D_{m,n,r}$ then $(p,n)=1$.

Therefore, for any positive integer i, we can find an integer k_i with $1 \le k_i \le p^i, (k_i, p^i)=1$ and $k_i n \equiv m \bmod p^i$.

As $1 \le k_i$ and $m < n$, we know that $0 < k_i n - m$, and so there must be a positive integer l_i with $(l_i, n)=1$ and $k_i n - m = l_i p^i$. Furthermore, $l_i < n$.

Returning to our expression for $a_{r,h}$, we have

$$v_p(a_{r,h}) = v_p\left(\binom{r}{h}\right) + v_p(C_{r,h}) - v_p(B_{r,h}).$$

It is well-known that

$$v_p\left(\binom{r}{h}\right) = \sum_{i=1}^{\infty}\left(\left\lfloor \frac{r}{p^i}\right\rfloor - \left\lfloor \frac{h}{p^i}\right\rfloor - \left\lfloor \frac{r-h}{p^i}\right\rfloor\right).$$

From the first expression in Lemma 3.1 with $u=1$ and $v=h$,

$$v_p(B_{r,h}) = \sum_{i=1}^{\infty}\left(\left\lfloor \frac{h-k_i}{p^i}\right\rfloor - \left\lfloor \frac{-k_i}{p^i}\right\rfloor\right).$$

From the second expression in Lemma 3.1 with $u=-r$ and $v=-r+h-1$,

$$v_p(C_{r,h}) = \sum_{i=1}^{\infty}\left(\left\lfloor \frac{r+k_i}{p^i}\right\rfloor - \left\lfloor \frac{r+k_i-h}{p^i}\right\rfloor\right).$$

Thus, we want to determine when

$$\left\lfloor \frac{r}{p^i}\right\rfloor - \left\lfloor \frac{h}{p^i}\right\rfloor - \left\lfloor \frac{r-h}{p^i}\right\rfloor - \left\lfloor \frac{h-k_i}{p^i}\right\rfloor + \left\lfloor \frac{-k_i}{p^i}\right\rfloor + \left\lfloor \frac{r+k_i}{p^i}\right\rfloor - \left\lfloor \frac{r+k_i-h}{p^i}\right\rfloor \tag{2}$$

is negative.

This will suffice for the purpose of proving this proposition since, as we shall show shortly, the expression in (2) can never be less than -1.

We now show that if $p^i > nr$, then the expression in (2) cannot be negative. This will establish the last statement in the Proposition.

Since $0 \le h \le r < p^i$ for such i, the first three terms in (2) are 0. Furthermore, the same inequalities for h and r along with the fact that $k_i > 0$ show that the sum of the last two terms cannot be negative.

We saw above that $k_i n - m = l_i p^i$ for a positive integer l_i. So it follows that $k_i n \ge p^i + m > nr \ge nh$. In particular, $k_i > h$. Furthermore, $1 \le k_i \le p^i$. Therefore, $\lfloor (h-k_i)/p^i \rfloor$ and $\lfloor -k_i/p^i \rfloor$, are both equal to -1, so the sum of the remaining terms in (2) is also zero.

This establishes the last statement in the Proposition.

Moreover, if $p^i > nr + m$, then

$$r + k_i < \frac{p^i - m}{n} + \frac{(n-1)p^i + m}{n} < p^i.$$

And so the expression in (2) is always 0 for such i. We will use this fact in the proof of Lemmas 3.3 and 3.4 below.

For any positive integer i, we can write h and r uniquely as

$$h = h_{i1}p^i + h_{i0} \quad \text{and} \quad r = r_{i1}p^i + r_{i0},$$

where $0 \leq h_{i0}, r_{i0} < p^i$.

With this notation, we see that

$$\begin{aligned} \left\lfloor \frac{r}{p^i} \right\rfloor - \left\lfloor \frac{h}{p^i} \right\rfloor - \left\lfloor \frac{r-h}{p^i} \right\rfloor &= - \left\lfloor \frac{r_{i0}-h_{i0}}{p^i} \right\rfloor \\ \left\lfloor \frac{h-k_i}{p^i} \right\rfloor - \left\lfloor \frac{-k_i}{p^i} \right\rfloor &= h_{i1} + 1 + \left\lfloor \frac{h_{i0}-k_i}{p^i} \right\rfloor \quad \text{and} \\ \left\lfloor \frac{r+k_i}{p^i} \right\rfloor - \left\lfloor \frac{r+k_i-h}{p^i} \right\rfloor &= h_{i1} + \left\lfloor \frac{r_{i0}+k_i}{p^i} \right\rfloor - \left\lfloor \frac{r_{i0}+k_i-h_{i0}}{p^i} \right\rfloor. \end{aligned} \tag{3}$$

The first relation holds since $1 \leq h_{i0}, r_{i0} < p^i$ and so $\lfloor h_{i0}/p^i \rfloor = \lfloor r_{i0}/p^i \rfloor = 0$. The second relation holding since $1 \leq k_i \leq p^i$ and so $\lfloor -k_i/p^i \rfloor = -1$.

The last two quantities can only have the values h_{i1} or $h_{i1}+1$, so if the expression in (2) is to be negative then the first quantity here must be zero, since it is never negative, the second must be $h_{i1}+1$ and the third must be h_{i1}. This information also substantiates our claim above that the expression in (2) is always at least -1.

Since $0 \leq h_{i0}, r_{i0} < p^i$, the first quantity in (3) is zero if and only if

$$r_{i0} \geq h_{i0}. \tag{4}$$

The second quantity in (3) is $h_{i1}+1$ if and only if

$$h_{i0} \geq k_i. \tag{5}$$

Finally, if the last quantity in (3) is h_{i1}, then $\lfloor (r_{i0}+k_i)/p^i \rfloor = \lfloor (r_{i0}+k_i-h_{i0})/p^i \rfloor$. From (5), we find that $r_{i0}+k_i-h_{i0} \leq r_{i0} < p^i$, so $r_{i0}+k_i < p^i$ also. Hence

$$0 < \frac{r_{i0}+k_i}{p^i} < 1, \tag{6}$$

the left-hand inequality being strict since $k_i > 0$.

From (4), we have $k_i \leq r_{i0}+k_i-h_{i0}$, while from (5) and (6), it follows that $r_{i0}+k_i-h_{i0} < p^i - h_{i0} \leq p^i - k_i$. Combining these inequalities, we find that

$$\frac{k_i}{p^i} \leq \frac{r_{i0}+k_i-h_{i0}}{p^i} < 1 - \frac{k_i}{p^i}.$$

In addition, from (4) and (5), we know that $k_i \le r_{i0}$ and from (6), $r_{i0} \le p^i - k_i - 1$, so

$$r_{i1} + \frac{k_i}{p^i} \le \frac{r}{p^i} \le r_{i1} + 1 - \frac{k_i+1}{p^i}.$$

Substituting $(l_i p^i + m)/n = k_i$ into this expression completes the proof of the Proposition. □

It will be helpful for applications to present a slightly weaker but more immediately applicable result on the prime divisors of $D_{m,n,r}$. With that in mind, we state the following.

Lemma 3.3. (a) *Let r be a positive integer. If $p|D_{m,n,r}$, then*

$$v_p\left(D_{m,n,r}\right) \le \left\lfloor \frac{\log(nr)}{\log p} \right\rfloor.$$

(b) *If p is a prime number greater than $(nr)^{1/2}$ which is a divisor of $D_{m,n,r}$, then $p^2 \nmid D_{m,n,r}$ and for some $1 \le l < n/2$ with $(l,n)=1$, $lp \equiv -m \bmod n$, and*

$$\frac{nr+m+n}{nA+n-l} \le p \le \frac{nr-m}{nA+l}, \tag{7}$$

for some non-negative integer A. Moreover, every such prime greater than $(nr+m)^{1/2}$ is a divisor of $D_{m,n,r}$.

Remark. The result in (a) is best possible. E.g., $D_{2,3,17}$ is divisible by 49 and $\lfloor \log(3\cdot 17)/(\log 7) \rfloor = 2$. This example also shows that neither of the statements in Lemma 3.4 holds here. Furthermore, 5 divides some of the $D_{2,3,10}$, so the congruence conditions in Lemma 3.4 do not hold in general either.

Proof. (a) This follows immediately from the last statement in Proposition 3.2.

(b) Again from the last statement in Proposition 3.2 and our lower bound for p, we need only consider $i=1$.

From the inequality on $r \bmod p$ in Proposition 3.2, we can write

$$Ap + \frac{lp+m}{n} \le r \le Ap + \frac{(n-l)p-m-n}{n}, \tag{8}$$

for some non-negative A. This provides our upper and lower bounds for p in part (b), which suffices to prove the first statement in part (b).

To prove the second statement, we will show that these primes divide the denominator of the leading coefficient of $Y_r(z)$. So we let the quantity denoted by h in the proof of Proposition 3.2 be r. Using the arguments to derive (3) in the proof of Proposition 3.2, (2) simplifies to

$$-1 - \left\lfloor \frac{r_{i0}-k_i}{p^i} \right\rfloor + \left\lfloor \frac{r_{i0}+k_i}{p^i} \right\rfloor - \left\lfloor \frac{k_i}{p^i} \right\rfloor = -1 - \left\lfloor \frac{r_{i0}-k_i}{p^i} \right\rfloor + \left\lfloor \frac{r_{i0}+k_i}{p^i} \right\rfloor,$$

where $r \equiv r_{i0} \bmod p^i$.

Therefore, as we saw in the proof of Proposition 3.2,

$$v_p(a_{r,r}) = \sum_{i=1}^{\infty} \left(-1 - \left\lfloor \frac{r_{i0} - k_i}{p^i} \right\rfloor + \left\lfloor \frac{r_{i0} + k_i}{p^i} \right\rfloor \right).$$

Notice that for $i \geq 2$, $p^i > nr + m$, so, as we saw in the proof of Proposition 3.2, the summands for such i are zero and can be ignored.

For $i = 1$, from (8), $(lp + m)/n \leq r_{i0} \leq ((n - l)p - m - n)/n$. From the relationship between k_1 and l_1 given in the proof of Proposition 3.2, we also have $k_1 = (lp + m)/n$. Therefore, $0 \leq \lfloor r_{10} - k_1 \rfloor, \lfloor r_{10} + k_1 \rfloor \leq p - 1$ and the summand for $i = 1$ is -1. Hence $v_p(a_{r,r}) = -1$, so p divides the denominator of $a_{r,r}$ precisely once, completing the proof of the Lemma. □

As n gets larger, the structure of the denominator becomes more complicated and the above is the best that we can do. However, in the case of $m = 1$ and $n = 3, 4$ or 6, we can obtain a sharper result which will be used in this paper.

Lemma 3.4. *Let $m = 1$ and $n = 3, 4$ or 6.*
(a) *Let r be a positive integer. If $p | D_{m,n,r}$ then $p \equiv n - 1 \bmod n$ and*

$$v_p\left(D_{m,n,r}\right) \leq \left\lfloor \frac{\log(nr)}{2 \log p} + \frac{1}{2} \right\rfloor.$$

(b) *If p is a prime number greater than $(nr)^{1/3}$ which is a divisor of $D_{m,n,r}$ then $p \equiv n - 1 \bmod n$, $p^2 \nmid D_{m,n,r}$ and*

$$\frac{nr + n + 1}{nA + n - 1} \leq p \leq \frac{nr - 1}{nA + 1}, \tag{9}$$

for some non-negative integer A. Moreover, every such prime greater than $(nr + 1)^{1/2}$ is a divisor of $D_{m,n,r}$.

Remark. The result in (a) is best possible. D_{42} is divisible by 25 and $\lfloor \log(3 \cdot 42)/(2 \log 5) + 1/2 \rfloor = 2$. Similarly, D_{1042} is divisible by 125 and $\lfloor \log(3 \cdot 1042)/(2 \log 5) + 1/2 \rfloor = 3$. However, it is not true that $v_p(D_r) \geq 2$ for all $p \leq (3r)^{1/3}$ (e.g., $v_5(D_{43}) = 1$).

Remark. The second statement in (b) holds for all $p > (nr)^{1/3}$, and this is best possible as the example in the previous remark shows, however the proof is technical and lengthy. Furthermore, the result here suffices for our needs below.

Proof. (a) We apply Proposition 3.2. As we saw there, $(p, n) = 1$. For these values of n, the only integers less than n and relatively prime to n are 1 and $n - 1$.

If $p \equiv 1 \bmod n$ or if $p \equiv n-1 \bmod n$ and i is even, then we require $l_i \equiv n-1 \bmod n$ to satisfy $l_i p^i \equiv -1 \bmod n$. However, with this value of l_i,

$$\frac{(n-1)p^i+1}{n} = \frac{l_i p^i + m}{n} \leq r \bmod p^i \leq \frac{(n-l_i)p^i - m - n}{n} = \frac{p^i - n - 1}{n}$$

can never be satisfied.

If $p \equiv n-1 \bmod n$ and i is odd, then we can take $l_i = 1$. From the last statement in Proposition 3.2, $p^i \leq nr$, so the largest possible i is at most $\log(nr)/\log(p)$, a fact which completes the proof of part (a).

(b) The same argument as for $p \equiv n-1 \bmod n$ in the proof of part (a) shows that we need only consider $i = 1$ for $p > (nr)^{1/3}$.

The remainder of the proof is identical to the proof of Lemma 3.3(b). □

Lemma 3.5. *Let m, n and r be positive integers with $(m,n) = 1$. Define $\mu_n = \prod_{p|n} p^{1/(p-1)}$ and $s_{n,r} = \prod_{p|n} p^{v_p(r!)}$.*
(a) *Let d be a positive divisor of n. The numerators of the coefficients of the polynomials $X_{m,n,r}(1-dz)$ and $Y_{m,n,r}(1-dz)$ are divisible by d^r.*
(b) *The numerators of the coefficients of the polynomials $X_{m,n,r}(1-n\mu_n z)$ and $Y_{m,n,r}(1-n\mu_n z)$ are divisible by $n^r s_{n,r}$.*

Proof. (a) This is a variation on part (b) which will prove useful both here and elsewhere. Its proof is virtually identical to the proof of part (b).

(b) This is Proposition 5.1 of [5]. □

4. Proof of Theorem 2.3

(a) The bounds for $x \leq 10^{12}$ are determined through direct calculation. We coded the Sieve of Eratosthenes in Java and ran it, in segments of size 10^8, to determine all primes less than 10^{12} as well as upper and lower bounds for $\theta(x;k,l)$ and $\psi(x;k,l)$ for $x \leq 10^{12}$. The entire computation took approximately $182,000$ seconds.

As Ramaré and Rumely note, considerable roundoff error can arise in the sum of so many floating point numbers. We handled this issue in a similar way to them. We multiply each log by 10^6, round the resulting number down to the greatest integer less than the number as a lower bound and round it up to the least integer greater than the number as an upper bound. We then sum these integers and store the sums in variables of type long, which have a maximum positive value of $2^{63}-1 = 9.233\ldots \cdot 10^{18}$ – a number greater than our sums. This is more crude than Ramaré and Rumely's method, but sufficiently accurate for our needs here.

In addition to just establishing the desired inequalities, we also compute, and have stored,
(i) our upper and lower bounds for $\theta(10^8 i;k,l)$ and $\psi(10^8 i;k,l)$,
(ii) $\pi(10^8 i,k,l)$,

(iii) $\displaystyle\min_{x\in(10^8(i-1),10^8 i]} \frac{\theta(x;k,l)-x/\varphi(k)}{\sqrt{x}}$, $\displaystyle\min_{x\in(10^8(i-1),10^8 i]} \frac{\psi(x;k,l)-x/\varphi(k)}{\sqrt{x}}$
and
(iv) $\displaystyle\max_{x\in(10^8(i-1),10^8 i]} \frac{\theta(x;k,l)-x/\varphi(k)}{\sqrt{x}}$, $\displaystyle\max_{x\in(10^8(i-1),10^8 i]} \frac{\psi(x;k,l)-x/\varphi(k)}{\sqrt{x}}$,
for $i = 1, ..., 10,000$.

(b) The bounds in part (b) are obtained by applying Theorem 5.1.1 of [13] with the L-function zero information calculated by Michael Rubinstein [14]. We include details of the values used in Table 3, where we round all quantities up by one in the seventh significant decimal (sixth decimal for $\tilde{A}_\chi$, $\tilde{B}_\chi$, $\tilde{C}_\chi$, $\tilde{D}_\chi$ for the sake of space).

Note that for these values of k, there is only one character, χ, for each d.

For the computation of $\tilde{A}_\chi$, we followed the advice of Ramaré and Rumely [13, p. 414] regarding the evaluation of their K_1 and K_2. Using Simpson's rule with an interval size of 0.001 (along with their Lemma 4.2.4), we bound from above the integral for $K_n(z,w)$ in their equation (4.2.4) for $u = w \dots 1000$. We then apply their Lemma 4.2.3 with $w = 1000$, which is sufficiently large to provide a good upper bound.

This provides us with an upper bound for $\epsilon(\psi, x, k)$.

Using the authors' upper bound for $\epsilon(\theta, x, k)$ on page 420 of [13], we see that our results holds for $x \geq x_0$.

Proceeding as above, we found agreement with the data that Ramaré and Rumely present in their Table 1 for $k = 1, 3$ and 4.

k	1	3		4	
m	14	14		14	
δ	$6.289071 \cdot 10^{-7}$	$1.256642 \cdot 10^{-6}$		$1.798450 \cdot 10^{-6}$	
$A(m,\delta)$	$1.082027 \cdot 10^{91}$	$6.691384 \cdot 10^{86}$		$4.425147 \cdot 10^{84}$	
$\tilde{R}$	$2.721552 \cdot 10^{-11}$	$9.085095 \cdot 10^{-11}$		$1.207835 \cdot 10^{-10}$	
$\epsilon(\psi,x,k)$	$3.613190 \cdot 10^{-5}$	$7.097148 \cdot 10^{-5}$		$1.001340 \cdot 10^{-4}$	
d	1	1	3	1	4
H_χ	8000000.365	4000000.042	4000000.413	2800000.0623	2800000.340
$\tilde{A}_\chi$	$5.81243 \cdot 10^{-98}$	$8.94572 \cdot 10^{-94}$	$9.83501 \cdot 10^{-94}$	$1.27527 \cdot 10^{-91}$	$1.43730 \cdot 10^{-91}$
$\tilde{B}_\chi$	$9.09392 \cdot 10^{-103}$	$2.85005 \cdot 10^{-98}$	$3.04716 \cdot 10^{-98}$	$5.86164 \cdot 10^{-96}$	$6.37182 \cdot 10^{-96}$
$\tilde{C}_\chi$	$7.30396 \cdot 10^{-98}$	$1.13798 \cdot 10^{-93}$	$1.23103 \cdot 10^{-93}$	$1.63333 \cdot 10^{-91}$	$1.80646 \cdot 10^{-91}$
$\tilde{D}_\chi$	$1.14495 \cdot 10^{-102}$	$3.63318 \cdot 10^{-98}$	$3.82113 \cdot 10^{-98}$	$7.52419 \cdot 10^{-96}$	$8.02375 \cdot 10^{-96}$
$\tilde{E}_\chi$	31.414915	28.3898896	33.1560902	26.8928884	32.8584828

TABLE 3. Data for the Proof of Theorem 2.3

Since $2.052818 < 0.0000186x^{1/2}$ for $x \geq 12.2 \cdot 10^9$, the stated inequalities for $\theta(x)$ and $\psi(x)$ holds for such x. Using the above sieve code, it is straightforward to calculate $\theta(x)$ and $\psi(x)$ for $x < 12.2 \cdot 10^9$. These calculations complete the proof of (b) for $\theta(x)$ and $\psi(x)$.

Similarly, $1.798158 < 0.0000351x^{1/2}$ for $x \geq 2.7 \cdot 10^9$ and a computation completes the proof of (b) for $k = 3$ and 6.

Finally, $1.780719 < 0.0000511x^{1/2}$ for $x \geq 2.7 \cdot 10^9$ and a computation completes the proof of (b) for $k = 4$..

5. Analytic properties of hypergeometric polynomials

Lemma 5.1. *Let r be a positive integer and define N_r to be the greatest common divisor of the numerators of the coefficients of $X_r(1-(a-b)x/a)$, where a, b and d are as defined in Theorem* 2.1.

(a) *We have*

$$\frac{1}{200} < \frac{0.29 D_r^2}{r^{1/6} 4^r}.$$

(b) *We have*

$$\frac{3^{dr} D_r}{N_r} < 1.61 \cdot 10^{39} e^{0.911r} \quad \textit{and} \quad \frac{(1/3) \cdots (r+1/3)}{r!} \frac{3^{dr} D_r}{N_r} < 1.176 \cdot 10^{40} e^{0.911r}.$$

Remark. These results are very close to best possible. Chudnovsky [5] has shown that $D_r \sim e^{\pi\sqrt{3}\,r/6} = e^{0.9068\ldots r}$ as $r \to \infty$.

Remark. We were able to calculate the D_r exactly for all $r \leq 2000$ (with two different methods using both Java and UBASIC 8.8). These actual values were equal to the values calculated using Proposition 3.2. This strengthens our belief that Proposition 3.2 captures the precise behaviour of the prime divisors of $D_{m,n,r}$ (at least for $m = 1, n = 3$).

Proof. We will establish both parts of this lemma via computation for r up to the point where Theorem 2.3 can be used to prove the lemma for all larger r.

(a) We computed the quantity on the right-hand side for all $r \leq 2000$, as part of the computation for part (b). We found that its minimum is $0.00501\ldots$, which occurs at $r = 13$.

From the second statement in Lemma 3.4(b), we know that if p is a prime congruent to 2 mod 3 with $(3r+4)/2 \leq p \leq 3r-1$, then $p | D_r$. Since we may now assume that $r > 2000$, we know that $(3r+4)/2 > 3000$.

From Theorem 2.3 and a bit of computation, for $x > 3000$, we find that $|\theta(x; 3, 2) - x/2| < 0.011x$, so the product of the primes congruent to 2 mod 3 in that interval is at least $e^{0.7r-1.511}$. Therefore, $D_r/4^r > e^{0.014r-1.511}$. Since $r^{1/6} = e^{(\log r)/6} < e^{0.0007r}$ for $r \geq 2000$, the desired result easily follows.

(b) Here the computation needs to include much larger values of r, so we need to proceed more carefully.

We break the computation into several parts.

(1) The computation of the factorial and factorial-like product on the left-hand side of the second inequality. We shall see below that the product of these terms grows quite slowly and they have a simple form, so this computation is both easy and fast.

(2) The computation of $3^{dr}/N_r$. From Lemma 3.5, we find that if $d = 0$ then $(3, N_r) = 1$, if $d = 1$ then $3^r | N_r$ and if $d = 3/2$ then $3^{r+v_3(r!)} | N_r$. For $d = 3/2$, one can often do better, by directly calculating the numerators of the coefficients of $X_r(1 - 3\sqrt{3}\,x)$, by means of equations (5.2)–(5.4) in the proof of Chudnovsky's Proposition 5.1 [5].

Directly calculating N_r is substantially more time-consuming than calculating $3^{r+v_3(r!)}$, so we always calculate $3^{r+v_3(r!)}$, continue with calculating D_r and only perform the direct calculation of N_r if the size of $3^{3r/2} D_r / 3^{r+v_3(r!)}$ warrants it.

(3) The computation of the contribution to D_r from the small primes, that is those less than $\sqrt[3]{3r}$, using Proposition 3.2.

To speed up this part of the calculation, and the following parts, the primes and their logarithms do not have to be recalculated for each r. Instead, we calculate and store the first million primes congruent to 2 mod 3 (the last one being 32,441,957) and their logarithms before we start the calculations for any of the r's.

(4) The computation of the contribution to D_r of all primes from $\sqrt[3]{3r}$ to $(3r - 1)/(3A(r) + 1)$ for some non-negative integer $A(r)$, which depends only on r. Again, we use Proposition 3.2 as well as the cached primes and their logarithms here.

(5) The computation of the contribution to D_r from the remaining larger primes.

From Lemma 3.4(b), we can see that for any non-negative integer A, the contribution to D_r from the primes satisfying (9) changes, as we increment r, by at most the addition of the log of one prime, if there is a prime congruent to 2 mod 3 between $3(r - 1)/(3A + 1)$ and $3r/(3A + 1)$, and the subtraction of another, if there is a prime congruent to 2 mod 3 between $3(r-1)/(3A+2)$ and $3r/(3A+2)$. This fact makes it very quick to compute the contribution from these intervals for r from the contribution from these intervals for $r - 1$ — much quicker than recomputing them directly. So we incorporate this strategy here: for each $i < A(r)$, we store the smallest and largest primes in these intervals along with the sum of the logarithms of the primes, $p \equiv 2 \bmod 3$, in these intervals.

Again, we use the cached primes and their logarithms for the intervals that lie within the cache.

In this manner, we proceeded to estimate the size of the required quantities for all $r \leq 200,000,000$. This computation took approximately 89,000 seconds.

The maximum of $3^{dr} D_r / \left(N_r e^{0.911r} \right)$ occurs at $r = 19,946$ and is less than $1.61 \cdot 10^{39}$, while the maximum of $(1/3) \cdots (r+1/3) 3^{dr} D_r / \left(N_r e^{0.911r} r! \right)$ also occurs at $r = 19,946$ and is less than $1.176 \cdot 10^{40}$.

For $r > 200 \cdot 10^6$, we can use the analytic estimates in Theorem 2.3.

From Lemma 3.5, we know that $3^{dr}N_r^{-1} \leq 3^{r/2-v_3(r!)}$. In addition, $r/2-v_3(r!) \leq (\log r)/(\log 3)+0.5$ and

$$\frac{(1/3)\cdots(r+1/3)}{r!} \leq \frac{4}{9}\exp\left(\int_1^r \frac{dx}{3x}\right) \leq \frac{4r^{1/3}}{9},$$

for $r \geq 1$, so

$$\frac{3^{dr}}{N_r} < 1.8r \quad \text{and} \quad \frac{3^{dr}}{N_r}\frac{(1/3)\cdots(r+1/3)}{r!} < 0.8r^{4/3}. \tag{10}$$

We divide the prime divisors of D_r into two sets, according to their size. We let $D_{r,s}$ denote the contribution to D_r from primes less than $(3r)^{1/3}$ and let $D_{r,l}$ denote the contribution from the remaining, larger, primes.

From Lemma 3.4(a), we know that

$$D_{r,s} \leq \prod_{\substack{p<(3r)^{1/3}\\ p\equiv 2 \bmod 3}} p^{\lfloor \log(3r)/(2\log(p))+1/2\rfloor}.$$

Now $\lfloor x/2+1/2\rfloor \leq 3/2\lfloor x/3\rfloor+1$, so

$$D_{r,s} \leq \exp\left\{\frac{3\psi\left(\sqrt[3]{3r};3,2\right)}{2}+\theta\left(\sqrt[3]{3r};3,2\right)\right\}.$$

From Theorem 2.3, and some calculation, we find that $\theta(x;3,2)$, $\psi(x;3,2) < 0.51x$, so

$$D_{r,s} < \exp\left(1.28\sqrt[3]{3r}\right). \tag{11}$$

From (10) and (11), we know that

$$\frac{D_{r,s}3^{dr}}{N_r} < e^{0.000006r} \quad \text{and} \quad \frac{D_{r,s}3^{dr}}{N_r}\frac{(1/3)\cdots(r+1/3)}{r!} < e^{0.000006r}, \tag{12}$$

for $r > 200\cdot 10^6$.

We next consider $D_{r,l}$.

From Lemma 3.4(b), we see that for any positive integer N satisfying $3r/(3N+2) \geq (3r)^{1/3}$, we have

$$D_{r,l} \leq \exp\left\{\sum_{A=0}^{N}\theta(3r/(3A+1);3,2)-\sum_{A=0}^{N-1}\theta(3r/(3A+2);3,2)\right\}.$$

Let $t_+(x)$ denote the maximum of 0.5000351 and $\theta(y;3,2)/y$ for all $y \geq x$ and let $t_-(x)$ denote the minimum of 0.4999649 and $\theta(y;3,2)/y$ for all $y \geq x$. With the choice $N=200$, we can write

$$D_{r,l} \leq \exp\left\{3r\left(\sum_{A=0}^{200}\frac{t_+(600\cdot 10^6/(3A+1))}{3A+1}-\sum_{A=0}^{199}\frac{t_-(600\cdot 10^6/(3A+2))}{3A+2}\right)\right\}$$

since $r > 200 \cdot 10^6$.

With Theorem 2.3(b), we calculate the necessary values of $t_+(x)$ and $t_-(x)$ and find that

$$D_{r,l} < e^{0.910993r},$$

for $r > 200 \cdot 10^6$.

Combining this inequality with (12) yields

$$\frac{3^{dr} D_r}{N_r} < e^{0.911r} \quad \text{and} \quad \frac{3^{dr}}{N_r} \frac{(1/3)\cdots(r+1/3)}{r!} D_r < e^{0.911r},$$

for $r > 200 \cdot 10^6$.

This completes the proof of the lemma. □

We now need to define our sequence of approximations to $(a/b)^{1/3}$ and find an upper bound on their size.

We start with bounds on the size of the polynomials.

Lemma 5.2. *Let m, n and r be positive integers with $m \le n/2$ and let z be any real number satisfying $0 \le z \le 1$. Then*

$$(1+z)^r \le Y_{m,n,r}(z) \le \left(1 + z^{1/2}\right)^{2r}. \tag{13}$$

Remark. The upper bound is best possible as can be seen by considering z near 0.

For hypergeometric applications, we are particularly interested in z near 1, where it appears that the upper bound could be sharpened to

$$4^{-r} \frac{(2r)!}{r!} \frac{\Gamma(1-m/n)}{\Gamma(r+1-m/n)} \left(1 + z^{1/2}\right)^{2r},$$

although we have been unable to prove this. This is an equality for $z = 1$. In the case of $m = 1$ and $n = 3$, this extra factor is about $0.8r^{-1/6}$.

Proof. We start by proving the upper bound.

We can write

$$\left(1 + z^{1/2}\right)^{2r} = \sum_{k=0}^{2r} \binom{2r}{k} z^{k/2} \quad \text{and}$$

$$Y_r(z) = \sum_{k=0}^{r} a_k z^k = \sum_{k=0}^{r} \binom{r}{k} \frac{(r-k+1+m/n)\cdots(r+m/n)}{(1-m/n)\cdots(k-m/n)} z^k.$$

We shall show that

$$\binom{r}{k} \frac{(r-k+1+m/n)\cdots(r+m/n)}{(1-m/n)\cdots(k-m/n)} z^k \le \binom{2r}{2k-1} z^{k-1/2} + \binom{2r}{2k} z^k.$$

This will prove that $Y_r(z) \le \left(1 + z^{1/2}\right)^{2r}$.

Since $0 \le z \le 1$, it suffices to show that

(14)
$$a_k = \binom{r}{k}\frac{(r-k+1+m/n)\cdots(r+m/n)}{(1-m/n)\cdots(k-m/n)} \le \binom{2r}{2k-1} + \binom{2r}{2k} = b_k.$$

We demonstrate this by induction.

For $k=0$, (14) holds since a_0 and b_0 are both equal to 1. So we can assume that (14) holds for some k.

Notice that

$$a_{k+1} = \frac{(r-k)(r-k+m/n)}{(k+1-m/n)(k+1)}a_k \quad \text{and}$$
$$b_{k+1} = \frac{(r-k)(2r-2k+1)}{(k+1)(2k+1)}b_k.$$

Thus
$$\frac{a_{k+1}}{b_{k+1}} = \frac{(r-k+m/n)}{(r-k+1/2)}\frac{(k+1/2)}{(k+1-m/n)}\frac{a_k}{b_k}.$$

Since $m \le n/2$, it is apparent that $(r-k+m/n)/(r-k+1/2) \le 1$ and that $(k+1/2)/(k+1-m/n) \le 1$. Since we have assumed that $a_k/b_k \le 1$, it is also true that $a_{k+1}/b_{k+1} \le 1$, which completes the proof of (14) and hence the upper bound for $Y_{m,n,r}(z)$.

To establish the lower bound, we again compare coefficients. It is clear that $a_0 = \binom{r}{0}$ and that $a_k \ge \binom{r}{k}$ for $1 \le k \le r$. Since $0 \le z \le 1$, the lower bound holds. □

Lemma 5.3. *Let r be a positive integer, a and b be positive integers with $b < a$. Put*

$$p_r = \frac{a^r D_r}{N_r}X_r(b/a) \quad \textit{and} \quad q_r = \frac{a^r D_r}{N_r}Y_r(b/a).$$

Then p_r and q_r are integers with $p_r q_{r+1} \ne p_{r+1} q_r$ and

(15)
$$\frac{D_r}{N_r}(a+b)^r \le q_r < 1.61 \cdot 10^{39}\left\{e^{0.911}3^{-d}\left(a^{1/2}+b^{1/2}\right)^2\right\}^r.$$

Proof. The first assertion is just a combination of our definitions of D_r and N_r along with an application of Lemma 3.5, while the second one is equation (16) in Lemma 4 of [2].

We now prove the upper bound for q_r.

From Lemma 5.2,

$$a^r Y_r(b/a) \le \left(a^{1/2}+b^{1/2}\right)^{2r}.$$

The upper bound for q_r now follows from Lemma 5.1(b).

The lower bound for q_r is an immediate consequence of the lower bound for $Y_{m,n,r}(z)$ in Lemma 5.2. □

The next lemma contains the relationship that allows the hypergeometric method to provide good sequences of rational approximations.

Lemma 5.4. *For any positive integers m, n and r with $(m,n) = 1$ and for any real number z satisfying $0 < z < 1$,*

$$z^{m/n} X_{m,n,r}(z) - Y_{m,n,r}(z) = (z-1)^{2r+1} R_{m,n,r}(z). \tag{16}$$

Proof. This is (4.2) of [5] with $\nu = m/n$. □

We next determine how close these approximations are to $(a/b)^{1/3}$.

Lemma 5.5. *Let a, b and r be positive integers with $b < a$. Then*

$$\frac{a-b}{200aq_r} < \left| q_r(a/b)^{1/3} - p_r \right| < \frac{1.176 \cdot 10^{40}(a-b)}{b} \left\{ e^{0.911} 3^{-d} \left(a^{1/2} - b^{1/2} \right)^2 \right\}^r. \tag{17}$$

Proof. Using our definitions of p_r, q_r and $R_r(z)$ and the equality expressed in Lemma 5.4, we find that

$$\begin{aligned} \left| q_r(a/b)^{1/3} - p_r \right| &= \frac{a^r D_r}{N_r} \left(\frac{a}{b} \right)^{1/3} \left(\frac{a-b}{a} \right)^{2r+1} \frac{(1/3)\cdots(r+1/3)}{(r+1)\cdots(2r+1)} \\ &\quad \times {}_2F_1\left(r+2/3, r+1; 2r+2; (a-b)/a \right). \end{aligned}$$

Since $(a-b)/a$ and the coefficients of this hypergeometric function are all positive, we have ${}_2F_1\left(r+2/3, r+1; 2r+2; (a-b)/a \right) > 1$.

Using the same arguments as in the proof of Lemma 5.1, we can also show that

$$\frac{(1/3)\cdots(r+1/3)}{(r+1)\cdots(2r+1)} > \frac{0.29}{4^r r^{1/6}}.$$

Combining these inequalities with the lower bound for q_r in Lemma 5.3, we obtain

$$\left| q_r(a/b)^{1/3} - p_r \right| > \left(\frac{D_r}{N_r} \right)^2 \frac{0.29(a-b)^{2r}(1+b/a)^r}{4^r r^{1/6}} \frac{a-b}{aq_r} \tag{18}$$

Recall that N_r is the greatest common factor of the numerators of the coefficients of $X_r\left(1-(a-b)z/a\right)$. Since $X_r(z)$ is a monic polynomial, $N_r \leq (a-b)^r$. The desired lower bound for $\left| q_r(a/b)^{1/3} - p_r \right|$ now follows from (18) and Lemma 5.1(a).

To obtain the upper bound, we apply Euler's integral representation for the hypergeometric function, we have

$$\begin{aligned} \left| q_r(a/b)^{1/3} - p_r \right| &= \frac{D_r a^r}{N_r} \left(1 - \frac{b}{a} \right)^{2r+1} \frac{(1/3)\cdots(r+1/3)}{r!} \left(\frac{a}{b} \right)^{1/3} \\ &\quad \times \left| \int_0^1 t^r (1-t)^r \left(1 - \frac{(a-b)t}{a} \right)^{-r-2/3} dt \right|. \end{aligned}$$

Easton (see the proof of his Lemma 8) showed that

$$\left| \int_0^1 t^r (1-t)^r \left(1 - \frac{(a-b)t}{a} \right)^{-r-2/3} dt \right| \leq (a/b)^{2/3} \left\{ a \left(a^{1/2} + b^{1/2} \right)^{-2} \right\}^r .$$

The lemma now follows from a little algebra and Lemma 5.1(b). □

6. A diophantine lemma

Finally, we state a lemma which will be used to determine an effective irrationality measure from these approximations.

Lemma 6.1. *Let $\theta \in \mathbb{R}$. Suppose that there exist $k_0, l_0 > 0$ and $E, Q > 1$ such that for all $r \in \mathbb{N}$, there are rational integers p_r and q_r with $|q_r| < k_0 Q^r$ and $|q_r \theta - p_r| \leq l_0 E^{-r}$ satisfying $p_r q_{r+1} \neq p_{r+1} q_r$. Then for any rational integers p and q with $p/q \neq p_i/q_i$ for any positive integer i and $|q| \geq 1/(2l_0)$ we have*

$$\left| \theta - \frac{p}{q} \right| > \frac{1}{c|q|^{\kappa+1}}, \text{ where } c = 2k_0(2l_0E)^{\kappa} \text{ and } \kappa = \frac{\log Q}{\log E}.$$

Proof. In the proof of Lemma 2.8 of [4], it is clearly noted that this is true. The extra Q which appears in the expression for c in the statement of Lemma 2.8 of [4] arises only from consideration of the case $p/q = p_i/q_i$ for some positive integer i. □

7. Proof of theorem 2.1

By the lower bound in Lemma 5.5, we need only prove Theorem 2.1 for those rational numbers $p/q \neq p_i/q_i$ for any positive integer i.

All that is required is a simple application of Lemma 6.1 using Lemmas 5.3 and 5.5 to provide the values of k_0, l_0, E and Q.

From these last two lemmas, we can choose $k_0 = 1.61 \cdot 10^{39}, l_0 = 1.176 \cdot 10^{40}(a-b)/b$, $E = e^{-0.911} 3^d \left(a^{1/2} - b^{1/2} \right)^{-2}$ and $Q = e^{0.911} \cdot 3^{-d} \left(a^{1/2} + b^{1/2} \right)^2$.

Lemma 5.3 assures us that $p_r q_{r+1} \neq p_{r+1} q_r$. In addition, $Q \geq e^{0.911} 3^{-1.5} (\sqrt{2}+1)^2 > 2.78 > 1$ and the condition $a > b$ shows that $l_0 > 0$. If $E > 1$ then we can use Lemma 6.1.

The quantity c in Lemma 6.1 is

$$3.22 \cdot 10^{39} \left\{ \frac{2.36 \cdot 10^{40} \cdot 3^d \left(a^{1/2} + b^{1/2} \right)}{e^{0.911} b \left(a^{1/2} - b^{1/2} \right)} \right\}^{\kappa}.$$

Under the assumptions that a and b are positive integers with $b < a, E > 1$ and $\kappa < 2$, one can show, by means of calculation and arguments from multivariable calculus, that $3^d e^{-0.911} (\sqrt{a} + \sqrt{b})/(b(\sqrt{a} - \sqrt{b})) < 1.822$, the maximum occurring for $a = 14$ and $b = 11$. So we can simplify the expression above, bounding it above by

$$3.22 \cdot 10^{39} \left(4.3 \cdot 10^{40}\right)^{\kappa}.$$

By the lower bound in Lemma 5.5 for the p_i/q_i's, we know that the c_1 in Theorem 2.1 will be a constant times a. Furthermore, we know that, with the exception of $a = 5$ and $b = 4$ (a case which we will return to below), $a \geq 6$ is required in order that $E > 1$ and $\kappa < 2$. So we can introduce a factor of $a/6$ into our expression for c above, obtaining

$$\begin{aligned} 3.22 \cdot 10^{39} \left(4.3 \cdot 10^{40}\right)^{\kappa} &< \frac{10^{40} a}{3.1 \cdot 6} \left(4.3 \cdot 10^{40}\right)^{\kappa} \\ &< 10^{40} a \left(\frac{4.3 \cdot 10^{40}}{\sqrt{18.6}} \right)^{\kappa} \\ &< 10^{40(\kappa+1)} a, \end{aligned}$$

since $\kappa < 2$.

This leaves the case of $a = 5$ and $b = 4$. We obtain a much better result for $\sqrt[3]{a/b}$ in the course of proving Corollary 2.2 below for $n = 10$, since $\sqrt[3]{5/4} = \sqrt[3]{10}/2$.

The condition that $E > 1$ (so that $a/2 < b < a$) along with Liouville's theorem shows that Theorem 2.1 is also true if $\kappa \geq 2$.

By these estimates and Lemma 6.1 we now know that Theorem 2.1 holds once $|q| \geq 1/(2l_0) > b/\left(2.36 \cdot 10^{40}(a-b)\right)$. There is a simple argument we can use to deal with q's of smaller absolute value.

If p/q did not satisfy (1), then $\left|(a/b)^{1/3} - p/q\right| < 1/\left(2q^2\right)$ would certainly hold and p/q would be a convergent in the continued fraction expansion of $(a/b)^{1/3}$.

Since $b < a$, it follows that $3b^{2/3} < a^{2/3} + (ab)^{1/3} + b^{2/3}$. As a consequence, $3b^{2/3}\left(a^{1/3} - b^{1/3}\right) < a - b$, or, more conveniently,

$$\left(\frac{a}{b}\right)^{1/3} - 1 = \frac{a^{1/3} - b^{1/3}}{b^{1/3}} < \frac{a-b}{3b}.$$

So we know that the continued fraction expansion of $(a/b)^{1/3}$ begins $[1; x, \ldots]$ where $x \geq \lfloor 3b/(a-b) \rfloor$. Therefore $p_0 = q_0 = 1$ (here p_0/q_0 is the 0-th convergent in the continued fraction expansion of $(a/b)^{1/3}$), while $q_1 \geq \lfloor 3b/(a-b) \rfloor$ and it is certainly true that $q_1 \geq b/(2.36 \cdot 10^{40}(a-b))$.

Hence $p/q = 1$, in which case $a/b \geq (b+1)/b$ and $E > 1$ imply that $(a/b)^{1/3} - 1 > 1/(4b) \geq 1/(8a)$ and (1) holds.

This completes the proof of the Theorem 2.1.

8. Proof of corollary 2.2

To prove Corollary 2.2, we first need to obtain obtain a lower bound for $|\sqrt[3]{n} - p/q|$ from the irrationality measure we have for the appropriate $\sqrt[3]{a/b}$. There are two different ways of doing this.

(i) If $\sqrt[3]{a/b}$ is of the form $s\sqrt[3]{n}/t$ then, from Theorem 2.1, we obtain

$$\left| \frac{s\sqrt[3]{n}}{t} - \frac{sp}{tq} \right| > \frac{1}{c_1 |tq|^{\kappa+1}}.$$

and, as a consequence,

$$\left| \sqrt[3]{n} - \frac{p}{q} \right| > \frac{1}{sc_1 t^{\kappa} |q|^{\kappa+1}}.$$

Let us look at the case of $n = 2$ to see how we proceed here. We have $a = 128, b = 125, s = 4$ and $t = 5$. From Theorem 2.1, we have $c_1 = 2 \cdot 10^{97}$ and $\kappa = 1.4321$, so

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{10^{-99}}{|q|^{2.4321}},$$

by the above reasoning.

We wrote a program in Java to calculate the first $500,000$ partial fractions and bounds from below the convergents in the continued-fraction expansion of $\sqrt[3]{2}$. For this, we used the algorithm described by Lang and Trotter [10] which uses only integer-arithmetic and does not require any truncated approximations to $\sqrt[3]{2}$.

The $500,000$-th convergent is greater than $10^{257,000}$ and it is easy to verify that

$$\frac{10^{-99}}{|q|^{2.4321}} > \frac{0.25}{|q|^{2.4325}}$$

for all q whose absolute value is larger than that. Thus, it only remains to check that the desired inequality is satisfied for all q whose absolute value is at most the denominator of the $500,000$-th convergent.

Rather than actually checking directly to see if

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{0.25}{|q|^{2.4325}}$$

held for all these the convergents of $\sqrt[3]{2}$, we simply looked at the partial fractions in the following way.

From the theory of continued-fractions, one can show that

$$\frac{1}{(a_{i+1}+2)\, q_i^2} < \left| \alpha - \frac{p_i}{q_i} \right|,$$

where a_{i+1} is the $i+1$-st partial fraction in the continued-fraction expansion of α while p_i/q_i is the i-th convergent.

As we see in Table 4, the largest partial fraction found for $\sqrt[3]{2}$ was $a_{484708} = 4,156,269$. Therefore, the corollary holds for $|q| > 9 \cdot 10^{13} > ((4156269+2)/4)^{(1/0.4325)}$. Now a direct check among the smaller convergents completes the proof of the corollary for $n = 2$ (the constant $c_2 = 0.25$ arises here).

We proceeded in the same way for n = 9, 10, 18, 19, 20, 22, 25, 28, 30, 36, 43, 44, 49, 57, 65, 66, 67, 68, 70, 76, 83 and 84.

(ii) The other possibility is that $\sqrt[3]{a/b}$ is of the form $s/(t\sqrt[3]{n})$. In this case, we use the fact that $|1/x - 1/y| = |(x-y)/(xy)|$ and find that

$$\left| \sqrt[3]{n} - \frac{q}{p} \right| > \frac{\sqrt[3]{n}}{sc_1|p|t^\kappa|q|^\kappa}.$$

We can assume that $|\sqrt[3]{n} - p/q| < 0.5$ and so

$$\left| \sqrt[3]{n} - \frac{q}{p} \right| > \frac{\sqrt[3]{n}}{sc_1 t^\kappa (\sqrt[3]{n}+1/2)^\kappa |p|^{\kappa+1}}.$$

We then proceed in the same way as in the previous case.

It is in this way that we prove the Corollary for n = 3, 4, 5, 6, 7, 11, 12, 13, 15, 17, 26, 31, 37, 39, 41, 42, 50, 52, 58, 60, 61, 62, 63, 78, 90, 91, 98 and 100.

n	a	b	i	a_i
2	$2\cdot 4^3$	5^3	484, 708	4, 156, 269
3	3^2	2^3	13, 628	738, 358
4	$2\cdot 4^3$	5^3	485, 529	8, 312, 539
5	239645788^3	$5\cdot 140145707^3$	266, 405	3, 494, 436
6	467^3	$6\cdot 257^3$	238, 114	466, 540
7	44^3	$7\cdot 23^3$	274, 789	12, 013, 483
9	9	2^3	97, 298	1, 063, 588
10	$5\cdot 13^3$	$4\cdot 14^3$	371, 703	1, 097, 381
11	25022^3	$11\cdot 11251^3$	217, 358	1, 352, 125
12	$9\cdot 29^3$	$4\cdot 38^3$	34, 767	1, 185, 798
13	57^3	$13\cdot 37^3$	55, 205	1, 406, 955
15	5^2	$3\cdot 2^3$	245, 733	1, 571, 507
17	18^3	$17\cdot 7^3$	169, 765	1, 536, 142
18	$9\cdot 29^3$	$4\cdot 38^3$	300, 238	3, 143, 844
19	$19\cdot 3^3$	8^3	138, 226	521, 398
20	$20\cdot 7^3$	19^3	72, 509	1, 840, 473
22	$11\cdot 5^3$	$4\cdot 7^3$	232, 141	595, 645
25	239645788^3	$5\cdot 140145707^3$	20, 862	2, 449, 303
26	3^3	26	252, 311	1, 722, 109
28	28	3^3	275, 575	1, 654, 773
30	10	9	228, 793	197, 558
31	22^3	$31\cdot 7^3$	205, 544	1, 643, 436
36	467^3	$6\cdot 257^3$	238, 549	2, 799, 247
37	10^3	$37\cdot 3^3$	494, 731	6, 591, 064
39	$39^2\cdot 2^3$	23^3	309, 275	483, 161
41	100^3	$41\cdot 29^3$	321, 697	417, 960, 093
42	49	$6\cdot 2^3$	408, 968	409, 489
43	$43\cdot 2^3$	7^3	227, 706	1, 359, 766
44	$44\cdot 2^3$	7^3	260, 709	370, 994
49	44^3	$7\cdot 23^3$	273, 736	1, 716, 211
50	$20\cdot 7^3$	19^3	54, 577	2, 055, 429
52	$2\cdot 2^3$	13	379, 989	3, 958, 641
57	$57\cdot 33^3$	127^3	110, 601	847, 651
58	$4\cdot 2^3$	29	172, 932	139, 963
60	$2\cdot 2^3$	15	44, 247	461, 876
61	4^3	61	76, 517	3, 405, 348
62	$4\cdot 2^3$	31	400, 816	330, 326
63	4^3	63	168, 229	2, 664, 200
65	65	4^3	183, 363	16, 950, 688
66	33	$4\cdot 2^3$	179, 933	589, 781
67	67	4^3	419, 845	937, 766
68	17	$2\cdot 2^3$	121, 095	1, 059, 335
70	35	$4\cdot 2^3$	376, 116	582, 245
76	$19\cdot 1111^3$	$2\cdot 2353^3$	300, 013	575, 574
78	47^3	$78\cdot 11^3$	421, 553	1, 145, 724
83	$83\cdot 58^3$	253^3	431, 244	434, 543
84	$84\cdot 33856^3$	148273^3	236, 330	5, 018, 560
90	$3\cdot 3^3$	$10\cdot 2^3$	43, 615	314, 175
91	9^3	$91\cdot 2^3$	123, 567	416, 579
98	28	3^3	274, 960	23, 166, 836
100	$5\cdot 13^3$	$4\cdot 14^3$	336, 362	1, 383, 591

TABLE 4. Data for the Proof of Corollary 2.2

References

[1] A. BAKER, *Rational approximations to certain algebraic numbers.* Proc. London. Math. Soc. (3) **14** (1964), 385–398.
[2] A. BAKER, *Rational approximations to* $\sqrt[3]{2}$ *and other algebraic numbers.* Quart. J. Math. Oxford **15** (1964), 375–383.
[3] M. BENNETT, *Explicit lower bounds for rational approximation to algebraic numbers.* Proc. London Math. Soc. **75** (1997), 63–78.
[4] JIANHUA CHEN, P. VOUTIER, *Complete solution of the diophantine equation* $X^2 + 1 = dY^4$ *and a related family of quartic Thue equations.* J. Number Theory **62** (1997), 71–99.
[5] G. V. CHUDNOVSKY, *The method of Thue-Siegel.* Annals of Math. **117** (1983), 325–383.
[6] D. EASTON, *Effective irrationality measures for certain algebraic numbers.* Math. Comp. **46** (1986), 613–622.
[7] N. I. FEL'DMAN, YU. V. NESTERENKO, *Number Theory IV: Transcendental Numbers.* Encyclopaedia of Mathematical Sciences **44**, Springer, 1998.
[8] A. HEIMONEN, *Effective irrationality measures for some values of Gauss hypergeometric function.* Department of Mathematics, University of Oulu preprint, March 1996.
[9] A. N. KOROBOV, *Continued fractions and Diophantine approximation.* Candidate's Dissertation, Moscow State University, 1990.
[10] S. LANG, H. TROTTER, *Continued-fractions for some algebraic numbers.* J. reine angew. Math. **255** (1972), 112–134.
[11] G. LETTL, A. PETHŐ, P. M. VOUTIER, *Simple Familiies of Thue Inequalities.* Trans. Amer. Math. Soc. **351** (1999), 1871–1894.
[12] E. M. NIKISHIN, *Arithmetic properties of the Markov function for the Jacobi weight.* Anal. Math. **8** (1982), 39–46.
[13] O. RAMARÉ, R. RUMELY, *Primes in arithmetic progressions.* Math. Comp. **65** (1996), 397–425.
[14] M. RUBINSTEIN, `http://pmmac03.math.uwaterloo.ca/~mrubinst/L_function_public/ZEROS`
[15] A. TOGBE, P. M. VOUTIER, P. G. WALSH, *Solving a family of Thue equations with an application to the equation* $x^2 - Dy^4 = 1$. Acta. Arith. **120** (2005), 39–58.

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