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An arithmetic formula of Liouville

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Résumé. Une preuve élémentaire est donnée d'une formule arithmétique qui fut présentée mais non pas prouvée par Liouville. Une application de cette formule donne une formule pour le nombre de représentations d'un nombre entier positif comme étant la somme de douze nombres triangulaires.

Abstract. An elementary proof is given of an arithmetic formula, which was stated but not proved by Liouville. An application of this formula yields a formula for the number of representations of a positive integer as the sum of twelve triangular numbers.

This paper is dedicated to the memory of Joseph Liouville (1809-1882).

1. Introduction

In his famous series of eighteen articles, Liouville [3] gave without proof numerous arithmetic formulae. These formulae are summarized in Dickson's *History of the Theory of Numbers* [1, Volume 2, Chapter XI], where references to proofs are given. The formula we are interested in involves a sum over sextuples (a, b, c, x, y, z) of positive odd integers satisfying $ax + by + cz = n$, where n is a fixed positive odd integer. It was stated without proof by Liouville in [3, sixth article, p. 331] and reproduced by Dickson [1, Volume 2, formula (M), p. 332].

Let n be a positive odd integer, and let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c))$$

$$= \frac{1}{8} \sum_{d \mid n} (d^2-1)F(d) - 3 \sum_{\substack{ax<n \\ a,x \text{ odd}}} \sigma(o(n-ax))F(a). \tag{1.1}$$

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In (1.1) $o(n)$ denotes the odd part of the positive integer n, that is $o(n) = n/2^s$, where $2^s \parallel n$. None of the authors referred to by Dickson give a proof of this result, although Nasimoff [5] indicates how an analytic proof can be given. Nor have we been able to locate a proof in the mathematical literature. In this paper we give what we believe to be the first proof of this result. Our proof is entirely elementary using nothing more than the rearrangement of terms in finite sums. Preliminary results are proved in Sections 2-5. Any omitted details can be found in [4]. The proof of (1.1) is given in Section 6. An application of (1.1) to triangular numbers is given in Section 7.

2. The equation $kx + ey = n$ and the quantities L_1, L_2, L_3, M_1, M_2, M_3

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $k, n \in \mathbb{N}$. In this section we are interested in the number of solutions $(x, e, y) \in \mathbb{N}^3$ of the equation

$$kx + ey = n, \tag{2.1}$$

where k, n are fixed positive integers and the unknowns x, e, y are required to satisfy certain inequalitites and certain congruences modulo 2. Throughout this paper all congruences are taken modulo 2. Thus, for example, the sum

$$\sum_{\substack{kx+ey=n \\ e<k \\ x\equiv a, e\equiv b, y\equiv c}} 1$$

counts the number of triples $(x, e, y) \in \mathbb{N}^3$ satisfying the equation $kx+ey = n$ with $e < k$ and $x \equiv a \pmod 2, e \equiv b \pmod 2, y \equiv c \pmod 2$, where a, b, c are fixed integers and k, n are fixed positive integers.

Definition 2.1.

$$L_1(a,b,c) = \sum_{\substack{kx+ey=n \\ e<k \\ x\equiv a, e\equiv b, y\equiv c}} 1, \quad L_2(a,b,c) = \sum_{\substack{kx+ey=n \\ e>k \\ x\equiv a, e\equiv b, y\equiv c}} 1,$$

$$L_3(a,b,c) = \sum_{\substack{kx+ey=n \\ e=k \\ x\equiv a, e\equiv b, y\equiv c}} 1, \quad M_1(a,b,c) = \sum_{\substack{kx+ey=n \\ x<y \\ x\equiv a, e\equiv b, y\equiv c}} 1,$$

$$M_2(a,b,c) = \sum_{\substack{kx+ey=n \\ x>y \\ x\equiv a, e\equiv b, y\equiv c}} 1, \quad M_3(a,b,c) = \sum_{\substack{kx+ey=n \\ x=y \\ x\equiv a, e\equiv b, y\equiv c}} 1.$$

The following result follows immediately from Definition 2.1.

Lemma 2.1.

$L_1(a,b,c) + L_2(a,b,c) + L_3(a,b,c) = M_1(a,b,c) + M_2(a,b,c) + M_3(a,b,c).$

Next we evaluate $L_3(a,b,c)$. (Recall that all congruences are taken modulo 2.)

Lemma 2.2.

$$L_3(a,b,c) = \begin{cases} 0, \ \textit{if } k \nmid n \textit{ or } k \mid n, k \not\equiv b \textit{ or } k \mid n, k \equiv b, \dfrac{n}{k} \not\equiv a+c, \\ \dfrac{1}{2}\left(\dfrac{n}{k} - 1\right) - \dfrac{1}{4}(-1)^a(1+(-1)^{n/k}), \ \textit{otherwise.} \end{cases}$$

Proof. From

$$L_3(a,b,c) = \sum_{\substack{kx+ey=n \\ e=k \\ x\equiv a, e\equiv b, y\equiv c}} 1 = \sum_{\substack{x+y=n/k \\ x\equiv a, k\equiv b, y\equiv c}} 1,$$

we easily obtain the asserted evaluation of $L_3(a,b,c)$.

$M_3(a,b,c)$ can be calculated similarly.

Lemma 2.3.

$$M_3(a,b,c) = \begin{cases} 0, \ \textit{if } \ a \not\equiv c \ \textit{or } \ a \equiv c \ \textit{and } n \not\equiv a(k+b), \\ \displaystyle\sum_{\substack{d\mid n, d>k \\ d\equiv k+b \\ n/d\equiv a}} 1, \ \textit{otherwise} \ . \end{cases}$$

Our next lemma gives a useful relationship between $L_2(a,b,c)$ and $M_2(a,b,c)$.

Lemma 2.4.

$$M_2(a,b,c) = L_2(a+c, k+b, c).$$

Proof. We have

$$M_2(a,b,c) = \sum_{\substack{kx+ey=n \\ x>y \\ x\equiv a, e\equiv b, y\equiv c}} 1 = \sum_{\substack{k(x+y)+ey=n \\ x\equiv a+c, \ e\equiv b, \ y\equiv c}} 1 = \sum_{\substack{kx+(k+e)y=n \\ x\equiv a+c, \ e\equiv b, \ y\equiv c}} 1$$

$$= \sum_{\substack{kx+ey=n \\ e>k \\ x\equiv a+c, \ e\equiv k+b, \ y\equiv c}} 1 = L_2(a+c, k+b, c),$$

as asserted.

For positive integers n and k we set

$$\delta(n,k) = \begin{cases} 1, & \text{if } k \mid n, \\ 0, & \text{if } k \nmid n. \end{cases}$$

The next result is a simple consequence of Lemmas 2.1, 2.2 and 2.3.

Lemma 2.5. *If k and n are positive odd integers then*

$$L_1(1,1,0) - M_1(1,1,0) = L_2(1,0,0) - L_2(1,1,0) - \left(\frac{(n/k)-1}{2}\right)\delta(n,k).$$

3. The equation $kx+ey=n$ and the quantities $R_1, R_2, R_3, S_1, S_2, S_3$

In this section we introduce the quantities R_1, R_2, R_3, S_1, S_2, S_3 formed from L_1, L_2, L_3, M_1, M_2, M_3 respectively by taking the sums over e rather than 1.

Definition 3.1.

$$R_1(a,b,c) = \sum_{\substack{kx+ey=n \\ e<k \\ x\equiv a, e\equiv b, y\equiv c}} e, \quad R_2(a,b,c) = \sum_{\substack{kx+ey=n \\ e>k \\ x\equiv a, e\equiv b, y\equiv c}} e,$$

$$R_3(a,b,c) = \sum_{\substack{kx+ey=n \\ e=k \\ x\equiv a, e\equiv b, y\equiv c}} e, \quad S_1(a,b,c) = \sum_{\substack{kx+ey=n \\ x<y \\ x\equiv a, e\equiv b, y\equiv c}} e,$$

$$S_2(a,b,c) = \sum_{\substack{kx+ey=n \\ x>y \\ x\equiv a, e\equiv b, y\equiv c}} e, \quad S_3(a,b,c) = \sum_{\substack{kx+ey=n \\ x=y \\ x\equiv a, e\equiv b, y\equiv c}} e.$$

We now give relationships between R_1, R_2, R_3, S_1, S_2, S_3. These relationships are very similar to those we saw in Section 2 for L_1, L_2, L_3, M_1, M_2, M_3. The following result follows immediately from Definition 3.1.

Lemma 3.1.

$$R_1(a,b,c) + R_2(a,b,c) + R_3(a,b,c) = S_1(a,b,c) + S_2(a,b,c) + S_3(a,b,c).$$

The sums $R_3(a,b,c)$ and $S_3(a,b,c)$ are easily evaluated.

Lemma 3.2.

$$R_3(a,b,c) = \begin{cases} 0, & \text{if } k \nmid n \text{ or } k \mid n, k \not\equiv b \text{ or } k \mid n, k\equiv b, \dfrac{n}{k} \not\equiv a+c, \\ \dfrac{1}{2}(n-k) - \dfrac{k}{4}(-1)^a(1+(-1)^{n/k}), & \text{otherwise.} \end{cases}$$

Lemma 3.3.

$$S_3(a,b,c) = \begin{cases} 0, & \text{if } a \not\equiv c \text{ or } a \equiv c \text{ and } n \not\equiv a(k+b), \\ \displaystyle\sum_{\substack{d\mid n, d>k \\ d\equiv k+b \\ n/d\equiv a}} (d-k), & \text{if } a\equiv c \text{ and } n \equiv a(k+b). \end{cases}$$

The following lemma gives a relationship between $R_2(a,b,c)$ and $S_2(a,b,c)$. The proof is similar to that of Lemma 2.4.

Lemma 3.4.

$$S_2(a,b,c) = R_2(a+c,k+b,c) - kL_2(a+e,k+b,c).$$

The values of $R_3(1,1,0)$ and $S_3(1,1,0)$ when k and n are both odd follow easily from Lemmas 3.2 and 3.3.

Lemma 3.5. *If k and n are positive odd integers then*

$$R_3(1,1,0) = \binom{\frac{n-k}{2}}{}\delta(n,k), \quad S_3(1,1,0) = 0.$$

4. Solutions of $kx + by + cz = n$ (k, n fixed odd integers)

Throughout this section k and n are restricted to be positive odd integers. We are concerned with solutions $(x,b,y,c,z) \in \mathbb{N}^5$ of

$$kx + by + cz = n$$

with b, c, x odd and y, z even. For such solutions we have $b+c \neq k$, $b-c \neq k$ and $x \neq z$, so that

$$\sum_{\substack{kx+by+cz=n \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 = \sum_{\substack{kx+by+cz=n \\ b+c<k \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1$$

and

$$\sum_{\substack{kx+by+cz=n \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 = \sum_{\substack{kx+by+cz=n \\ b-c<k,\ x<z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b-c>k,\ x<z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1$$
$$+ \sum_{\substack{kx+by+cz=n \\ b-c<k,\ x>z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b-c>k,\ x>z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1.$$

With this in mind, we find it useful to define the following sums.

Definition 4.1.

$$A_1 = \sum_{\substack{kx+by+cz=n \\ b+c<k \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1, \qquad A_2 = \sum_{\substack{kx+by+cz=n \\ b+c>k \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1.$$

$$B_1 = \sum_{\substack{kx+by+cz=n \\ b-c<k,\ x<z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1, \qquad B_2 = \sum_{\substack{kx+by+cz=n \\ b-c>k,\ x<z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1,$$

$$B_3 = \sum_{\substack{kx+by+cz=n \\ b-c<k,\ x>z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1, \qquad B_4 = \sum_{\substack{kx+by+cz=n \\ b-c>k,\ x>z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1.$$

Immediately from Definition 4.1 we obtain

Lemma 4.1.

$$A_1 + A_2 = B_1 + B_2 + B_3 + B_4.$$

It is also convenient to define another sum.

Definition 4.2.

$$C = \sum_{\substack{kx+by+cz=n \\ x<y<z \\ b,c,x \text{ odd, } y,z \text{ even}}} 1.$$

We now give various relations between A_1, A_2, B_1, B_2, B_3, B_4 and C with the ultimate goal of determining the quantity $A_1 - 2B_1 + 2C$, see Lemma 4.7.

Lemma 4.2.

$$B_2 = \sum_{\substack{kx+by+cz=n \\ y<x<z \\ b,c,x \text{ odd, } y,z \text{ even}}} 1.$$

Using Lemma 4.2 we obtain

Lemma 4.3.

$$2C = B_1 - \frac{1}{2} S_1(1,0,0).$$

Lemma 4.4.

$$B_4 = \sum_{\substack{kx+by+cz=n \\ y<z<x \\ b,c,x \text{ odd, } y,z \text{ even}}} 1.$$

Appealing to Lemmas 4.2 and 4.4 we obtain

Lemma 4.5.

$$B_2 + B_4 = B_3 - \frac{1}{2} S_2(1,0,0).$$

Lemma 4.6.

$$A_2 = 2B_3 + \frac{1}{2} R_2(1,0,0).$$

Proof. From Definition 4.1 we obtain

$$\begin{aligned} A_2 &= \sum_{\substack{kx+by+cz=n \\ b+c>k \\ b,c,x \text{ odd},\, y,z \text{ even}}} 1 \\ &= \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y>z \\ b,c,x \text{ odd},\, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y<z \\ b,c,x \text{ odd},\, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y=z \\ b,c,x \text{ odd},\, y,z \text{ even}}} 1 \\ &= 2 \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y>z \\ b,c,x \text{ odd},\, y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y=z \\ b,c,x \text{ odd},\, y,z \text{ even}}} 1, \end{aligned}$$

where we used the transformation $(b, y, c, z) \to (c, z, b, y)$ in the last step.

We now examine the above two sums. Firstly, we have

$$\begin{aligned} \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y=z \\ b,c,x \text{ odd},\, y,z \text{ even}}} 1 &= \sum_{\substack{kx+(b+c)y=n \\ b+c>k \\ b,c,x \text{ odd},\, y \text{ even}}} 1 \\ &= \sum_{e \text{ even}} \frac{e}{2} \sum_{\substack{kx+ey=n \\ e>k \\ x \text{ odd},\, y \text{ even}}} 1 \\ &= \frac{1}{2} \sum_{\substack{kx+ey=n \\ e>k \\ x\equiv 1, e\equiv 0, y\equiv 0}} e \\ &= \frac{1}{2} R_2(1,0,0), \end{aligned}$$

by Definition 3.1, and secondly,

$$\begin{aligned} \sum_{\substack{kx+by+cz=n \\ b+c>k,\ y>z \\ b,c,x \text{ odd},\, y,z \text{ even}}} 1 &= \sum_{\substack{kx+b(z+y')+cz=n \\ b+c>k \\ b,c,x \text{ odd},\, y',z \text{ even}}} 1 \qquad (y = z + y') \\ &= \sum_{\substack{kx+by+(b+c)z=n \\ b+c>k \\ b,c,x \text{ odd},\, y,z \text{ even}}} 1 \qquad (y = y') \\ &= \sum_{\substack{kx+by+ez=n \\ e=b+c, e>k \\ b,c,x \text{ odd},\, e,y,z \text{ even}}} 1 \end{aligned}$$

$$= \sum_{\substack{kx+by+ez=n \\ e>b, e>k \\ b,x \text{ odd}, \ e,y,z \text{ even}}} 1$$

$$= \sum_{\substack{kx+by+(k+c')z=n \\ k+c'>b \\ b,c',x \text{ odd}, \ y,z \text{ even}}} 1 \qquad (e = k + c')$$

$$= \sum_{\substack{k(x+z)+by+cz=n \\ b-c<k \\ b,c,x \text{ odd}, \ y,z \text{ even}}} 1 \qquad (c = c')$$

$$= \sum_{\substack{kx'+by+cz=n \\ b-c<k, x'>z \\ b,c,x' \text{ odd}, \ y,z \text{ even}}} 1 \qquad (x' = x + z)$$

$$= B_3,$$

by Definition 4.1. This completes the proof of the lemma.

Appealing to Lemmas 4.3, 4.1, 4.5 and 4.6, we obtain

Lemma 4.7.

$$A_1 - 2B_1 + 2C = -\frac{1}{2}R_2(1,0,0) - \frac{1}{2}S_1(1,0,0) - \frac{1}{2}S_2(1,0,0).$$

5. Solutions of $ax + by + cz = n$ (n fixed odd integer)

Throughout this section k and n are positive odd integers. We are interested in the solutions $(a, x, b, y, c, z) \in \mathbb{N}^6$ of the equation

$$ax + by + cz = n$$

with a, b, c, x, y, z odd and $a + b + c = k$ or $a + b - c = k$ or $a - b - c = k$. We denote the number of such solutions by U, V and W respectively.

Definition 5.1.

$$U = \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd}}} 1, \qquad V = \sum_{\substack{ax+by+cz=n \\ a+b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1, \qquad W = \sum_{\substack{ax+by+cz=n \\ a-b-c=k \\ a,b,c,x,y,z \text{ odd}}} 1.$$

In the next three lemmas we relate U, V, W to quantities of the previous three sections.

Lemma 5.1.

$$U = \frac{(k^2-1)}{8}\delta(n,k) + \frac{3}{2}(kL_1(1,1,0) - R_1(1,1,0)) + 3A_1.$$

Proof. We have

$$U = \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd}}} 1 = \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ x\neq y, y\neq z, z\neq x}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ \text{exactly 2 of } x,y,z \text{ equal}}} 1 + \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ x=y=z}} 1.$$

We begin by considering the first sum on the right hand side of the above expression. The condition $x \neq y, y \neq z, z \neq x$, comprises six possibilities, namely, $x < y < z$, $x < z < y$, $y < x < z$, $y < z < x$, $z < x < y$ and $z < y < x$. Simple transformations of the summation variables show that these six cases yield the same contribution to the sum. Thus

$$\begin{aligned}
\sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ x\neq y, y\neq z, z\neq x}} 1 &= 6 \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ x<y<z \\ a,b,c,x,y,z \text{ odd}}} 1 \\
&= 6 \sum_{\substack{(k-b-c)x+by+cz=n \\ b+c<k \\ x<y<z \\ b,c,x,y,z \text{ odd}}} 1 \\
&= 6 \sum_{\substack{kx+b(y-x)+c(z-x)=n \\ b+c<k \\ x<y<z \\ b,c,x,y,z \text{ odd}}} 1 \\
&= 6 \sum_{\substack{kx+by'+cz'=n \\ b+c<k \\ y'<z' \\ b,c,x \text{ odd},\ y',z' \text{ even}}} 1 \qquad (y' = y-x, z' = z-x) \\
&= \frac{6}{2} \sum_{\substack{kx+by+cz=n \\ b+c<k \\ y\neq z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 \\
&= 3 \sum_{\substack{kx+by+cz=n \\ b+c<k \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 - 3 \sum_{\substack{kx+by+cz=n \\ b+c<k \\ y=z \\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 \\
&= 3A_1 - 3 \sum_{\substack{kx+(b+c)y=n \\ b+c<k \\ b,c,x \text{ odd},\ y \text{ even}}} 1 \qquad \text{(by Definition 4.1)}
\end{aligned}$$

$$= 3A_1 - 3 \sum_{\substack{kx+ey=n \\ e<k \\ x \text{ odd}, y,e \text{ even}}} \frac{e}{2}$$

$$= 3A_1 - \frac{3}{2} R_1(1,0,0),$$

by Definition 3.1.

Next we consider the second sum on the right hand side of the expression for U. There are six different cases in which exactly two of x, y, and z are equal. Simple transformations of the summation variables show that the cases $x = y < z$, $x = z < y$, and $y = z < x$, lead to the same sum, and similarly for the cases $x < y = z$, $y < x = z$, and $z < y = x$. As above we obtain

$$\sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ \text{exactly 2 of } x,y,z \text{ equal}}} 1 = \frac{3}{2}\left(kL_1(1,1,0) - R_1(1,1,0) + R_1(1,0,0)\right).$$

Finally, we consider the third sum on the right hand side of the expression for U. We have

$$\sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd} \\ x=y=z}} 1 = \sum_{\substack{(a+b+c)x=n \\ a+b+c=k \\ a,b,c,x \text{ odd}}} 1 = \delta(n,k) \sum_{\substack{a+b+c=k \\ a,b,c \text{ odd}}} 1 = \delta(n,k)\left(\frac{k^2-1}{8}\right).$$

Putting these three sums together, we obtain

$$\begin{aligned} U &= 3A_1 - \frac{3}{2}R_1(1,0,0) + \frac{3}{2}\left(kL_1(1,1,0) - R_1(1,1,0) + R_1(1,0,0)\right) \\ &\quad + \delta(n,k)\left(\frac{k^2-1}{8}\right) \\ &= \frac{(k^2-1)}{8}\delta(n,k) + \frac{3}{2}(kL_1(1,1,0) - R_1(1,1,0)) + 3A_1, \end{aligned}$$

as asserted.

Lemma 5.2.

$$V = \frac{1}{2}(kM_1(1,1,0) + S_1(1,1,0)) + 2B_1.$$

Proof. We have

$$
\begin{aligned}
V = \sum_{\substack{ax+by+cz=n\\ a+b-c=k\\ a,b,c,x,y,z \text{ odd}}} 1 &= \sum_{\substack{ax+by+cz=n\\ a+b-c=k\\ x>y\\ a,b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{ax+by+cz=n\\ a+b-c=k\\ x<y\\ a,b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{ax+by+cz=n\\ a+b-c=k\\ x=y\\ a,b,c,x,y,z \text{ odd}}} 1 \\
&= 2 \sum_{\substack{ax+by+cz=n\\ a+b-c=k\\ x<y\\ a,b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{ax+by+cz=n\\ a+b-c=k\\ x=y\\ a,b,c,x,y,z \text{ odd}}} 1 \\
&= 2 \sum_{\substack{(k-b+c)x+by+cz=n\\ b-c<k\\ x<y\\ b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{(a+b)x+cz=n\\ a+b-c=k\\ a,b,c,x,z \text{ odd}}} 1 \\
&= 2 \sum_{\substack{kx+b(y-x)+c(z+x)=n\\ b-c<k\\ x<y\\ b,c,x,y,z \text{ odd}}} 1 + \sum_{\substack{ex+cz=n\\ e-c=k\\ e=a+b\\ a,b,c,x,z \text{ odd},\ e \text{ even}}} 1 \\
&= 2 \sum_{\substack{kx+by'+c(z+x)=n\\ b-c<k\\ b,c,x,z \text{ odd},\ y' \text{ even}}} 1 + \sum_{\substack{ex+cz=n\\ e=c+k\\ c,x,z \text{ odd},\ e \text{ even}}} \frac{e}{2} \\
&= 2 \sum_{\substack{kx+by+cz'=n\\ b-c<k,\ z'>x\\ b,c,x \text{ odd},\ y,z' \text{ even}}} 1 + \sum_{\substack{(c+k)x+cz=n\\ c,x,z \text{ odd}}} \frac{c+k}{2} \\
&= 2 \sum_{\substack{kx+by+cz=n\\ b-c<k,\ z>x\\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 + \sum_{\substack{kx+cy=n\\ y>x\\ c,x \text{ odd},\ y \text{ even}}} \frac{k+c}{2} \\
&= 2B_1 + \frac{k}{2} \sum_{\substack{kx+ey=n\\ x<y\\ e,x \text{ odd},\ y \text{ even}}} 1 + \frac{1}{2} \sum_{\substack{kx+ey=n\\ x<y\\ e,x \text{ odd},\ y \text{ even}}} e \\
&= 2B_1 + \frac{1}{2}(kM_1(1,1,0) + S_1(1,1,0)),
\end{aligned}
$$

as required.

Lemma 5.3.

$$W = \frac{1}{2} S_1(1,0,0) + 2C.$$

Proof. We have

$$
\begin{aligned}
W &= \sum_{\substack{ax+by+cz=n\\ a-b-c=k\\ a,b,c,x,y,z \text{ odd}}} 1\\
&= \sum_{\substack{(k+b+c)x+by+cz=n\\ b,c,x,y,z \text{ odd}}} 1\\
&= \sum_{\substack{kx+b(x+y)+c(x+z)=n\\ b,c,x,y,z \text{ odd}}} 1\\
&= \sum_{\substack{kx+by'+cz'=n\\ y'>x,\ z'>x\\ b,c,x \text{ odd},\ y',z' \text{ even}}} 1\\
&= \sum_{\substack{kx+by+cz=n\\ y>x,\ z>x,\ y<z\\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n\\ y>x,\ z>x,\ y>z\\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 + \sum_{\substack{kx+by+cz=n\\ y>x,\ z>x, y=z\\ b,c,x \text{ odd},\ y,z \text{ even}}} 1\\
&= 2 \sum_{\substack{kx+by+cz=n\\ x<y<z\\ b,c,x \text{ odd},\ y,z \text{ even}}} 1 + \sum_{\substack{kx+(b+c)y=n\\ y>x\\ b,c,x \text{ odd},\ y \text{ even}}} 1\\
&= 2C + \sum_{\substack{kx+ey=n\\ y>x\\ e=b+c\\ b,c,x \text{ odd},\ e,y \text{ even}}} 1 \qquad \text{(by Definition 4.2)}\\
&= 2C + \sum_{\substack{kx+ey=n\\ x<y\\ x \text{ odd},\ e,y \text{ even}}} \frac{e}{2}\\
&= 2C + \frac{1}{2} S_1(1,0,0),
\end{aligned}
$$

as claimed.

Lemma 5.4.

$$U - 3V + 3W = \frac{(k^2-1)}{8}\delta(n,k) - 3 \sum_{\substack{kx<n\\ x \text{ odd}}} \sigma(o(n-kx)).$$

Proof. Applying Lemmas 5.1, 5.2 and 5.3 we find

$$
\begin{aligned}
&U - 3V + 3W \\
&= \frac{(k^2-1)}{8}\delta(n,k) + \frac{3}{2}(kL_1(1,1,0) - R_1(1,1,0)) + 3A_1 \\
&\quad - 3\left(\frac{1}{2}(kM_1(1,1,0) + S_1(1,1,0)) + 2B_1\right) + 3\left(\frac{1}{2}S_1(1,0,0) + 2C\right) \\
&= \frac{(k^2-1)}{8}\delta(n,k) + \frac{3}{2}\Big(k(L_1(1,1,0) - M_1(1,1,0)) - R_1(1,1,0) \\
&\quad - S_1(1,1,0) + S_1(1,0,0)\Big) + 3(A_1 - 2B_1 + 2C).
\end{aligned}
$$

Appealing to Lemmas 2.5 and 4.7, we obtain

$$
\begin{aligned}
&U - 3V + 3W \\
&= \frac{(k^2-1)}{8}\delta(n,k) + \frac{3}{2}\Big(k(L_2(1,0,0) - L_2(1,1,0) - \left(\frac{(n/k)-1}{2}\right)\delta(n,k)) \\
&\quad - R_1(1,1,0) - S_1(1,1,0) + S_1(1,0,0)\Big) \\
&\quad - \frac{3}{2}\Big(R_2(1,0,0) + S_2(1,0,0) + S_1(1,0,0)\Big) \\
&= \frac{(k^2-1)}{8}\delta(n,k) + \frac{3}{2}\Big(- (S_2(1,0,0) + kL_2(1,1,0)) - (R_2(1,0,0) \\
&\quad - kL_2(1,0,0)) - \left(\frac{n-k}{2}\right)\delta(n,k) - R_1(1,1,0) - S_1(1,1,0)\Big).
\end{aligned}
$$

Appealing to Lemmas 3.4 and 3.5 we obtain

$$
\begin{aligned}
&U - 3V + 3W \\
&= \frac{(k^2-1)}{8}\delta(n,k) + \frac{3}{2}\Big(- R_2(1,1,0) - S_2(1,1,0) - R_3(1,1,0) \\
&\quad - R_1(1,1,0) - S_1(1,1,0)\Big) \\
&= \frac{(k^2-1)}{8}\delta(n,k) - \frac{3}{2}\Big(R_1(1,1,0) + R_2(1,1,0) + R_3(1,1,0) \\
&\quad + S_1(1,1,0) + S_2(1,1,0) + S_3(1,1,0)\Big) \\
&= \frac{(k^2-1)}{8}\delta(n,k) - 3\sum_{\substack{kx+ey=n \\ x\equiv 1, e\equiv 1, y\equiv 0}} e
\end{aligned}
$$

$$= \frac{(k^2-1)}{8}\delta(n,k) - 3\sum_{\substack{kx<n \\ x \text{ odd}}} \sum_{\substack{ey=n-kx \\ e \text{ odd, } y \text{ even}}} e$$

$$= \frac{(k^2-1)}{8}\delta(n,k) - 3\sum_{\substack{kx<n \\ x \text{ odd}}} \sum_{\substack{e|n-kx \\ e \text{ odd}}} e$$

as asserted.

6. Proof of Liouville's formula

We now use the results of Sections 2-5 to prove Liouville's formula.

Theorem 6.1. *Let n be a positive odd integer, and let $F : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c))$$

$$= \frac{1}{8}\sum_{d|n}(d^2-1)F(d) - 3\sum_{\substack{ax<n \\ a,x \text{ odd}}} \sigma(o(n-ax))F(a).$$

Proof. First we have

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c))$$

$$= \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a+b+c) + \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a-b-c)$$

$$- \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a+b-c) - \sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a-b+c).$$

Next we look at each of these sums individually. We have

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a+b+c) = \sum_{\substack{k\in\mathbb{N} \\ k \text{ odd}}} F(k) \sum_{\substack{ax+by+cz=n \\ a+b+c=k \\ a,b,c,x,y,z \text{ odd}}} 1 = \sum_{\substack{k\in\mathbb{N} \\ k \text{ odd}}} F(k)U,$$

and, as F is odd,

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a-b-c) = \sum_{\substack{k\in\mathbb{N} \\ k \text{ odd}}} F(k)\left(W-V\right),$$

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a+b-c) = \sum_{\substack{k\in\mathbb{N} \\ k \text{ odd}}} F(k)\left(V-W\right),$$

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} F(a-b+c) = \sum_{\substack{k\in\mathbb{N} \\ k \text{ odd}}} F(k)\,(V-W)\,.$$

Therefore by Lemma 5.4

$$\sum_{\substack{ax+by+cz=n \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c)+F(a-b-c)-F(a+b-c)-F(a-b+c))$$

$$= \sum_{\substack{k\in\mathbb{N} \\ k \text{ odd}}} F(k)(U-3V+3W)$$

$$= \sum_{\substack{k\in\mathbb{N} \\ k \text{ odd}}} F(k)\left(\frac{(k^2-1)}{8}\delta(n,k) - 3\sum_{\substack{kx<n \\ x \text{ odd}}} \sigma(o(n-kx))\right)$$

$$= \sum_{d|n} \frac{(d^2-1)}{8} F(d) - 3\sum_{\substack{ax<n \\ x,a \text{ odd}}} \sigma(o(n-ax))F(a),$$

as required.

In the next section we give an application of Theorem 6.1.

7. Sums of twelve triangular numbers

There are many applications of Theorem 6.1. We just give one application to triangular numbers. The triangular numbers are the nonnegative integers

$$T_k = \frac{1}{2}k(k+1), \qquad k\in\mathbb{N}\cup\{0\}.$$

Let

$$\Delta = \{T_k \mid k=0,1,2,3,\ldots\} = \{0,1,3,6,10,15,\ldots\},$$

and for $m\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$, let

$$\delta_m(n) = \operatorname{card}\{(t_1,\ldots,t_m)\in\Delta^m \mid n = t_1+\cdots+t_m\}.$$

Then $\delta_m(n)$ counts the number of representations of n as the sum of m triangular numbers. The evaluation of $\delta_4(n)$ was known to Legendre, namely

$$\delta_4(n) = \sigma(2n+1), \qquad n\in\mathbb{N}\cup\{0\}. \tag{7.1}$$

Ono, Robins and Wahl [6, p. 80] have proved this result using modular forms. An elementary proof has been given by Huard, Ou, Spearman and Williams [2, p. 259]. Using Theorem 6.1 and the above formula for $\delta_4(n)$, we obtain a new evaluation of $\delta_{12}(n)$.

Theorem 7.1. *Let n be a positive integer. Then*

$$\delta_{12}(n) = \frac{1}{192}\sigma_5(2n+3) - \frac{1}{192}\sigma_3(2n+3) - \frac{1}{8}\sum_{\substack{n_1+2^s n_2=2n+3 \\ s\geq 1,\ n_1,n_2 \text{ odd}}} \sigma_3(n_1)\sigma(n_2).$$

Proof. By (7.1) we have

$$\begin{aligned}\delta_{12}(n) &= \sum_{\substack{m_1+m_2+m_3=n \\ m_1,m_2,m_3\geq 0}} \delta_4(m_1)\delta_4(m_2)\delta_4(m_3) \\ &= \sum_{\substack{m_1+m_2+m_3=n \\ m_1,m_2,m_3\geq 0}} \sigma(2m_1+1)\sigma(2m_2+1)\sigma(2m_3+1) \\ &= \sum_{\substack{n_1+n_2+n_3=2n+3 \\ n_1,n_2,n_3 \text{ odd}}} \sigma(n_1)\sigma(n_2)\sigma(n_3).\end{aligned}$$

We apply Theorem 6.1 with $F(x) = x^3$, so that $F(a+b+c) + F(a-b-c) - F(a+b-c) - F(a-b+c) = 24abc$, and n replaced by $2n+3$. The left-hand side of Theorem 6.1 gives

$$\begin{aligned}&\sum_{\substack{ax+by+cz=2n+3 \\ a,b,c,x,y,z \text{ odd}}} (F(a+b+c)+F(a-b-c)-F(a+b-c)-F(a-b+c)) \\ &\qquad = \sum_{\substack{ax+by+cz=2n+3 \\ a,b,c,x,y,z \text{ odd}}} 24abc \\ &\qquad = 24 \sum_{\substack{n_1+n_2+n_3=2n+3 \\ n_1,n_2,n_3 \text{ odd}}} \sigma(n_1)\sigma(n_2)\sigma(n_3) \\ &\qquad = 24\delta_{12}(n).\end{aligned}$$

The right hand side gives

$$\begin{aligned}&\frac{1}{8}\sum_{d\mid 2n+3}(d^2-1)d^3 - 3\sum_{\substack{ax<2n+3 \\ a,x \text{ odd}}} \sigma(o(2n+3-ax))a^3 \\ &\qquad = \frac{1}{8}\sigma_5(2n+3) - \frac{1}{8}\sigma_3(2n+3) - 3\sum_{\substack{n_1+2^s n_2=2n+3 \\ s\geq 1,\ n_1,n_2 \text{ odd}}} \sigma(n_2)\sigma_3(n_1).\end{aligned}$$

Equating the left-hand side and the right-hand side, we obtain the required assertion.

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