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Binary quadratic forms and Eichler orders

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Résumé. Pour tout ordre d'Eichler $\mathcal{O}(D,N)$ de niveau N dans une algèbre de quaternions indéfinie de discriminant D, il existe un groupe Fuchsien $\Gamma(D,N) \subseteq \mathrm{SL}(2,\mathbb{R})$ et une courbe de Shimura $X(D,N)$. Nous associons à $\mathcal{O}(D,N)$ un ensemble $\mathcal{H}(\mathcal{O}(D,N))$ de formes quadratiques binaires ayant des coefficients semi-entiers quadratiques et developpons une classification des formes quadratiques primitives de $\mathcal{H}(\mathcal{O}(D,N))$ pour rapport à $\Gamma(D,N)$. En particulier nous retrouvons la classification des formes quadratiques primitives et entières de $\mathrm{SL}(2,\mathbb{Z})$. Un domaine fondamental explicite pour $\Gamma(D,N)$ permet de caractériser les $\Gamma(D,N)$ formes réduites.

Abstract. For any Eichler order $\mathcal{O}(D,N)$ of level N in an indefinite quaternion algebra of discriminant D there is a Fuchsian group $\Gamma(D,N) \subseteq \mathrm{SL}(2,\mathbb{R})$ and a Shimura curve $X(D,N)$. We associate to $\mathcal{O}(D,N)$ a set $\mathcal{H}(\mathcal{O}(D,N))$ of binary quadratic forms which have semi-integer quadratic coefficients, and we develop a classification theory, with respect to $\Gamma(D,N)$, for primitive forms contained in $\mathcal{H}(\mathcal{O}(D,N))$. In particular, the classification theory of primitive integral binary quadratic forms by $\mathrm{SL}(2,\mathbb{Z})$ is recovered. Explicit fundamental domains for $\Gamma(D,N)$ allow the characterization of the $\Gamma(D,N)$-reduced forms.

1. Preliminars

Let $H = \left(\frac{a,b}{\mathbb{Q}}\right)$ be the quaternion $\mathbb{Q}$-algebra of basis $\{1,i,j,ij\}$, satisfying $i^2 = a, j^2 = b, ji = -ij$, $a,b \in \mathbb{Q}^*$. Assume H is an indefinite quaternion algebra, that is, $H \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}(2,\mathbb{R})$. Then the discriminant D_H of H is the product of an even number of different primes $D_H = p_1 \cdots p_{2r} \geq 1$ and we can assume $a > 0$. Actually, a discriminant D determines a quaternion algebra H such that $D_H = D$ up to isomorphism. Let us denote by $\mathrm{n}(\omega)$ the reduced norm of $\omega \in H$.

Fix any embedding $\Phi : H \hookrightarrow \mathrm{M}(2,\mathbb{R})$. For simplicity we can keep in mind the embedding given at the following lemma.

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Lemma 1.1. *Let* $H = \left(\frac{a,b}{\mathbb{Q}}\right)$ *be an indefinite quaternion algebra with* $a > 0$*. An embedding* $\Phi : H \hookrightarrow \mathrm{M}(2, \mathbb{R})$ *is obtained by:*

$$\Phi(x + yi + zj + tij) = \begin{pmatrix} x + y\sqrt{a} & z + t\sqrt{a} \\ b(z - t\sqrt{a}) & x - y\sqrt{a} \end{pmatrix}.$$

Given $N \geq 1$, $\gcd(D, N) = 1$, let us consider an Eichler order of level N, that is a $\mathbb{Z}$-module of rank 4, subring of H, intersection of two maximal orders. By Eichler's results it is unique up to conjugation and we denote it by $\mathcal{O}(D, N)$.

Consider $\Gamma(D, N) := \Phi(\{\omega \in \mathcal{O}(D, N)^* \mid \mathrm{n}(\omega) > 0\}) \subseteq \mathrm{SL}(2, \mathbb{R})$ a group of quaternion transformations. This group acts on the upper complex half plane $\mathcal{H} = \{x + \iota y \in \mathbb{C} \mid y > 0\}$. We denote by $X(D, N)$ the canonical model of the Shimura curve defined by the quotient $\Gamma(D, N)\backslash\mathcal{H}$, cf. [Shi67], [AAB01].

For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{R})$ we denote by $\mathcal{P}(\gamma)$ the set of fixed points in $\mathbb{C}$ of the transformation defined by $\gamma(z) = \frac{az+b}{cz+d}$.

Let us denote by $\mathcal{E}(H, F)$ the set of embeddings of a quadratic field F into the quaternion algebra H. Assume there is an embedding $\varphi \in \mathcal{E}(H, F)$. Then, all the quaternion transformations in $\Phi(\varphi(F^*)) \subset \mathrm{GL}(2, \mathbb{R})$ have the same set of fixed points, which we denote by $\mathcal{P}(\varphi)$. In the case that F is an imaginary quadratic field it yields to complex multiplication points, since $\mathcal{P}(\varphi) \cap \mathcal{H}$ is just a point, $z(\varphi)$.

Now, we take in account the arithmetic of the orders. Let us consider the set of optimal embeddings of quadratic orders Λ into quaternion orders $\mathcal{O}$,

$$\mathcal{E}^*(\mathcal{O}, \Lambda) := \{\varphi \mid \varphi : \Lambda \hookrightarrow \mathcal{O},\ \varphi(F) \cap \mathcal{O} = \varphi(\Lambda)\}.$$

Any group $G \leq \mathrm{Nor}(\mathcal{O})$ acts on $\mathcal{E}^*(\mathcal{O}, \Lambda)$, and we can consider the quotient $\mathcal{E}^*(\mathcal{O}, \Lambda)/G$. Put $\nu(\mathcal{O}, \Lambda; G) := \sharp\, \mathcal{E}^*(\mathcal{O}, \Lambda)/G$. We will also use the notation $\nu(D, N, d, m; G)$ for an Eichler order $\mathcal{O}(D, N) \subseteq H$ of level N and the quadratic order of conductor m in $F = \mathbb{Q}(\sqrt{d})$, which we denote $\Lambda(d, m)$.

Since further class numbers in this paper will be related to this one, we include next theorem (cf. [Eic55]). It provides the well-known relation between the class numbers of local and global embeddings, and collects the formulas for the class number of local embeddings given in [Ogg83] and [Vig80] in the case $G = \mathcal{O}^*$. Consider ψ_p the multiplicative function given by $\psi_p(p^k) = p^k(1 + \frac{1}{p})$, $\psi_p(a) = 1$ if $p \nmid a$. Put $h(d, m)$ the ideal class number of the quadratic order $\Lambda(d, m)$.

Theorem 1.2. *Let* $\mathcal{O} = \mathcal{O}(D, N)$ *be an Eichler order of level* N *in an indefinite quaternion* $\mathbb{Q}$*-algebra* H *of discriminant* D*. Let* $\Lambda(d, m)$ *be the quadratic order of conductor* m *in* $\mathbb{Q}(\sqrt{d})$*. Assume that* $\mathcal{E}(H, \mathbb{Q}(\sqrt{d})) \neq \emptyset$

and $\gcd(m,D)=1$. *Then,*

$$\nu(D,N,d,m;\mathcal{O}^*) = h(d,m)\prod_{p|DN}\nu_p(D,N,d,m;\mathcal{O}^*).$$

The local class numbers of embeddings $\nu_p(D,N,d,m;\mathcal{O}^*)$, *for the primes* $p|DN$, *are given by*

(i) *If* $p|D$, *then* $\nu_p(D,N,d,m;\mathcal{O}^*) = 1-\left(\frac{D_F}{p}\right)$.

(ii) *If* $p \parallel N$, *then* $\nu_p(D,N,d,m;\mathcal{O}^*)$ *is equal to* $1+\left(\frac{D_F}{p}\right)$ *if* $p\nmid m$, *and equal to* 2 *if* $p|m$.

(iii) *Assume* $N=p^r u_1$, *with* $p\nmid u_1$, $r\geq 2$. *Put* $m=p^k u_2$, $p\nmid u_2$.

  (a) *If* $r\geq 2k+2$, *then* $\nu_p(D,N,d,m;\mathcal{O}^*)$ *is equal to* $2\psi_p(m)$ *if* $\left(\frac{D_F}{p}\right)=1$, *and equal to* 0 *otherwise.*

  (b) *If* $r=2k+1$, *then* $\nu_p(D,N,d,m;\mathcal{O}^*)$ *is equal to* $2\psi_p(m)$ *if* $\left(\frac{D_F}{p}\right)=1$, *equal to* p^k *if* $\left(\frac{D_F}{p}\right)=0$, *and equal to* 0 *if* $\left(\frac{D_F}{p}\right)=-1$.

  (c) *If* $r=2k$, *then* $\nu_p(D,N,d,m;\mathcal{O}^*) = p^{k-1}\left(p+1+\left(\frac{D_F}{p}\right)\right)$.

  (d) *If* $r\leq 2k-1$, *then* $\nu_p(D,N,d,m;\mathcal{O}^*)$ *is equal to* $p^{k/2}+p^{k/2-1}$ *if* k *is even, and equal to* $2p^{k-1/2}$ *if* k *is odd.*

2. Classification theory of binary forms associated to quaternions

Given $\alpha=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{M}(2,\mathbb{R})$, we put $f_\alpha(x,y) := cx^2+(d-a)xy-by^2$. It is called the binary quadratic form associated to α.

For a binary quadratic form $f(x,y) := Ax^2+Bxy+Cy^2 = (A,B,C)$, we consider the associated matrix $A(f)=\begin{pmatrix} A & B/2\\ B/2 & C\end{pmatrix}$, and the determinants $\det_1(f)=\det A(f)$ and $\det_2(f)=2^2\det A(f) = -(B^2-4AC)$. Denote by $\mathcal{P}(f)$ the set of solutions in $\mathbb{C}$ of $Az^2+Bz+C=0$. If f is (positive or negative) definite, then $\mathcal{P}(f)\cap\mathcal{H}$ is just a point which we denote by $\tau(f)$.

The proof of the following lemma is straightforward.

Lemma 2.1. *Let* $\alpha\in\mathrm{M}(2,\mathbb{R})$.

(i) *For all* $\lambda,\mu\in\mathbb{Q}$, *we have* $f_{\lambda\alpha}=\lambda f_\alpha$ *and* $f_{\alpha+\mu\,\mathrm{Id}} = f_\alpha$; *in particular,* $\mathcal{P}(f_{\lambda\alpha+\mu\,\mathrm{Id}}) = \mathcal{P}(f_\alpha)$.

(ii) $z\in\mathbb{C}$ *is a fixed point of* α *if and only if* $z\in\mathcal{P}(f_\alpha)$, *that is,* $\mathcal{P}(f_\alpha)=\mathcal{P}(\alpha)$.

(iii) *Let* $\gamma\in\mathrm{GL}(2,\mathbb{R})$. *Then* $A(f_{\gamma^{-1}\alpha\gamma}) = (\det\gamma^{-1})\gamma^t A(f_\alpha)\gamma$; *in particular, if* $\gamma\in\mathrm{SL}(2,\mathbb{R})$, $z\in\mathcal{P}(f_\alpha)$ *if and only if* $\gamma^{-1}(z)\in\mathcal{P}(f_{\gamma^{-1}\alpha\gamma})$.

Definition 2.2. For a quaternion $\omega\in H^*$, we define the binary quadratic form associated to ω as the binary quadratic form $f_{\Phi(\omega)}$.

Given a quaternion algebra H denote by H_0 the pure quaternions. By using lemma 2.1 it is enough to consider the binary forms associated to pure quaternions:

$$\mathcal{H}(a,b)=\{f_{\Phi(\omega)}:\omega\in H_0\},\quad \mathcal{H}(\mathcal{O})=\{f_{\Phi(\omega)}:\omega\in\mathcal{O}\cap H_0\}.$$

Definition 2.3. Let $\mathcal{O}$ be an order in a quaternion algebra H. We define the denominator $m_{\mathcal{O}}$ of $\mathcal{O}$ as the minimal positive integer such that $m_{\mathcal{O}}\cdot\mathcal{O}\subseteq\mathbb{Z}[1,i,j,ij]$. Then the ideal $(m_{\mathcal{O}})$ is the conductor of $\mathcal{O}$ in $\mathbb{Z}[1,i,j,ij]$.

Properties for these binary forms are collected in the following proposition, easy to be verified.

Proposition 2.4. *Consider an indefinite quaternion algebra* $H=\left(\frac{a,b}{\mathbb{Q}}\right)$, *and an order* $\mathcal{O}\subseteq H$. *Fix the embedding* Φ *as in lemma 1.1. Then:*

(i) *There is a bijective mapping* $H_0\to\mathcal{H}(a,b)$ *defined by* $\omega\mapsto f_{\Phi(\omega)}$. *Moreover* $\det_1(f_{\Phi(\omega)})=\mathrm{n}(\omega)$.

(ii) $$\begin{aligned}\mathcal{H}(a,b)&=\{(b(\lambda_2+\lambda_3\sqrt{a}),\,\lambda_1\sqrt{a},-\lambda_2+\lambda_3\sqrt{a})\mid\lambda_1,\lambda_2,\lambda_3\in\mathbb{Q}\}\\&=\{(b\beta',\alpha,-\beta)\mid\alpha,\beta\in\mathbb{Q}(\sqrt{a}),\,\mathrm{tr}(\alpha)=0\}.\end{aligned}$$

(iii) *the binary quadratic forms of* $\mathcal{H}(\mathcal{O})$ *have coefficients in* $\mathbb{Z}\left[\frac{1}{m_{\mathcal{O}}},\sqrt{a}\right]$

Given a quaternion order $\mathcal{O}$ and a quadratic order Λ, put

$$\mathcal{H}(\mathcal{O},\Lambda):=\{f\in\mathcal{H}(\mathcal{O}):\det_1(f)=-D_\Lambda\}.$$

Remark that an imaginary quadratic order yields to consider definite binary quadratic forms, and a real quadratic order yields to indefinite binary forms.

Given $\omega\in\mathcal{O}\cap H_0$, consider $F_\omega=\mathbb{Q}(\sqrt{d})$, $d=-\mathrm{n}(\omega)$. Then $\varphi_\omega(\sqrt{d})=\omega$ defines an embedding $\varphi_\omega\in\mathcal{E}(H,F_\omega)$. By considering $\Lambda_\omega:=\varphi_\omega^{-1}(\mathcal{O})\cap F_\omega$, we have $\varphi_\omega\in\mathcal{E}^*(\mathcal{O},\Lambda_\omega)$. Therefore, by construction, it is clear that $\mathcal{P}(f_{\Phi(\omega)})=\mathcal{P}(\Phi(\omega))=\mathcal{P}(\varphi_\omega)$. In particular, if we deal with quaternions of positive norm, we obtain definite binary forms, imaginary quadratic fields and a unique solution $\tau(f_{\Phi(\omega)})=z(\varphi_\omega)\in\mathcal{H}$. The points corresponding to these binary quadratic forms are in fact the complex multiplication points.

Theorem 4.53 in [AB04] states a bijective mapping $\mathfrak{f}$ from the set $\mathcal{E}(\mathcal{O},\Lambda)$ of embeddings of a quadratic order Λ into a quaternion order $\mathcal{O}$ onto the set $\mathcal{H}(\mathbb{Z}+2\mathcal{O},\Lambda)$ of binary quadratic forms associated to the orders $\mathbb{Z}+2\mathcal{O}$ and Λ. By using optimal embeddings, a definition of primitivity for the forms in $\mathcal{H}(\mathbb{Z}+2\mathcal{O},\Lambda)$ was introduced. We denote by $\mathcal{H}^*(\mathbb{Z}+2\mathcal{O},\Lambda)$ the corresponding subset of $(\mathcal{O},\Lambda)$-primitive binary forms. Then equivalence of embeddings yields to equivalence of forms.

Corollary 2.5. *Given orders* $\mathcal{O}$ *and* Λ *as above, for any* $G\subseteq\mathcal{O}^*$ *consider* $\Phi(G)\subseteq\mathrm{GL}(2,\mathbb{R})$. *There is a bijective mapping between* $\mathcal{E}^*(\mathcal{O},\Lambda)/G$ *and* $\mathcal{H}^*(\mathbb{Z}+2\mathcal{O},\Lambda)/\Phi(G)$.

Fix $\mathcal{O} = \mathcal{O}(D,N)$, $\Lambda = \Lambda(d,m)$ and $G = \mathcal{O}^*$. We use the notation $\mathrm{h}(D,N,d,m) := \sharp\mathcal{H}^*(\mathbb{Z}+2\mathcal{O}(D,N),\Lambda(d,m))/\Gamma_{\mathcal{O}^*}$. Thus, $\mathrm{h}(D,N,d,m) = \nu(D,N,d,m;\mathcal{O}^*)$, which can be computed explicitly by Eichler results (cf. theorem 1.2).

3. Generalized reduced binary forms

Fix an Eichler order $\mathcal{O}(D,N)$ in an indefinite quaternion algebra H. Consider the associated group $\Gamma(D,N)$ and the Shimura curve $X(D,N)$.

For a quadratic order $\Lambda(d,m)$, consider the set $\mathcal{H}^*(\mathbb{Z}+2\mathcal{O}(2p,N),\Lambda)$ of binary quadratic forms. As above, for a definite binary quadratic form $f = Ax^2+Bxy+Cy^2$, denote by $\tau(f)$ the solution of $Az^2+Bz+C=0$ in $\mathcal{H}$.

Definition 3.1. Fix a fundamental domain $\mathcal{D}(D,N)$ for $\Gamma(D,N)$ in $\mathcal{H}$. Make a choice about the boundary in such a way that every point in $\mathcal{H}$ is equivalent to a unique point of $\mathcal{D}(D,N)$. A binary form $f \in \mathcal{H}^*(\mathbb{Z}+2\mathcal{O}(D,N),\Lambda)$ is called $\Gamma(D,N)$-reduced form if $\tau(f) \in \mathcal{D}(D,N)$.

Theorem 3.2. *The number of positive definite* $\Gamma(D,N)$*-reduced forms in* $\mathcal{H}^*(\mathbb{Z}+2\mathcal{O}(D,N),\Lambda(d,m))$ *is finite and equal to* $h(D,N,d,m)$*.*

Proof. We can assume $d < 0$, in order $\mathcal{H}^*(\mathbb{Z}+2\mathcal{O}(D,N),\Lambda(d,m))$ consists on definite binary forms. By lemma 2.1 (iii), we have that $\Gamma(D,N)$-equivalence of forms yields to $\Gamma(D,N)$-equivalence of points. Note that $\tau(f) = \tau(-f)$, but f is not $\Gamma(D,N)$-equivalent to $-f$. Thus, in each class of $\Gamma(D,N)$-equivalence of forms there is a unique reduced binary form.

Consider $G = \{\omega \in \mathcal{O}^* \mid \mathrm{n}(\omega) > 0\}$ in order to get $\Phi(G) = \Gamma(D,N)$. The group G has index 2 in $\mathcal{O}^*$ and the number of classes of $\Gamma(D,N)$-equivalence in $\mathcal{H}^*(\mathbb{Z}+2\mathcal{O}(D,N),\Lambda(d,m))$ is $2\,\mathrm{h}(D,N,d,m)$. In that set, positive and negative definite forms were included, thus the number of classes of positive definite forms is exactly $\mathrm{h}(D,N,d,m)$. □

4. Non-ramified and small ramified cases

Definition 4.1. Let H be a quaternion algebra of discriminant D. We say that H is nonramified if $D=1$, that is $H \simeq \mathrm{M}(2,\mathbb{Q})$. We say H is small ramified if $D = pq$; in this case, we say it is of type A if $D = 2p$, $p \equiv 3 \mod 4$, and we say it is of type B if $D_H = pq$, $q \equiv 1 \mod 4$ and $\left(\frac{p}{q}\right) = -1$. It makes sense because of the following statement.

Proposition 4.2. *For $H = \left(\frac{p,q}{\mathbb{Q}}\right)$, p, q primes, exactly one of the following statements holds:*

(i) *H is nonramified.*
(ii) *H is small ramified of type A.*
(iii) *H is small ramified of type B.*

We are going to specialize above results for reduced binary forms for each one of these cases.

4.1. Nonramified case. Consider $H = \mathrm{M}(2, \mathbb{Q})$ and take the Eichler order

$$\mathcal{O}_0(1, N) := \left\{ \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix} \right) \mid a, b, c, d \in \mathbb{Z} \right\}.$$

Then $\Gamma(1, N) = \Gamma_0(N)$ and the curve $X(1, N)$ is the modular curve $X_0(N)$.

To unify results with the ramified case, it is also interesting to work with the Eichler order $\mathcal{O}(1, N) := \mathbb{Z}\left[1, \frac{j+ij}{2}, N\frac{(-j+ij)}{2}, \frac{1-i}{2}\right]$ in the nonramified quaternion algebra $\left(\frac{1,-1}{\mathbb{Q}}\right)$.

Proposition 4.3. *Consider the Eichler order $\mathcal{O} = \mathcal{O}_0(1, N) \subseteq \mathrm{M}(2, \mathbb{Q})$ and the quadratic order $\Lambda = \Lambda(d, m)$. Then:*

(i) $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}, \Lambda) \simeq \{f = (Na, b, c) \mid a, b, c \in \mathbb{Z}, \det_2(f) = -D_\Lambda\}$.
(ii) *The $(\mathcal{O}, \Lambda)$-primitivity condition is* $\gcd(a, b, c) = 1$.
(iii) *If $d < 0$, the number of $\Gamma_0(N)$-reduced positive definite primitive binary quadratic forms in $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}, \Lambda)$ is equal to $h(1, N, d, m)$.*

For $N = 1$, the well-known theory on reduced integer binary quadratic forms is recovered. In particular, the class number of $\mathrm{SL}(2, \mathbb{Z})$-equivalence is $h(d, m)$.

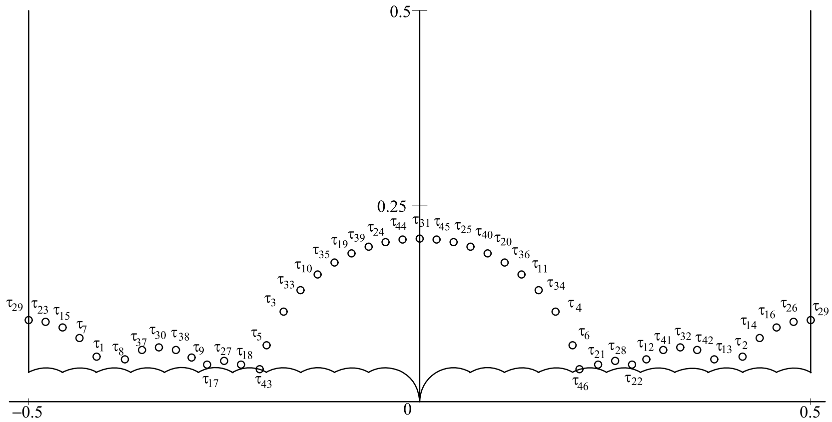
For $N > 1$, a general theory of reduced binary forms is obtained. For N equal to a prime, let us fix the symmetrical fundamental domain

$$\mathcal{D}(1, N) = \left\{ z \in \mathcal{H} \mid |\operatorname{Re}(z)| \leq 1/2, \left|z - \tfrac{k}{N}\right| > \tfrac{1}{N},\ k \in \mathbb{Z},\ 0 < |k| \leq \tfrac{N-1}{2} \right\}$$

given at [AB04]; a detailed construction can be found in [Als00]. Then a positive definite binary form $f = (Na, b, c)$, $a > 0$, is $\Gamma_0(N)$-reduced if and only if $|b| \leq Na$ and $|\tau(f) - \frac{k}{N}| > \frac{1}{N}$ for $k \in \mathbb{Z}$, $0 < |k| \leq \frac{N-1}{2}$. Figure 4.1 shows the 46 points corresponding to reduced binary forms in $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}_0(1, 23), \Lambda)$ for $D_\Lambda = 7, 11, 19, 23, 28, 43, 56, 67, 76, 83, 88, 91, 92$, which occurs in an special graphical position. In fact these points are exactly the special complex multiplication points of $X(1, 23)$, characterized by the existence of elements $\alpha \in \Lambda(d, m)$ of norm DN (cf. [AB04]). The table describes the $n = h(1, 23, d, m)$ inequivalent points for each quadratic order $\Lambda(d, m)$.

Note that for these symmetrical domains it is easy to implement an algorithm to decide if a form in this set is reduced or not, by using isometric circles.

FIGURE 4.1. The points $\tau(f)$ for some f reduced binary forms corresponding to quadratic orders $\Lambda(d, m)$ in a fundamental domain for $X(1, 23)$.



(d,m)	n	$\tau(f)$
(-7, 1)	2	$\left\{\tau_1 = \frac{-19+\sqrt{7}\iota}{46},\ \tau_2 = \frac{19+\sqrt{7}\iota}{46}\right\}$
(-7, 2)	2	$\left\{\tau_3 = \frac{-4+\sqrt{7}\iota}{23},\ \tau_4 = \frac{4+\sqrt{7}\iota}{23}\right\}$
(-11, 1)	2	$\left\{\tau_5 = \frac{-9+\sqrt{11}\iota}{46},\ \tau_6 = \frac{9+\sqrt{11}\iota}{46}\right\}$
(-14, 1)	8	$\left\{\tau_7 = \frac{-20+\sqrt{14}\iota}{46},\ \tau_8 = \frac{-26+\sqrt{14}\iota}{69},\ \tau_9 = \frac{-20+\sqrt{14}\iota}{69},\ \tau_{10} = \frac{-3+\sqrt{14}\iota}{23},\ \tau_{11} = \frac{3+\sqrt{14}\iota}{23},\ \tau_{12} = \frac{20+\sqrt{14}\iota}{69},\ \tau_{13} = \frac{26+\sqrt{14}\iota}{69},\ \tau_{14} = \frac{20+\sqrt{14}\iota}{46}\right\}$
(-19, 1)	2	$\left\{\tau_{15} = \frac{-21+\sqrt{19}\iota}{46},\ \tau_{16} = \frac{21+\sqrt{19}\iota}{46}\right\}$
(-19, 2)	6	$\left\{\tau_{17} = \frac{-25+\sqrt{19}\iota}{92},\ \tau_{18} = \frac{-21+\sqrt{19}\iota}{92},\ \tau_{19} = \frac{-2+\sqrt{19}\iota}{23},\ \tau_{20} = \frac{2+\sqrt{19}\iota}{23},\ \tau_{21} = \frac{21+\sqrt{19}\iota}{92},\ \tau_{22} = \frac{25+\sqrt{19}\iota}{92}\right\}$
(-22, 1)	4	$\left\{\tau_{23} = \frac{-22+\sqrt{22}\iota}{46},\ \tau_{24} = \frac{-1+\sqrt{22}\iota}{23},\ \tau_{25} = \frac{1+\sqrt{22}\iota}{23},\ \tau_{26} = \frac{22+\sqrt{22}\iota}{46}\right\}$
(-23, 1)	3	$\left\{\tau_{27} = \frac{-23+\sqrt{23}\iota}{92},\ \tau_{28} = \frac{23+\sqrt{23}\iota}{92},\ \tau_{29} = \frac{-23+\sqrt{23}\iota}{46} \sim \frac{23+\sqrt{23}\iota}{46}\right\}$
(-23, 2)	3	$\left\{\tau_{30} = \frac{-23+\sqrt{23}\iota}{69},\ \tau_{31} = \frac{\sqrt{23}\iota}{23},\ \tau_{32} = \frac{23+\sqrt{23}\iota}{69}\right\}$
(-43, 1)	2	$\left\{\tau_{33} = \frac{-7+\sqrt{43}\iota}{46},\ \tau_{34} = \frac{7+\sqrt{43}\iota}{46}\right\}$
(-67, 1)	2	$\left\{\tau_{35} = \frac{-5+\sqrt{67}\iota}{46},\ \tau_{36} = \frac{5+\sqrt{67}\iota}{46}\right\}$
(-83, 1)	6	$\left\{\tau_{37} = \frac{-49+\sqrt{83}\iota}{138},\ \tau_{38} = \frac{-43+\sqrt{83}\iota}{138},\ \tau_{39} = \frac{-3+\sqrt{83}\iota}{46},\ \tau_{40} = \frac{3+\sqrt{83}\iota}{46},\ \tau_{41} = \frac{43+\sqrt{83}\iota}{138},\ \tau_{42} = \frac{49+\sqrt{83}\iota}{138}\right\}$
(-91, 1)	4	$\left\{\tau_{43} = \frac{-47+\sqrt{91}\iota}{230},\ \tau_{44} = \frac{-1+\sqrt{91}\iota}{46},\ \tau_{45} = \frac{1+\sqrt{91}\iota}{46},\ \tau_{46} = \frac{47+\sqrt{91}\iota}{230}\right\}$

4.2. Small ramified case of type A. Let us consider $H_A(p) := \left(\frac{p,-1}{\mathbb{Q}}\right)$ and the Eichler order $\mathcal{O}_A(2p,N) := \mathbb{Z}\left[1, i, Nj, \frac{1+i+j+ij}{2}\right]$, for $N \mid \frac{p-1}{2}$, N square-free. The elements in the group $\Gamma_A(2p,N)$ are $\gamma = \frac{1}{2}\begin{pmatrix} \alpha & \beta \\ -\beta' & \alpha' \end{pmatrix}$ such that $\alpha, \beta \in \mathbb{Z}[\sqrt{p}]$, $\alpha \equiv \beta \equiv \alpha\sqrt{p} \mod 2$, $\det\gamma = 1$, $N | \left(\operatorname{tr}(\beta) - \frac{\beta-\beta'}{\sqrt{p}}\right)$. We denote by $X_A(2p,N)$ the Shimura curve of type A defined by $\Gamma_A(2p,N)$.

Proposition 4.4. *Consider the Eichler order $\mathcal{O}_A(2p,N)$ and the quadratic order $\Lambda = \Lambda(d,m)$.*

(i) *The set $\mathcal{H}(\mathbb{Z} + 2\mathcal{O}_A(2p,N), \Lambda)$ of binary forms is equal to*

$$\{f = (a + b\sqrt{p},\, 2c\sqrt{p},\, a - b\sqrt{p}) : \quad a,b,c \in \mathbb{Z},\ \ a \equiv b \equiv c \mod 2, \\ N \mid (a+b), \quad \det_1(f) = -D_\Lambda\}.$$

(ii) *The $(\mathcal{O}_A(2p,N), \Lambda)$-primitivity condition for these binary quadratic forms is* $\gcd\left(\frac{c+b}{2}, \frac{a+b}{2N}, b\right) = 1$.

(iii) *If $d < 0$, the number of $\Gamma_A(2p,N)$-reduced positive definite primitive binary forms in $\mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}_A(2p,N), \Lambda)$ is equal to $h(2p,N,d,m)$.*

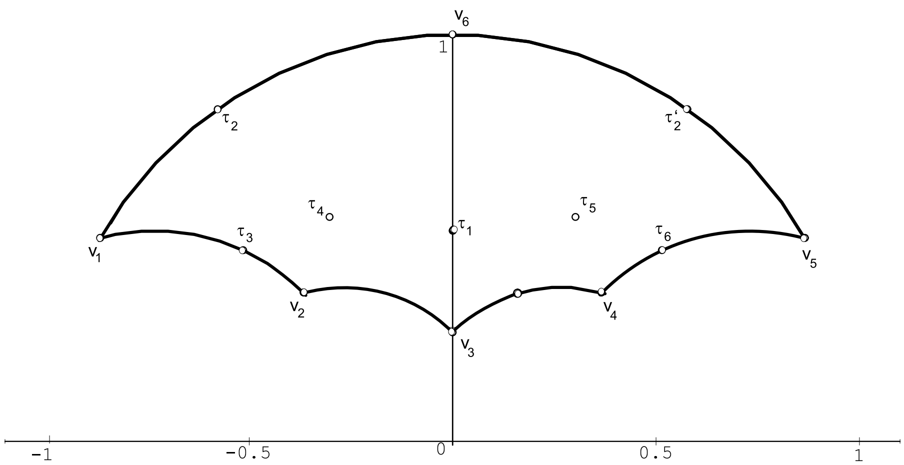
For example, consider the fundamental domain $\mathcal{D}(6,1)$ for the Shimura curve $X_A(6,1)$ in the Poincaré half plane defined by the hyperbolic polygon of vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ at figure 4.2 (cf. [AB04]). The table contains the corresponding reduced binary quadratic forms $f \in \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}_A(6,1), \Lambda(d,1))$ and the associated points $\tau(f)$ for $\det_1(f) = 4, 3, 24, 40$, that is $d = -1, -3, -6, -10$. Since the vertices are elliptic points of order 2 or 3, they are the associated points to forms of determinant 4 or 3, respectively. We put $n = h(6,1,d,1)$ the number of such reduced forms for each determinant.

4.3. Small ramified case of type B. Consider $H_B(p,q) := \left(\frac{p,q}{\mathbb{Q}}\right)$ and the Eichler order $\mathcal{O}_B(pq,N) := \mathbb{Z}\left[1, Ni, \frac{1+j}{2}, \frac{i+ij}{2}\right]$, where $N | \frac{q-1}{4}$, N square-free and $\gcd(N,p) = 1$. Then the group of quaternion transformations is

$$\Gamma_B(pq,N) = \left\{\gamma = \frac{1}{2}\begin{pmatrix} \alpha & \beta \\ q\beta' & \alpha' \end{pmatrix} : \ \alpha, \beta \in \mathbb{Z}[\sqrt{p}], \quad \alpha \equiv \beta \mod 2, \\ N \mid \frac{\alpha - \alpha' - \beta + \beta'}{2\sqrt{p}},\ \det\gamma = 1\right\}.$$

We denote by $X_B(pq,N)$ the corresponding Shimura curve of type B.

FIGURE 4.2. Reduced binary forms in a fundamental domain for $X_A(6,1)$.



$\det_1(f)$	n	f	$\tau(f)$
3	2	$(3+\sqrt{3})x^2 + 2\sqrt{3}xy + (3-\sqrt{3})y^2$	$v_2 = \frac{1-\sqrt{3}}{2}(1-\iota)$
		$(3+\sqrt{3})x^2 - 2\sqrt{3}xy + (3-\sqrt{3})y^2$	$v_4 = \frac{-1+\sqrt{3}}{2}(1-\iota)$
4	2	$4x^2 + 4\sqrt{3}xy + 4y^2$	$v_1 = \frac{-\sqrt{3}+\iota}{2} \sim v_3 \sim v_5$
		$2x^2 + 2y^2$	$v_6 = \iota$
24	2	$(6+2\sqrt{3})x^2 - (-6+2\sqrt{3})y^2$	$\tau_1 = \frac{(\sqrt{6}-\sqrt{2})\iota}{2}$
		$6x^2 - 4\sqrt{3}xy + 6y^2$	$\tau_2 = \frac{-\sqrt{3}+\sqrt{6}\iota}{3} \sim \tau_2'$
40	4	$(10+2\sqrt{3})x^2 + 8\sqrt{3}xy - (-10+2\sqrt{3})y^2$	$\tau_3 = \frac{3-5\sqrt{3}}{11} + \frac{5\sqrt{10}-\sqrt{30}}{22}\iota$
		$(8+2\sqrt{3})x^2 + 4\sqrt{3}xy - (-8+2\sqrt{3})y^2$	$\tau_4 = \frac{3-4\sqrt{3}}{13} + \frac{4\sqrt{10}-\sqrt{30}}{22}\iota$
		$(8+2\sqrt{3})x^2 - 4\sqrt{3}xy - (-8+2\sqrt{3})y^2$	$\tau_5 = \frac{-3+4\sqrt{3}}{13} + \frac{4\sqrt{10}-\sqrt{30}}{22}\iota$
		$(10+2\sqrt{3})x^2 - 8\sqrt{3}xy - (-10+2\sqrt{3})y^2$	$\tau_6 = \frac{-3+5\sqrt{3}}{11} + \frac{5\sqrt{10}-\sqrt{30}}{22}\iota$

Proposition 4.5. *Consider the Eichler order $\mathcal{O}_B(pq,N)$ in $H_B(p,q)$ and the quadratic order $\Lambda = \Lambda(d,m)$.*

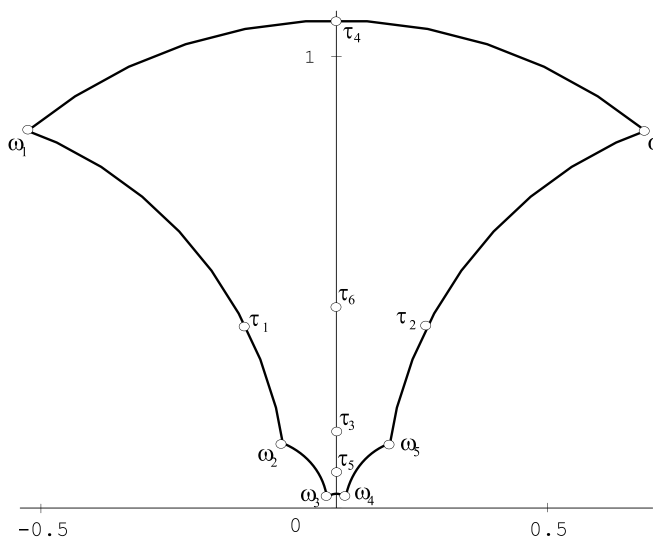
(i) *The set $\mathcal{H}(\mathbb{Z}+2\mathcal{O}_B(pq,N),\Lambda)$ of binary forms contains precisely the forms $f = (q(a+b\sqrt{p}),\ 2c\sqrt{p},\ -a+b\sqrt{p})$ where $a,b,c \in \mathbb{Z}$, $2N|(c-b)$ and $\det_1(f) = -D_\Lambda$.*

(ii) *The $(\mathcal{O}_B(pq,N),\Lambda)$-primitivity condition for these binary quadratic forms in (i) is $\gcd\left(a,b,\frac{c-b}{2N}\right) = 1$.*

(iii) *If $d<0$, the number of $\Gamma_B(pq,N)$-reduced positive definite primitive binary forms in $\mathcal{H}^*(\mathbb{Z}+2\mathcal{O}_B(pq,N),\Lambda)$ is equal to $h(pq,N,d,m)$.*

In figure 4.3 we show a fundamental domain for $\Gamma_B(10,1)$ given by the hyperbolic polygon of vertices $\{w_1,w_2,w_3,w_4,w_5,w_6\}$. All the vertices are elliptic points of order 3; thus they are the associated points to binary

FIGURE 4.3. Reduced binary forms in a fundamental domain for $X_B(10,1)$.



$\det_1(f)$	n	f	$\tau(f)$
3	4	$(-5+5\sqrt{2})x^2+2\sqrt{2}xy+(1+\sqrt{2})y^2$	$w_1 = \frac{-\sqrt{2}+\sqrt{3}\iota}{5(-1+\sqrt{2})} \sim w_3$
		$(5+5\sqrt{2})x^2+2\sqrt{2}xy+(-1+\sqrt{2})y^2$	$w_2 = \frac{-\sqrt{2}+\sqrt{3}\iota}{5(1+\sqrt{2})}$
		$(35+25\sqrt{2})x^2-2\sqrt{2}xy+(-7+5\sqrt{2})y^2$	$w_4 = \frac{\sqrt{2}+\sqrt{3}\iota}{5(7+5\sqrt{2})} \sim w_6$
		$(5+5\sqrt{2})x^2+2\sqrt{2}xy+(-1+\sqrt{2})y^2$	$w_5 = \frac{\sqrt{2}+\sqrt{3}\iota}{5(1+\sqrt{2})}$
8	2	$5\sqrt{2}x^2+2\sqrt{2}xy+\sqrt{2}y^2$	$\tau_1 = \frac{-1+2\iota}{5}$
		$(5\sqrt{2}x^2+2\sqrt{2}xy+\sqrt{2}y^2$	$\tau_2 = \frac{1+2\iota}{5}$
20	2	$(10+10\sqrt{2})x^2+(-2+2\sqrt{2})y^2$	$\tau_3 = \frac{(\sqrt{10}-\sqrt{5})\iota}{5}$
		$(-10+10\sqrt{2})x^2+(2+2\sqrt{2})y^2$	$\tau_4 = \frac{(\sqrt{10}+\sqrt{5})\iota}{5}$
40	2	$(40+30\sqrt{2})x^2+(-8+6\sqrt{2})y^2$	$\tau_5 = \frac{(3\sqrt{5}-2\sqrt{10})\iota}{5}$
		$10\sqrt{2}x^2+2\sqrt{2}y^2$	$\tau_6 = \frac{\sqrt{5}\iota}{5}$

forms of determinant 3. We also represent the points corresponding to reduced binary quadratic forms f with $\det_1(f) = 40$, which correspond to special complex points. The table also contains the explicit reduced definite positive binary forms and the corresponding points for determinants 8 and 20.

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