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Extremal values of Dirichlet L-functions in the half-plane of absolute convergence

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Résumé. On démontre que, pour tout θ réel, il existe une infinité de $s = \sigma + it$ avec $\sigma \to 1+$ et $t \to +\infty$ tel que

$$\mathrm{Re}\,\{\exp(i\theta)\log L(s,\chi)\} \geq \log \frac{\log\log\log t}{\log\log\log\log t} + O(1).$$

La démonstration est basée sur une version effective du théorème de Kronecker sur les approximations diophantiennes.

Abstract. We prove that for any real θ there are infinitely many values of $s = \sigma + it$ with $\sigma \to 1+$ and $t \to +\infty$ such that

$$\mathrm{Re}\,\{\exp(i\theta)\log L(s,\chi)\} \geq \log \frac{\log\log\log t}{\log\log\log\log t} + O(1).$$

The proof relies on an effective version of Kronecker's approximation theorem.

1. Extremal values

Extremal values of the Riemann zeta-function in the half-plane of absolute convergence were first studied by H. Bohr and Landau [1]. Their results rely essentially on the diophantine approximation theorems of Dirichlet and Kronecker. Whereas everything easily extends to Dirichlet series with real coefficients of one sign (see [7], §9.32) the question of general Dirichlet series is more delicate. In this paper we shall establish quantitative results for Dirichlet L-functions.

Let q be a positive integer and let χ be a Dirichlet character mod q. As usual, denote by $s = \sigma + it$ with $\sigma, t \in \mathbb{R}, i^2 = -1$, a complex variable. Then the Dirichlet L-function associated to the character χ is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the product is taken over all primes p; the Dirichlet series, and so the Euler product, converge absolutely in the half-plane $\sigma > 1$. Denote by

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χ_0 the principal character mod q, i.e., $\chi_0(n) = 1$ for all n coprime with q. Then

$$L(s,\chi_0) = \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right). \tag{1}$$

Thus we may interpret the well-known Riemann zeta-function $\zeta(s)$ as the Dirichlet L-function to the principal character χ_0 mod 1. Furthermore, it follows that $L(s,\chi_0)$ has a simple pole at $s = 1$ with residue 1. On the other side, any $L(s,\chi)$ with $\chi \neq \chi_0$ is regular at $s = 1$ with $L(1,\chi) \neq 0$ (by Dirichlet's analytic class number-formula). Since $L(s,\chi)$ is non-vanishing in $\sigma > 1$, we may define the logarithm (by choosing any one of the values of the logarithm). It is easily shown that for $\sigma > 1$

$$\log L(s,\chi) = \sum_p \sum_{k\geq 1} \frac{\chi(p)^k}{kp^{ks}} = \sum_p \frac{\chi(p)}{p^s} + O(1). \tag{2}$$

Obviously, $|\log L(s,\chi)| \leq L(\sigma,\chi_0)$ for $\sigma > 1$. However

Theorem 1.1. *For any $\epsilon > 0$ and any real θ there exists a sequence of $s = \sigma + it$ with $\sigma > 1$ and $t \to +\infty$ such that*

$$Re\,\{\exp(i\theta)\log L(s,\chi)\} \geq (1-\epsilon)\log L(\sigma,\chi_0) + O(1).$$

In particular,

$$\liminf_{\sigma>1,t\geq 1} |L(s,\chi)| = 0 \qquad \text{and} \qquad \limsup_{\sigma>1,t\geq 1} |L(s,\chi)| = \infty.$$

In spite of the non-vanishing of $L(s,\chi)$ the absolute value takes arbitrarily small values in the half-plane $\sigma > 1$!

The proof follows the ideas of H. Bohr and Landau [1] (resp. [8], §8.6) with which they obtained similar results for the Riemann zeta-function (answering a question of Hilbert). However, they argued with Dirichlet's *homogeneous* approximation theorem for growth estimates of $|\zeta(s)|$ and with Kronecker's *inhomogeneous* approximation theorem for its reciprocal. We will unify both approaches.

Proof. Using (2) we have for $x \geq 2$

$$\begin{aligned}\text{(3)}\quad &\mathrm{Re}\,\{\exp(i\theta)\log L(s,\chi)\}\\ &\geq \sum_{p\leq x}\frac{\chi_0(p)}{p^\sigma}\mathrm{Re}\,\{\exp(i\theta)\chi(p)p^{-it}\} - \sum_{p>x}\frac{\chi_0(p)}{p^\sigma} + O(1).\end{aligned}$$

Denote by $\varphi(q)$ the number of prime residue classes mod q. Since the values $\chi(p)$ are $\varphi(q)$-th roots of unity if p does not divide q, and equal to zero otherwise, there exist integers λ_p (uniquely determined mod $\varphi(q)$) with

$$\chi(p) = \begin{cases} \exp\left(2\pi i \frac{\lambda_p}{\varphi(q)}\right) & \text{if} \quad p \not| q, \\ 0 & \text{if} \quad p|q. \end{cases}$$

Hence,

$$\mathrm{Re}\left\{\exp(i\theta)\chi(p)p^{-it}\right\} = \cos\left(t \log p - 2\pi \frac{\lambda_p}{\varphi(q)} - \theta\right).$$

In view of the unique prime factorization of the integers the logarithms of the prime numbers are linearly independent. Thus, Kronecker's approximation theorem (see [8], §8.3, resp. Theorem 3.2 below) implies that for any given integer ω and any x there exist a real number $\tau > 0$ and integers h_p such that

$$\left|\frac{\tau}{2\pi}\log p - \frac{\lambda_p}{\varphi(q)} - \frac{\theta}{2\pi} - h_p\right| < \frac{1}{\omega} \qquad \text{for all} \quad p \leq x. \tag{4}$$

Obviously, with $\omega \to \infty$ we get infinitely many τ with this property. It follows that

$$\cos\left(\tau \log p - 2\pi \frac{\lambda_p}{\phi(q)} - \theta\right) \geq \cos\left(\frac{2\pi}{\omega}\right) \qquad \text{for all} \quad p \leq x, \tag{5}$$

provided that $\omega \geq 4$. Therefore, we deduce from (3)

$$\mathrm{Re}\left\{\exp(i\theta)\log L(\sigma + i\tau, \chi)\right\} \geq \cos\left(\frac{2\pi}{\omega}\right) \sum_{p \leq x} \frac{\chi_0(p)}{p^\sigma} - \sum_{p > x} \frac{\chi_0(p)}{p^\sigma} + O(1),$$

resp.

$$\begin{aligned} &\mathrm{Re}\left\{\exp(i\theta)\log L(\sigma + i\tau, \chi)\right\} \\ &\qquad \geq \cos\left(\frac{2\pi}{\omega}\right) \log L(\sigma, \chi_0) - 2\sum_{p > x} \frac{1}{p^\sigma} + O(1) \end{aligned} \tag{6}$$

in view of (2). Obviously, the appearing series converges. Thus, sending ω and x to infinity gives the inequality of Theorem 1.1. By (1) we have

$$\log L(\sigma, \chi_0) = \log\left(\frac{1}{\sigma - 1} + O(1)\right) = \log \frac{1}{\sigma - 1} + o(1) \tag{7}$$

for $\sigma \to 1+$. Therefore, with $\theta = 0$, resp. $\theta = \pi$, and $\sigma \to 1+$ the further assertions of the theorem follow. □

The same method applies to other Dirichlet series as well. For example, one can show that the Lerch zeta-function is unbounded in the half-plane of absolute convergence:

$$\limsup_{\sigma>1,t\geq 1} \sum_{n=0}^{\infty} \frac{\exp(2\pi i\lambda n)}{(n+\alpha)^s} = +\infty$$

if $\alpha > 0$ is transcendental; note that in the case of transcendental α the Lerch zeta-function has zeros in $\sigma > 1$ (see [3] and [4]).

In view of Theorem 1.1 we have to ask for quantitative estimates. Let $\pi(x)$ count the prime numbers $p \leq x$. By partial summation,

$$\sum_{x<p\leq y} \frac{1}{p^\sigma} = \frac{\pi(y)}{y^\sigma} - \frac{\pi(x)}{x^\sigma} + \sigma \int_x^y \frac{\pi(u)}{u^{\sigma+1}} \mathrm{d}u.$$

The prime number theorem implies for $x \geq 2$

$$\sum_{x<p\leq y} \frac{1}{p^\sigma} \sim \left(\frac{y^{1-\sigma}}{\log y} - \frac{x^{1-\sigma}}{\log x}\right) + \sigma \int_x^y \frac{\mathrm{d}u}{u^\sigma \log u}.$$

By the second mean-value theorem,

$$\int_x^y \frac{\mathrm{d}u}{u^\sigma \log u} \mathrm{d}u = \frac{1}{\log \xi} \int_x^y \frac{\mathrm{d}u}{u^\sigma} = \frac{x^{1-\sigma} - y^{1-\sigma}}{(\sigma-1)\log \xi}$$

for some $\xi \in (x,y)$. Thus, substituting ξ by x and sending $y \to \infty$, we obtain the estimate

$$\sum_{x<p} \frac{1}{p^\sigma} \leq (1+o(1)) \frac{x^{1-\sigma}}{(\sigma-1)\log x}$$

as $x \to \infty$. This gives in (6)

$$\text{(8)} \quad \mathrm{Re}\,\{\exp(i\theta) \log L(\sigma+i\tau,\chi)\} \\ \geq \cos\left(\frac{2\pi}{\omega}\right) \log L(\sigma,\chi_0) - (2+o(1)) \frac{x^{1-\sigma}}{(\sigma-1)\log x} + O(1).$$

Substituting (7) in formula (8) yields

$$\mathrm{Re}\,\{\exp(i\theta) \log L(\sigma+i\tau,\chi)\} \\ \geq (1+O(\omega^{-2})) \log \frac{1}{\sigma-1} - (2+o(1)) \frac{x^{1-\sigma}}{(\sigma-1)\log x} + O(1).$$

Let

$$x = \exp\left(\frac{1}{\sigma-1} \log \frac{1}{\sigma-1}\right),$$

then x tends to infinity as $\sigma \to 1+$. We obtain for x sufficiently large

$$\text{(9)} \qquad \mathrm{Re}\,\{\exp(i\theta) \log L(\sigma+i\tau,\chi)\} \geq (1+O(\omega^{-2})) \log \frac{\log x}{\log\log x} + O(1).$$

The question is how the quantities ω, x and τ depend on each other.

2. Effective approximation

H. Bohr and Landau [2] (resp. [8], §8.8) proved the existence of a τ with $0 \leq \tau \leq \exp(N^6)$ such that

$$\cos(\tau \log p_\nu) < -1 + \frac{1}{N} \qquad \text{for} \quad \nu = 1, \ldots, N,$$

where p_ν denotes the ν-th prime number. This can be seen as a first effective version of Kronecker's approximation theorem, with a bound for τ (similar to the one in Dirichlet's approximation theorem). In view of (5) this yields, in addition with the easier case of bounding $|\zeta(s)|$ from below, the existence of infinite sequences $s_\pm = \sigma_\pm + it_\pm$ with $\sigma_\pm \to 1+$ and $t_\pm \to +\infty$ for which

$$(10) \qquad |\zeta(s_+)| \geq A \log\log t_+ \qquad \text{and} \qquad \frac{1}{|\zeta(s_-)|} \geq A \log\log t_-,$$

where $A > 0$ is an absolute constant. However, for Dirichlet L-functions we need a more general effective version of Kronecker's approximation theorem. Using the idea of Bohr and Landau in addition with Baker's estimate for linear forms, Rieger [6] proved the remarkable

Theorem 2.1. *Let $v, N \in \mathbb{N}, b \in \mathbb{Z}, 1 \leq \omega, U \in \mathbb{R}$. Let $p_1 < \ldots < p_N$ be prime numbers (not necessarily consecutive) and*

$$u_\nu \in \mathbb{Z}, \quad 0 < |u_\nu| \leq U, \quad \beta_\nu \in \mathbb{R} \qquad \textit{for} \quad \nu = 1, \ldots, N.$$

Then there exist $h_\nu \in \mathbb{Z}, 0 \leq \nu \leq N$, and an effectively computable number $C = C(N, p_N) > 0$, depending on N and p_N only, with

$$(11) \qquad \left| h_0 \frac{u_\nu}{v} \log p_\nu - \beta_\nu - h_\nu \right| < \frac{1}{\omega} \qquad \textit{for} \quad \nu = 1, \ldots, N$$

and $b \leq h_0 \leq b + (2Uv\omega)^C$.

We need C explicitly. Therefore we shall give a sketch of Rieger's proof and add in the crucial step a result on an explicit lower bound for linear forms in logarithms due to Waldschmidt [9].

Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$ and denote by $L_\mathbb{K}$ the set of logarithms of the elements of $\mathbb{K} \setminus \{0\}$, i.e.,

$$L_\mathbb{K} = \{\ell \in \mathbb{C} : \exp(\ell) \in \mathbb{K}\}.$$

If a is an algebraic number with minimal polynomial $P(X)$ over $\mathbb{Z}$, then define the absolute logarithmic height of a by

$$h(a) = \frac{1}{D} \int_0^1 \log |P(\exp(2\pi i \phi))| \mathrm{d}\phi;$$

note that $h(a) = \log a$ for integers $a \geq 2$. Waldschmidt proved

Theorem 2.2. *Let $\ell_\nu \in L_{\mathbb{K}}$ and $\beta_\nu \in \mathbb{Q}$ for $\nu = 1, \ldots, N$, not all equal zero. Define $a_\nu = \exp(\ell_\nu)$ for $\nu = 1, \ldots, N$ and*

$$\Lambda = \beta_0 + \beta_1 \log a_1 + \ldots + \beta_N \log a_N.$$

Let E, W and $V_\nu, 1 \leq \nu \leq N$, be positive real numbers, satisfying

$$W \geq \max_{1 \leq \nu \leq N} \{h(\beta_\nu)\},$$

$$\frac{1}{D} \leq V_1 \leq \ldots \leq V_N,$$

$$V_\nu \geq \max\left\{h(a_\nu), \frac{|\log a_\nu|}{D}\right\} \qquad \textit{for} \quad \nu = 1, \ldots, N$$

and

$$1 < E \leq \min\left\{\exp(V_1), \min_{1 \leq \nu \leq N}\left\{\frac{4DV_\nu}{|\log a_\nu|}\right\}\right\}.$$

Finally, define $V_\nu^+ = \max\{V_\nu, 1\}$ for $\nu = N$ and $\nu = N-1$, with $V_1^+ = 1$ in the case $N = 1$. If $\Lambda \neq 0$, then

$$|\Lambda| > \exp\Big(- c(N) D^{N+2} (W + \log(EDV_N^+)) \log(EDV_{N-1}^+) \times$$
$$\times (\log E)^{-N-1} \prod_{\nu=1}^{N} V_\nu \Big)$$

with $c(N) \leq 2^{8N+51} N^{2N}$.

This leads to

Theorem 2.3. *With the notation of Theorem 2.1 and under its assumptions there exists an integer h_0 such that (11) holds and*

$$b \leq h_0 \leq b + 2 + ((3\omega U(N+2)\log p_N)^4 + 2)^{N+2} \times$$
$$\times \exp\left(2^{8N+51} N^{2N} (1 + 2\log p_N)(1 + \log p_{N-1}) \prod_{\nu=2}^{N} \log p_\nu\right); \tag{12}$$

if p_N is the N-th prime number, then, for any $\epsilon > 0$ and N sufficiently large,

$$b \leq h_0 \leq b + (\omega U)^{(4+\epsilon)N} \exp\left(N^{(2+\epsilon)N}\right). \tag{13}$$

Proof. For $t \in \mathbb{R}$ define

$$f(t) = 1 + \exp(t) + \sum_{\nu=1}^{N} \exp\left(2\pi i \left(t \frac{u_\nu}{v} \log p_\nu - \beta_\nu\right)\right).$$

With $\gamma_{-1} := 0, \beta_{-1} := 0, \gamma_0 := 1, \beta_0 := 0$ and $\gamma_\nu := \frac{u_\nu}{v} \log p_\nu, 1 \le \nu \le N$, we have

$$f(t) = \sum_{\nu=-1}^{N} \exp(2\pi i(t\gamma_\nu - \beta_\nu)). \tag{14}$$

By the multinomial theorem,

$$f(t)^k = \sum_{\substack{j_\nu \ge 0 \\ j_{-1}+\ldots+j_N=k}} \frac{k!}{j_{-1}! \cdots j_N!} \exp\left(2\pi i \sum_{\nu=-1}^{N} j_\nu(t\gamma_\nu - \beta_\nu)\right).$$

Hence, for $0 < B \in \mathbb{R}$ and $k \in \mathbb{N}$

$$\begin{aligned} J &:= \int_b^{b+B} |f(t)|^{2k} \mathrm{d}t \\ &= \sum_{\substack{j_\nu \ge 0 \\ j_{-1}+\ldots+j_N=k}} \frac{k!}{j_{-1}! \cdots j_N!} \sum_{\substack{j_\nu' \ge 0 \\ j'_{-1}+\ldots+j'_N=k}} \frac{k!}{j'_{-1}! \cdots j'_N!} \\ &\quad \int_b^{b+B} \exp\left(2\pi i \left(\sum_{\nu=-1}^{N} (j_\nu - j'_\nu)\gamma_\nu t - \sum_{\nu=-1}^{N} (j_\nu - j'_\nu)\beta_\nu\right)\right) \mathrm{d}t. \end{aligned}$$

By the theorem of Lindemann

$$\sum_{\nu=-1}^{N} (j_\nu - j'_\nu)\gamma_\nu$$

vanishes if and only if $j_\nu = j'_\nu$ for $\nu = -1, 0, \ldots, N$. Thus, integration gives

$$\int_b^{b+B} \exp\left(2\pi i \left(\sum_{\nu=-1}^{N} (j_\nu - j'_\nu)\gamma_\nu t - \sum_{\nu=-1}^{N} (j_\nu - j'_\nu)\beta_\nu\right)\right) \mathrm{d}t = B$$

if $j_\nu = j'_\nu, \nu = -1, 0, \ldots, N$, and

$$\left|\int_b^{b+B} \exp\left(2\pi i \left(\sum_{\nu=-1}^{N} (j_\nu - j'_\nu)\gamma_\nu t - \sum_{\nu=-1}^{N} (j_\nu - j'_\nu)\beta_\nu\right)\right) \mathrm{d}t\right| \le \frac{1}{\pi} \left|\sum_{\nu=-1}^{N} (j_\nu - j'_\nu)\gamma_\nu\right|^{-1}$$

if $j_\nu \ne j'_\nu$ for some $\nu \in \{-1, 0, \ldots, N\}$. In the latter case there exists by Baker's estimate for linear forms an effectively computable constant A such that

$$\left|\sum_{\nu=-1}^{N} (j_\nu - j'_\nu)\gamma_\nu\right|^{-1} < A.$$

Setting $\beta_0 = j_0 - j_0', \beta_\nu = \frac{u_\nu}{v}(j_\nu - j_\nu')$ and $a_\nu = p_\nu$ for $\nu = 1, \ldots, N$, we have, with the notation of Theorem 2.2,

$$\Lambda = \sum_{\nu=-1}^{N} (j_\nu - j_\nu')\gamma_\nu.$$

We may take $E = 1$, $W = \log p_N$, $V_1 = 1$ and $V_\nu = \log p_\nu$ for $\nu = 2, \ldots, N$. If $N \geq 2$, Theorem 2.2 gives

$$|\Lambda| > \exp\left(-2^{8N+51}N^{2N}(1 + 2\log p_N)(1 + \log p_{N-1})\prod_{\nu=2}^{N}\log p_\nu\right).$$

Thus we may take

$$A = \exp\left(2^{8N+51}N^{2N}(1 + 2\log p_N)(1 + \log p_{N-1})\prod_{\nu=2}^{N}\log p_\nu\right). \tag{15}$$

Hence, we obtain

$$J \geq B \sum_{\substack{j_\nu \geq 0 \\ j_{-1}+\ldots+j_N=k}} \left(\frac{k!}{j_{-1}!\cdots j_N!}\right)^2 - \frac{A}{\pi} \sum_{\substack{j_\nu \geq 0 \\ j_{-1}+\ldots+j_N=k}} \frac{k!}{j_{-1}!\cdots j_N!} \sum_{\substack{j_\nu' \geq 0 \\ j_{-1}'+\ldots+j_N'=k}} \frac{k!}{j_{-1}'!\cdots j_N'!}. \tag{16}$$

Since

$$\sum_{\substack{j_\nu \geq 0 \\ j_{-1}+\ldots+j_N=k}} 1 \leq (k+1)^{N+2},$$

application of the Cauchy Schwarz-inequality to the first multiple sum and of the multinomial theorem to the second multiple sum on the right hand side of (16) yields

$$\begin{aligned} J &\geq \left(\frac{B}{(k+1)^{N+2}} - \frac{A}{\pi}\right)\left(\sum_{\substack{j_\nu \geq 0 \\ j_{-1}+\ldots+j_N=k}} \frac{k!}{j_{-1}!\cdots j_N!}\right)^2 \\ &\geq \left(\frac{B}{(k+1)^{N+2}} - \frac{A}{\pi}\right)(N+2)^{2k}. \end{aligned}$$

Setting $B = A(k+1)^{N+2}$ and with $\tau \in [b, b+B]$ defined by

$$|f(\tau)| = \max_{t\in[b,b+B]} |f(t)|,$$

we obtain

$$\frac{B(N+2)^{2k}}{2(k+1)^{N+2}} \leq J \leq B|f(\tau)|^{2k}.$$

This gives

$$(17)\qquad |f(\tau)| > N + 2 - 2\mu, \qquad \text{where} \qquad \mu := \frac{(N+2)^2 \log k}{3k};$$

note that $\mu < 1$ for $k \geq 11$. By definition

$$f(t) = 1 + \exp(2\pi i(t\gamma_\nu - \beta_\nu)) + \sum_{\substack{m=0 \\ m\neq\nu}}^{N} \exp(2\pi i(t\gamma_m - \beta_m)).$$

Therefore, using the triangle inequality,

$$|f(t)| \leq N + |1 + \exp(2\pi i(\tau\gamma_\nu - \beta_\nu))| \qquad \text{for} \quad \nu = 0, \ldots, N,$$

and arbitrary $t \in \mathbb{R}$. Thus, in view of (17)

$$|1 + \exp(2\pi i(\tau\gamma_\nu - \beta_\nu))| > 2 - 2\mu \qquad \text{for} \quad \nu = 0, \ldots, N.$$

If h_ν denotes the nearest integer to $\tau\gamma_\nu - \beta_\nu$, then

$$|\tau\gamma_\nu - \beta_\nu - h_\nu| < \sqrt{\frac{\mu}{2}} \qquad \text{for} \quad \nu = 0, \ldots, N.$$

For $\nu = 0$ this implies $|\tau - h_0| < \sqrt{\mu}$. Replacing τ by h_0 yields

$$|h_0\gamma_\nu - \beta_\nu h_\nu| < \sqrt{\mu}\left(1 + \max_{\nu=1,\ldots,N} |\gamma_\nu|\right) \qquad \text{for} \quad \nu = 1, \ldots, N.$$

Putting $k = [(3wU(N+2)\log p_N)^4] + 1$ we get

$$b - 1 \leq h_0 \leq b + 1 + B = b + 1 + A([(3\omega U(N+2)\log p_N)^4] + 2)^{N+2}.$$

Substituting (15) and replacing $b-1$ by b, the assertion of Theorem 2.1 follows with the estimate (12) of Theorem 2.3; (13) can be proved by standard estimates. □

3. Quantitative results

We continue with inequality (9). Let p_N be the N-th prime. Then, using Theorem 2.3 with $N = \pi(x), v = u_\nu = 1$, and

$$\beta_\nu = \frac{\lambda_{p_\nu}}{\varphi(q)} + \frac{\theta}{2\pi} \qquad \text{for} \quad \nu = 1, \ldots, N,$$

yields the existence of $\tau = 2\pi h_0$ with

$$(18)\qquad b \leq \frac{\tau}{2\pi} \leq b + \omega^{(4+\epsilon)N} \exp(N^{(2+\epsilon)N})$$

such that (4) holds, as N and x tend to infinity. We choose $\omega = \log\log x$, then the prime number theorem and (18) imply

$$\log x = \log N + O(\log\log N), \quad \log N \geq \log\log\log\tau + O(\log\log\log\log\tau).$$

Substituting this in (9) we obtain

Theorem 3.1. *For any real θ there are infinitely many values of $s = \sigma + it$ with $\sigma \to 1+$ and $t \to +\infty$ such that*

$$Re\,\{\exp(i\theta)\log L(s,\chi)\} \geq \log \frac{\log\log\log t}{\log\log\log\log t} + O(1).$$

Using the Phragmén-Lindelöf principle, it is even possible to get quantitative estimates on the abscissa of absolute convergence. We write $f(x) = \Omega(g(x))$ with a positive function $g(x)$ if

$$\liminf_{x\to\infty} \frac{|f(x)|}{g(x)} > 0;$$

hence, $f(x) = \Omega(g(x))$ is the negation of $f(x) = o(g(x))$. Then, by the same reasoning as in [8], §8.4, we deduce

$$L(1+it,\chi) = \Omega\left(\frac{\log\log\log t}{\log\log\log\log t}\right),$$

and

$$\frac{1}{L(1+it,\chi)} = \Omega\left(\frac{\log\log\log t}{\log\log\log\log t}\right).$$

However, the method of Ramachandra [5] yields better results. As for the Riemann zeta-function (10) it can be shown that

$$L(1+it,\chi) = \Omega(\log\log t), \qquad \text{and} \qquad \frac{1}{L(1+it,\chi)} = \Omega(\log\log t),$$

and further that, assuming Riemann's hypothesis, this is the right order (similar to [8], §14.8). Hence, it is natural to expect that also in the half-plane of absolute convergence for Dirichlet L-functions similar growth estimates as for the Riemann zeta-function (10) should hold. We give a heuristical argument. Weyl improved Kronecker's approximation theorem by

Theorem 3.2. *Let $a_1, \ldots, a_N \in \mathbb{R}$ be linearly independent over the field of rational numbers, and let γ be a subregion of the N-dimensional unit cube with Jordan volume Γ. Then*

$$\lim_{T\to\infty} \frac{1}{T} meas\{\tau \in (0,T)\,:\,(a_1 t, \ldots, a_N t) \in \gamma \bmod 1\} = \Gamma.$$

Since the limit does not depend on translations of the set γ, we do not expect any *deep* influence of the inhomogeneous part to our approximation problem (4) (though it is a question of the speed of convergence). Thus, we may conjecture that we can find a suitable $\tau \leq \exp(N^c)$ with some positive constant c instead of (13), as in Dirichlet's *homogeneous* approximation theorem. This would lead to estimates similar to (10).

We conclude with some observations on the density of extremal values of $\log L(s,\chi)$. First of all note that if

$$|L(1+i\tau,\chi)|^{\pm 1}\geq f(T)$$

holds for a subset of values $\tau\in[T,2T]$ of measure μT, where $f(T)$ is any function which tends with T to infinity, then

$$\int_T^{2T}|L(1+it,\chi)|^{\pm 2}\mathrm{d}t\geq \mu T f(T)^2.$$

In view of well-known mean-value formulae we have $\mu=0$, which implies

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{m}eas\{\tau\in[0,T]\,:\,|L(\sigma+i\tau)|^{\pm 1}\geq f(T)\}=0.$$

This shows that the set on which extremal values are taken is rather thin.

The situation is different for fixed $\sigma>1$. Let Q be the smallest prime p for which $\chi_0(p)\neq 0$. Then

$$|\log L(s,\chi)|\leq \log L(\sigma,\chi_0)=Q^{-\sigma}\left(1+O\left(\left(\frac{Q}{Q+1}\right)^{\sigma}\right)\right);$$

note that the right hand side tends to $0+$ as $\sigma\to+\infty$, and that $Q\leq q+1$.

Theorem 3.3. *Let $0<\delta<\frac{1}{2}$. Then, for arbitrary θ and fixed $\sigma>1$,*

$$\liminf_{M\to\infty}\frac{1}{M}\sharp\{m\leq M:(1-\delta)\log L(\sigma,\chi_0)-Re\{\exp(i\theta)\log L(\sigma+2\pi i m,\chi)\}\}$$
$$\geq Q^{-2\sigma}\left(1+\frac{24}{\sigma}\right)\geq \delta^{2Q^2+8}(2Q)^{-8Q^2-32}\exp\left(-2^{3Q^2+51}Q^{4Q^2+2}\right).$$

Proof. We omit the details. First, we may replace (2) by

$$\left|\log L(s,\chi)-\sum_p\frac{\chi(p)}{p^s}\right|\leq\sum_{p,k\geq 2}\frac{\chi_0(p)}{kp^{k\sigma}}.$$

This gives with regard to (8)

$$\mathrm{Re}\,\{\exp(i\theta)\log L(\sigma+2\pi i m,\chi)\}\geq(1-\delta)\log L(\sigma,\chi_0)-2\frac{x^{1-\sigma}}{\sigma-1}-8\frac{Q^{2-2\sigma}}{2^{\sigma}(\sigma-1)}$$

for some integer $h_0=m$, satisfying (12), where $N=\pi(x)$ and $\cos\frac{2\pi}{\omega}=1-\delta$. Putting $x=Q^2$, proves (after some simple computation) the theorem. □

For example, if χ is a character with odd modulus q, then the quantity of Theorem 3.3 is bounded below by

$$\geq\frac{\delta^{16}}{2^{128}\exp\left(2^{81}\right)}.$$

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